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Universidade de São Paulo

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V.G. Bagrov _ , M.C. Baldiotti y , D.M. Gitman z and V.V. Shamshutdinova x

Instituto de Física, Universidade de São Paulo, CP 66.318 05315-970, São Paulo, SP, Brasil

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Darboux transformation for two-level systems

V.G. Bagrov[*], M.C. Baldiotti[†], D.M. Gitman[‡] and V.V. Shamshutdinova[§]

Instituto de Física, Universidade de São Paulo,
Caixa Postal 66318-CEP, 05315-970 São Paulo, S.P., Brazil

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Abstract

We develop the Darboux procedure for the case of two-level systems. In particular, it is demonstrated that one can construct a Darboux intertwinning operator which preserves the form of the equations defining the two-level system, and only transforms the interaction potential. We apply the obtained Darboux transformation to known exact solutions of certain two-level systems. Thus, we find two classes of new exact solutions and the corresponding interaction potentials.

1 Introduction

It is well-known that some complex quantum systems with a discrete energy spectrum are situated in some special dynamical configuration in which only two stationary states are important. To describe such systems one can use appropriate models with two-level energy spectra. In a number of important cases, these two-level models in a time-dependent background are based on the Schrödinger equation ($\hbar = c = 1$) in $0+1$ dimensions,

$$i\frac{d\Psi}{dt} = \hat{H}\Psi\,,\ \Psi(t) = \left(\begin{array}{c}\psi_1(t)\\ \psi_2(t)\end{array}\right), \tag{1}$$

where the Hamiltonian $\hat{H}$ reads

$$\hat{H} = \left(\begin{array}{cc}\varepsilon & f(t)\\ f(t) & -\varepsilon\end{array}\right), \tag{2}$$

with ε a constant and $f(t)$ a real function of time (in what follows, we call it the interaction potential or just the potential). In the sequel, the equation (1) with the Hamiltonian (2) is called the two-level system.

The dynamics of such two-level models in time-dependent backgrounds possesses a wide range of applications, e.g., in quantum optics and in the semi-classical theory of laser. These systems can be helpful to describe the behavior of molecule beams that cross a cavity immerse in a time dependent magnetic or electric field, as well as the behavior of an atom under the action of the electric field of a laser (see, for example, [1]). Another important example is the use of two-level systems to describe resonance absorption and nuclear induction experiments [3]. The two-level systems with periodic (quasi-periodic)

*On leave from Tomsk State University and Tomsk Institute of High Current Electronics, Russia, e-mail: bagrov@phys.tsu.ru

†e-mail: baldiott@fma.if.usp.br

‡e-mail: gitman@fma.if.usp.br

§Tomsk State University

potentials $f(t)$ were studied by several authors. They considered various approximation methods for finding solutions of the equation (1), e.g. perturbative expansions [4], the method of averaging [2], and the rotating wave approximation method. For a review of these and other methods see [5].

So far a few cases have been known for which two-level systems admit exact solutions, see the pioneer work of Rabi [6], where a spatially homogeneous and time-dependent external magnetic field is analyzed and the work [7], where exact solutions of two-level systems were found for the following specific potentials:

$$f(t) = \frac{r_0}{\cosh \tau}, \; \tau = \frac{t}{T}, \tag{3}$$

$$f(t) = \frac{r_0}{T} \tanh \tau + \frac{r_1}{T}, \tag{4}$$

where r_0, r_1 and T are real constants.

In some circumstances, there exists the possibility to construct new exact solutions of certain differential equations (in particular, of eigenvalue problems) with the help of the Darboux transformation method [8, 9]. The idea of the Darboux transformation method is to find an operator (an intertwining operator) that relates solutions which correspond to different potentials. Thus, if one knows solutions for a certain potential, and the Darboux transformation can be found, there exists the possibility to construct solutions for another potential and, at the same time, determine the explicit form of this potential. The method was applied for the first time by Darboux to find solutions of the Sturm-Liouville problem. Applications of the Darboux transformations to Schrödinger-type equations can be found in the survey [10]. For the generalization of the method to sets of differential equations see e.g. [11].

In the present article, we develop the Darboux procedure to the case of two-level systems (Sect.II). We demonstrate that one can construct the Darboux intertwining operator which preserves the form of the defining equations of the two-level system, only transforming the interaction potentials $f(t)$. Then (Sect.III) we apply the obtained Darboux transformation to known exact solutions of the two-level system. Thus, we find two classes of new solutions and the corresponding new potentials that allow such solutions.

2 Darboux transformations

2.1 General

Let us consider two linear operators $\hat{h}_0$ and $\hat{h}_1$, and let us suppose that there exists an operator $\hat{L}$ such that

$$\hat{L}\hat{h}_0 = \hat{h}_1 \hat{L}. \tag{5}$$

We call $\hat{L}$ the intertwining operator for the operators $\hat{h}_0$ and $\hat{h}_1$. In this case, an eigenvector Ψ of the eigenvalue problem

$$\hat{h}_0 \Psi = \varepsilon \Psi, \tag{6}$$

generates an eigenvector

$$\Phi = \hat{L}\Psi \tag{7}$$

(sometimes trivial) of the eigenvalue problem

$$\hat{h}_1 \Phi = \varepsilon \Phi. \tag{8}$$

If Φ is not trivial, then both eigenvectors correspond to the same eigenvalues ε.

In the case where $\hat{h}_0$ and $\hat{h}_1$ are differential operators and $\hat{L}$ is a first order differential operator, the relation (7) is called a Darboux transformation.

Let us suppose that Ψ and Φ are column matrices with n time-dependent components, and $\hat{h}_0$ and $\hat{h}_1$ are $n \times n$ matrices whose elements can only contain first order derivatives. In this case, we have two linear sets of n first-order differential equations. Let us consider $\hat{h}_0$ and $\hat{h}_1$ of the following form [11]

$$\hat{h}_0 = \gamma \frac{d}{dt} + V_0(t) \;, \; \hat{h}_1 = \gamma \frac{d}{dt} + V_1(t) \;, \tag{9}$$

where $V_0(t)$ and $V_1(t)$ are matrix potentials and γ is a constant $n \times n$ matrix. In this case, we can suppose the following form for the intertwining operator:

$$\hat{L} = A(t)\frac{d}{dt} + B(t)\,, \tag{10}$$

where $A(t)$ and $B(t)$ are some time dependent $n \times n$ matrices. Then the initial equation (5) for $\hat{L}$ is equivalent to the following sets of equations for $A(t)$ and $B(t)$,

$$A\gamma - \gamma A = 0\,, \tag{11}$$

$$\gamma\dot{A} + \gamma B - B\gamma + V_1 A - AV_0 = 0\,, \tag{12}$$

$$\gamma\dot{B} + V_1 B - A\dot{V}_0 - BV_0 = 0\,, \tag{13}$$

where the dot denotes differentiation with respect to t and the derivative of a matrix is the derivative of each of its elements. Choosing A as a non-singular matrix, we can use (12) to define V_1,

$$V_1 = \left(AV_0 + B\gamma - \gamma B - \gamma\dot{A}\right)A^{-1}\,. \tag{14}$$

By substituting the above expression in (13) we obtain,

$$\gamma\dot{B} + \left(AV_0 + B\gamma - \gamma B - \gamma\dot{A}\right)A^{-1}B - A\dot{V}_0 - BV_0 = 0\,, \tag{15}$$

where A remains an arbitrary non-singular matrix restricted by (11). The equation (15) can be linearized and integrated by means of the substitution

$$B = -A\dot{U}U^{-1}\,, \tag{16}$$

where U is an arbitrary non-singular matrix. As a result, we get the relation

$$AU\dot{\Lambda}U^{-1} = 0\,, \tag{17}$$

where the matrix Λ obeys the relation

$$\gamma\dot{U} + V_0 U = U\Lambda \Longleftrightarrow \hat{h}_0 U = U\Lambda\,. \tag{18}$$

We can satisfy the equation (17), and consequently (13), by setting Λ to be a given constant ($\dot{\Lambda} = 0$) matrix. Then (18) is the equation which defines the matrix U. The matrix B (16), and the transformed potential V_1 (14) are expressed now in terms of the matrix U, the matrix Λ, and the matrix A as follows:

$$B = A\gamma^{-1}\left(V_0 - U\Lambda U^{-1}\right)\,, \tag{19}$$

$$V_1 = A\left(\gamma^{-1}V_0\gamma + U\Lambda U^{-1} - \gamma^{-1}U\Lambda U^{-1}\gamma\right)A^{-1} - \gamma\dot{A}A^{-1}\,. \tag{20}$$

Let us select Λ to be Hermitian and diagonal. Thus,

$$\Lambda_{rs} = \delta_{rs}\lambda_{(s)}\,, \tag{21}$$

where all the $\lambda_{(s)}$ are real. In such a case we get n equations of the form (6) for the n-component columns $\Psi_{(1)}, \Psi_{(2)}, ..., \Psi_{(n)}$,

$$\hat{h}_0\Psi_{(s)} = \lambda_{(s)}\Psi_{(s)}\,,\ \Psi_{(s)} = \begin{pmatrix} U_{1s} \\ U_{2s} \\ \vdots \\ U_{ns} \end{pmatrix},\ s = 1, 2, ..., n\,. \tag{22}$$

Therefore, we arrive at a remarkable result: whenever one knows any n different levels $(\lambda_{(s)},\ \Psi_{(s)})$ of the eigenvalue problem (6), one can construct the matrix U as

$$U = \left(\begin{array}{cccc} \Psi_{(1)} & \Psi_{(2)} & \cdots & \Psi_{(n)} \end{array} \right) . \tag{23}$$

Once the matrix U is constructed, and the matrix A is fixed, the matrix B and the potential V_1 are defined by the equations (19) and (20). The intertwining operator $\hat{L}$ assumes the form

$$\hat{L} = A\gamma^{-1}\left(\hat{h}_0 - U\Lambda U^{-1}\right) . \tag{24}$$

Now, let us take Ψ to be an eigenvector of (6) with the eigenvalue ε. If $\varepsilon \neq \lambda_{(s)}$, then we find an eigenvector Φ of the eigenvalue problem (8) that corresponds to the eigenvalue ε as follows

$$\Phi = \hat{L}\Psi = A\gamma^{-1}\left(\varepsilon - U\Lambda U^{-1}\right)\Psi . \tag{25}$$

One can see that for $\lambda_{(s)} = \varepsilon$ the intertwining operator annihilates the corresponding eigenvector $\hat{L}\Psi^{(\varepsilon)} = 0$.

It is easily verified that the transformation $\tilde{\Psi} = A\Psi$ of the vector Ψ is also a solution of the problem, so that we can take, without loss of generality, $A = I$, where I is the $n \times n$ unit matrix.

The procedure described above can be used iteratively, so that solutions to a new problem, with a new potential V_2, can be constructed in the same way from the solutions of $\hat{h}_1$, and so forth.

2.2 Darboux transformations for two-level system

In this Subsection, we adopt the above described Darboux procedure to two-level systems. The Schrödinger equation (1), with Hamiltonian (2) implies the following pair of equations

$$i\dot{\psi}_2 + f\psi_2 = \varepsilon\psi_1\,,\ i\dot{\psi}_1 - f\psi_1 = \varepsilon\psi_2 \tag{26}$$

for two time-dependent functions ψ_1 and ψ_2. This set of equations can be written in the form (6) with the operator $\hat{h}_0$ given by

$$\hat{h}_0 = \gamma\frac{d}{dt} + V_0(t)\ ,\ \gamma = i\sigma_1\,,\ V_0 = i\sigma_2 f(t)\ , \tag{27}$$

where σ_i are the Pauli matrices. Let us look for a intertwining operator that relates $\hat{h}_0$ to an operator $\hat{h}_1$ of a similar form

$$\hat{h}_1 = \gamma\frac{d}{dt} + V_1(t)\ ,\ \gamma = i\sigma_1\,,\ V_1 = i\sigma_2 g(t)\ . \tag{28}$$

Which means that the matrix potentials V_0 and V_1 obey some algebraic restrictions and the Darboux transformation has to respect these restrictions. In other words, we are looking for Darboux transformations that do not change the form of the equation of the two-level system. The existence of such transformations is a nontrivial fact which we are going to prove below.

Choosing $A = I$ in (10) and substituting the potential V_1 from (28) into the equations (12) and (13), we get

$$\sigma_1 B - B\sigma_1 + \sigma_2(g - f) = 0\,, \tag{29}$$

$$\sigma_1\dot{B} + \sigma_2 Bg - \sigma_2\dot{f} - B\sigma_2 f = 0\,. \tag{30}$$

Let us chose

$$B = \alpha(t) + i\beta(t)\sigma_3\,, \tag{31}$$

where $\alpha(t)$ and $\beta(t)$ are some real functions. Then, it follows from (29) that

$$g = f - 2\beta\,. \tag{32}$$

Substituting (31) and (32) into (30), we obtain two real equations

$$\dot{\alpha}+2\beta\left(\beta-f\right)=0\,,\ \dot{\beta}-\dot{f}-2\alpha\beta=0\,. \tag{33}$$

These equations imply

$$\frac{d}{dt}\left(\alpha^2+\left(\beta-f\right)^2\right)=0\Longrightarrow\alpha^2+\left(\beta-f\right)^2=R^2\,, \tag{34}$$

where R is a real constant.

Note that the above expression is satisfied if we choose

$$\alpha=R\cos\mu\left(t\right)\ ,\ \beta=f\left(t\right)+R\sin\mu\left(t\right)\ , \tag{35}$$

with $\mu\left(t\right)$ a real function . Substituting Eqs. (35) into (33), we obtain for the function $\mu\left(t\right)$ a nontrivial transcendental differential equation

$$\dot{\mu}=2\left(R\sin\mu+f\right)\,. \tag{36}$$

In what follows, we are going to find the functions α and β independently, without the need to solve the equation (36). Thus, at the same time, we find in an indirect way solutions for this latter equation.

It follows from (23) that the matrix U can be construct as

$$U=\left(\begin{array}{cc}\Psi & \Phi\end{array}\right)\,,\ \Psi^{(\lambda_1)}=\left(\begin{array}{c}\psi_1\\ \psi_2\end{array}\right)\,,\ \Phi^{(\lambda_2)}=\left(\begin{array}{c}\varphi_1\\ \varphi_2\end{array}\right)\,, \tag{37}$$

where $\Psi^{(\lambda_1)}$ and $\Phi^{(\lambda_2)}$ are solutions of the system (26), that is,

$$i\dot{\psi}_2+f\psi_2=\lambda_{(1)}\psi_1\,,\ i\dot{\psi}_1-f\psi_1=\lambda_{(1)}\psi_2\,, \tag{38}$$

$$i\dot{\varphi}_2+f\varphi_2=\lambda_{(2)}\varphi_1\ ,\ i\dot{\varphi}_1-f\varphi_1=\lambda_{(2)}\varphi_2\,. \tag{39}$$

Using Eq. (37), we may find

$$\begin{aligned}U\Lambda U^{-1}=&\frac{\lambda_{(1)}+\lambda_{(2)}}{2}+\frac{\lambda_{(1)}-\lambda_{(2)}}{2\Delta}\left[\left(\psi_2\varphi_2-\psi_1\varphi_1\right)\sigma_1-\right.\\ &\left.i\left(\psi_1\varphi_1+\psi_2\varphi_2\right)\sigma_2+\left(\psi_1\varphi_2+\psi_2\varphi_1\right)\sigma_3\right]\,,\end{aligned} \tag{40}$$

where $\Delta=\psi_1\varphi_2-\psi_2\varphi_1$. Substituting the above expression in (20), with $A=I$, gives

$$V_1=-i\sigma_2 f+\frac{\lambda_{(1)}-\lambda_{(2)}}{\Delta}\left[\left(\psi_1\varphi_2+\psi_2\varphi_1\right)\sigma_3-i\left(\psi_1\varphi_1+\psi_2\varphi_2\right)\sigma_2\right]\,. \tag{41}$$

In order to maintain the specific form (28) of the potential V_1, we must choose

$$Q=\psi_1\varphi_2+\psi_2\varphi_1=0\,. \tag{42}$$

On the other hand, the equations (38) imply

$$i\dot{Q}=\left(\lambda_{(1)}+\lambda_{(2)}\right)\left(\psi_1\varphi_1+\psi_2\varphi_2\right)\,. \tag{43}$$

Thus, we obtain

$$\lambda_{(1)}=-\lambda_{(2)}=\lambda\,. \tag{44}$$

Besides, notice that if $\Psi^{(\varepsilon)}$ is a solution of (26) with eigenvalue ε, then $\Psi^{(-\varepsilon)}=\sigma_3\Psi^{(\varepsilon)}$ is a solution of the same equation with the eigenvalue $-\varepsilon$. Therefore,

$$\Psi^{(-\varepsilon)}=\left(\begin{array}{c}\psi_1^{(\varepsilon)}\\ -\psi_2^{(\varepsilon)}\end{array}\right)\,. \tag{45}$$

Thus,

$$\Psi^{(\lambda)} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \ \Phi^{(-\lambda)} = \begin{pmatrix} \varphi_1 = \psi_1 \\ \varphi_2 = -\psi_2 \end{pmatrix}, \ \Delta = -2\psi_1\psi_2 \,. \tag{46}$$

Using the above result, we obtain

$$V_1 = \left[\lambda\left(\frac{\psi_1}{\psi_2} - \frac{\psi_2}{\psi_1}\right) - f\right] i\sigma_2 \,, \tag{47}$$

$$B = \frac{i\lambda}{2}\left(\frac{\psi_1}{\psi_2} + \frac{\psi_2}{\psi_1}\right) + i\left[f - \frac{\lambda}{2}\left(\frac{\psi_1}{\psi_2} - \frac{\psi_2}{\psi_1}\right)\right]\sigma_3 \,. \tag{48}$$

Comparing the expressions (47) and (48) with (28) and (31), respectively, we obtain explicit expressions for the functions α, β, and g,

$$\alpha = \frac{i\lambda}{2}\left(\frac{\psi_1}{\psi_2} + \frac{\psi_2}{\psi_1}\right), \ \beta = f - \frac{\lambda}{2}\left(\frac{\psi_1}{\psi_2} - \frac{\psi_2}{\psi_1}\right), \ g = f - 2\beta \,. \tag{49}$$

Taking into account the relation (34), we obtain $\lambda^2 = -R^2$, which implies $\lambda = -iR$. Thus, we arrive at

$$\alpha = \frac{R}{2}\left(\frac{\psi_1}{\psi_2} + \frac{\psi_2}{\psi_1}\right), \ \beta = f + i\frac{R}{2}\left(\frac{\psi_1}{\psi_2} - \frac{\psi_2}{\psi_1}\right). \tag{50}$$

Finally, we represent the explicit form of the intertwining operator, remembering that the operator $\hat{L}$ (10) will be used to produce eigenfunctions of $\hat{h}_1$ from eigenfunctions of $\hat{h}_0$ only. We write this operator as

$$\hat{L} = \gamma^{-1}\left(\hat{h}_0 - U\Lambda U^{-1}\right) = \alpha + i(\beta - f)\sigma_3 - i\hat{h}_0\sigma_1 \,. \tag{51}$$

As was mentioned above, with the knowledge of α and β, we, in fact, can find a solution of the transcendental equation (36).

We note once again that the new potential V_1 (47) has the same matrix form as the potential V_0 (27). Hence, all the relations between (37) and (44) continue to be valid, and the same procedure can be applied to the potential V_1: one starts from one eigenfunction of $\hat{h}_1$, which determines an independent one by (45), and one obtains solutions for a new problem, with a new potential $V_2 = i\sigma_2 g$ with g given by (49). This procedure can continuously be applied to provide an entire family of potentials and its respective solutions.

3 New exact solutions for two-level systems

In the following, we will apply the formalism developed in the previous Section to two-level systems in order to obtain two types of new solutions. Namely, below we consider an application of the Darboux transformations to exact solvable cases with the potentials (4).

3.1 The first case

Let V_0 in (27) be a constant matrix potential, $f = f_0 = \text{const.}$, such that

$$\hat{h}_0 = i\sigma_1\frac{d}{dt} + i\sigma_2 f_0 \,. \tag{52}$$

A general solution of (38), with the Hamiltonian (52) and eigenvalue $\lambda_{(1)} = -iR$, can be written as

$$\begin{aligned} &\psi_1 = i(f_0 + i\omega_0)p + iRq \,, \ \psi_2 = Rp + (f_0 + i\omega_0)q \,, \\ &p = p_0\exp(\omega_0 t) \,, \ q = q_0\exp(-\omega_0 t) \,, \ \omega_0^2 = R^2 - f_0^2 \,, \end{aligned} \tag{53}$$

where q_0 and p_0 are complex constants. Substituting these expressions in (50) gives

$$\alpha = \frac{R\omega_0}{2(Q+f_0)}\left(\frac{q}{p}-\frac{p}{q}\right), \ \beta = -\frac{\omega_0^2}{Q+f_0},$$
$$Q = \frac{R}{2}\left(\frac{p}{q}+\frac{q}{p}\right). \tag{54}$$

As a result, the function g (32), which defines the new matrix potential V_1 (28), assumes the form

$$g = f_0 + \frac{2\omega_0^2}{Q+f_0}. \tag{55}$$

In the case $R^2 > f_0^2$, with the choice $p_0/q_0 = \exp(2a)$, with a an arbitrary phase, we get

$$Q = R\cosh 2(\omega_0 t + a), \tag{56}$$

For $f_0 = 0$ this potential is a special case of (3), otherwise we have a new solution.

In the case $R^2 < f_0^2$, with the choice $p_0/q_0 = \exp(2ia)$, we obtain

$$Q = R\cos 2(|\omega_0|\, t + a).$$

Thus, from the knowledge of solutions of two-level systems with the potential $f_0 =$ const. we construct solutions of the same system with the potential g given by (55). Such solutions have the form

$$\Phi^{(\varepsilon)} = \hat{L}\Psi^{(\varepsilon)}, \tag{57}$$

where the components ψ_1 and ψ_2 of $\Psi^{(\varepsilon)}$ are given by (53) by substituting R for $i\varepsilon$ and p_0, q_0 for arbitrary new constants. Remark that these new solutions are expressed in terms of elementary functions.

Afterwards, we can use these new functions $\Phi^{(\varepsilon)}$ with eigenvalue $\varepsilon = -iR_1$ (and $R_1 \neq R$) to find new solutions and so on.

3.2 The second case

Now we assume that the function f in (27) has the form (4). In this case, solutions of (26) can be written as (see [7])

$$\psi_1 = (1-z)^\nu E\left[c_1 z^\mu F(a+1,b;c;z) + c_2 z^{-\mu} F(\bar{a}+1,\bar{b};\bar{c};z)\right],$$
$$\psi_2 = (1-z)^\nu \left[(r_0 - r_1 + 2i\mu)\, c_1 z^\mu F(a,b+1;c;z) + \right.$$
$$\left.(r_0 - r_1 - 2i\mu_0)\, c_2 z^{-\mu} F(\bar{a},\bar{b}+1;\bar{c};z)\right], \tag{58}$$

where r_0, r_1 and T are real constants present in the definition of f, and

$$z = \frac{1}{2}(1+\tanh\tau), \ a = \mu+\nu+ir_0, \ b = \mu+\nu-ir_0, \ \bar{a} = -\mu+\nu+ir_0,$$
$$\bar{b} = -\mu+\nu-ir_0, \ c = 1+2\mu, \ \bar{c} = 1-2\mu, \ E = \varepsilon T,$$

with c_1, c_2, μ and ν complex constants. If the following relations are satisfied

$$4\mu^2 + E^2 + (r_0 - r_1)^2 = 0,$$
$$4\nu^2 + E^2 + (r_0 + r_1)^2 = 0, \tag{59}$$

we can identify $F(a,b;c;z)$ with the hyper-geometrical function.

We are going to construct the operator $\hat{L}$ in the case where μ and ν are real. Therefore, making $E = -iR$ in (59), the reality condition will be satisfied if

$$R^2 > \max (r_0 \pm r_1)^2 .$$

In this case, we can write

$$\mu_0 = \frac{1}{2}\sqrt{R^2 - (r_0 - r_1)^2} , \; \nu_0 = \frac{1}{2}\sqrt{R^2 - (r_0 + r_1)^2} , \tag{60}$$

and the expressions (58) become

$$\begin{aligned} &\psi_1^{(0)} = -iR\,(1-z)^{\nu_0}\left(c_1 z^{\mu_0} F_0 + c_2 z^{-\mu_0} F_1\right) , \\ &\psi_2^{(0)} = (1-z)^{\nu_0}\left[(r_0 - r_1 + 2i\mu_0)\, c_1 z^{\mu_0} F_0^* + (r_0 - r_1 - 2i\mu_0)\, c_2 z^{-\mu_0} F_1^*\right] , \\ &F_0 = F\left(a_0 + 1, a_0^*; 1 + 2\mu_0; z\right) , \quad F_1 = F\left(\bar{a}_0 + 1, \bar{a}_0^*; 1 - 2\mu_0; z\right) , \\ &a_0 = \mu_0 + \nu_0 + ir_0 , \quad a_1 = -\mu_0 + \nu_0 + ir_0 . \end{aligned} \tag{61}$$

where the $*$ represents complex conjugation. The constants c_1 and c_2 will be chosen such that the relation

$$\frac{c_1}{c_2} = p^{2\mu_0}\left(r_0 - r_1 - 2i\mu_0\right) R^{-1} = p^{2\mu_0} e^{-2i\varphi_0} \tag{62}$$

is satisfied, where p is a new real constant and φ_0 is a constant phase defined, in agreement with (60), by the expression

$$\left(r_0 - r_1 + 2i\mu_0\right) R^{-1} = e^{2i\varphi_0} . \tag{63}$$

For such a choice of the constants c_1 and c_2, the solutions (61) assume the form

$$\begin{aligned} &\psi_1^{(0)} = -iR\,(1-z)^{\nu_0}\sqrt{c_1 c_2} A , \; \psi_2^{(0)} = R\,(1-z)^{\nu_0}\sqrt{c_1 c_2} A^* , \\ &A = (pz)^{\mu_0} e^{-i\varphi_0} F_0 + (pz)^{-\mu_0} e^{i\varphi_0} F_1 . \end{aligned} \tag{64}$$

Using the above solutions and the expression (49), the constants α and β are seen to be real, and they can be written as

$$\alpha = \frac{iR\left(A^{*2} - A^2\right)}{2TAA^*} , \; \beta = f + \frac{R\left(A^{*2} + A^2\right)}{2TAA^*} , \tag{65}$$

with f defined by (4). In this case, the Darboux transformation (51) provides exact solutions of the two-level problem with a potential given by

$$g = \frac{R\left(A^{*2} + A^2\right)}{T\left|A\right|^2} - \frac{r_0}{T}\tanh \tau - \frac{r_1}{T} . \tag{66}$$

We remark that the new potential, as well as the corresponding solutions, are expressed via the hypergeometric functions only.

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