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GENERAL QUADRATIC GAUGE THEORY. CONSTRAINT STRUCTURE, SYMMETRIES, AND PHYSICAL FUNCTIONS

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General quadratic gauge theory. Constraint structure, symmetries, and physical functions.

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Abstract

1 Introduction

It is well known that the many modern physical theories are formulated as gauge theories, that is theories with first-class constraints (FCC) in the Hamiltonian formulation. Majority of the latter theories are perturbative theories whose behavior is in essence determined by the quadratic part of the action and nonquadratic part is, in a sense, "small". Very often the constraint and gauge structure of the complete theory and its quadratic approximation is the same. Namely, constraints of the complete theory differ from linear constraints of the quadratic theory by "small" nonlinear terms, such that does not change the numbers of first-class and second-class constraints. The gauge transformations of the complete theory and its quadratic approximation have the same number of gauge parameters. The majority of properties of the complete gauge theory and of its quadratic approximation are the same. However, to derive and to prove these properties for a general gauge theory is sometimes a very complicated task. At the same time simplifications due to the quadratic approximation, allow one to represent simple derivations and illustrations of these properties. The aim of the present work is to consider a general quadratic gauge theory and to prove for such a theory a set of generic for any gauge theory properties. In particular, we establish the relation between the constraint structure of the theory and structure of its gauge transformations, we prove the famous Dirac conjecture, and identify definitions of physical functions as those which commute with first-class constraints and those which are gauge invariant on extremals. To fulfill such a program, we demonstrate the existence of so-called superspecial phase-space variables (Sect. 2), in which the quadratic Hamiltonian action takes a simple canonical form. On the base of such a representation, we analyze the functional

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arbitrariness in solutions of equations of motion of the quadratic gauge theory (Sect. 3), and derive a general structure of symmetries analyzing the symmetry equation (Sect. 4). In Sect. 5, we use these results to identify two definitions of physical functions and, thus, to prove the Dirac conjecture.

2 Superspecial phase-space variables

In this Section we are going to demonstrate that there exist so-called superspecial phase-space variables in which the total Hamiltonian for a general quadratic gauge theory takes a simple canonical form.

First, we recall [2] that there exists a canonical transformation from the initial phase-space variables $\eta = (q,p)$ to the special phase-space variables $\vartheta = (\omega, Q, \Omega)$ such that: The constraint surface is described by the equations $\Omega = 0$. The variables Ω consist of two groups: $\Omega = (\mathcal{P}, U)$, where U are all the SCC and $\mathcal{P}$ are all the FCC. At the same time $\mathcal{P}$ are momenta conjugate to the coordinates Q. Moreover, the special variables can be chosen such that $\Omega = \left(\Omega^{(1)}, \Omega^{(2\ldots)}\right)$, where $\Omega^{(1)}$ are primary and $\Omega^{(2\ldots)}$ are secondary constraints. Respectively, $\Omega^{(1)} = (\mathcal{P}^{(1)}, U^{(1)})$, $\Omega^{(2\ldots)} = (\mathcal{P}^{(2\ldots)}, U^{(2\ldots)})$; $\mathcal{P} = (\mathcal{P}^{(1)}, \mathcal{P}^{(2\ldots)})$, $U = (U^{(1)}, U^{(2\ldots)})$; $\mathcal{P}^{(1)}$ are primary FCC, $\mathcal{P}^{(2\ldots)}$ are secondary FCC, $U^{(1)}$ are primary SCC, $U^{(2)}$ are secondary SCC. The Hamiltonian action S_{H} of a general quadratic gauge theory has the following structure

$$\begin{aligned} &S_{\mathrm{H}}[\vartheta] = S_{\mathrm{ph}}[\omega] + S_{\mathrm{non-ph}}[\vartheta]\,, \quad \vartheta = (\vartheta, \lambda)\,, \\ &S_{\mathrm{ph}}[\omega] = \int \left[\omega_p \dot{\omega}_q - H_{\mathrm{ph}}(\omega)\right] dt\,, \\ &S_{\mathrm{non-ph}}[\vartheta] = \int \left[\mathcal{P}\dot{Q} + U_p \dot{U}_q - H^{(1)}_{\mathrm{non-ph}}(\vartheta)\right] dt\,, \end{aligned} \tag{1}$$

where

$$\begin{aligned} H^{(1)}_{\mathrm{non-ph}} &= (Q^{(1)}A + Q^{(2\ldots)}B + \omega C)\mathcal{P}^{(2\ldots)} + \mathcal{P}^{(2\ldots)} D \mathcal{P}^{(2\ldots)} \\ &+ \mathcal{P}^{(2\ldots)} E U^{(2\ldots)} + U^{(2\ldots)} G U^{(2\ldots)} + \lambda_{\mathcal{P}} \mathcal{P}^{(1)} + \lambda_U U^{(1)}\,, \end{aligned} \tag{2}$$

and A, B, C, E and G are some matrices (depending, in the general case, on time). Note that the special variables (ω, Q, Ω) may be chosen in more than one way. The equations of motion are

$$I = \frac{\delta S_{\mathrm{H}}}{\delta \vartheta} = 0 \Longrightarrow \begin{cases} \dot{\vartheta} = \{\vartheta, H^{(1)}\} \\ \Omega = 0 \end{cases},$$

where

$$H^{(1)} = H_{\mathrm{ph}} + H_{\mathrm{non-ph}}$$

is the total Hamiltonian. In what follows we call I and $O(I)$ the extremals.

We are going to demonstrate that the special phase-space variables can be chosen such that the non-physical part of the total Hamiltonian (2) takes a simple (canonical) form:

$$H^{(1)}_{\mathrm{non-ph}} = H^{(1)}_{\mathrm{FCC}} + H^{(1)}_{\mathrm{SCC}}\,, \tag{3}$$

where

$$H^{(1)}_{\mathrm{FCC}} = \sum_{a=1}^{\aleph_\chi}\left(\sum_{i=1}^{a-1} Q^{(i|a)}\mathcal{P}^{(i+1|a)} + \lambda^a_{\mathcal{P}}\mathcal{P}^{(1|a)}\right),$$
$$H^{(1)}_{\mathrm{SCC}} = U^{(2\ldots)}FU^{(2\ldots)} + \lambda_U U^{(1)}\,.$$

Here $(Q,\mathcal{P}) = \left(Q^{(i|a)},\mathcal{P}^{(i|a)}\right)$, $\lambda_{\mathcal{P}} = (\lambda^a_{\mathcal{P}})$, $a = 1,...,\aleph_\chi$, $i = 1,...,a$, F is a matrix, and $\aleph_\chi$ is the number of the stages of the Dirac procedure that is necessary to determine all the independent FCC. In what follows, we call such special phase-space variables the superspecial phase-space variables. In the superspecial phase-space variables, the consistency conditions for the primary FCC $\mathcal{P}^{(1|a)}$, $a > 1$, determine the secondary FCC $\mathcal{P}^{(2|\aleph_\chi)}$, and so on, creating the a-chain of FCC as follows $\mathcal{P}^{(1|a)} \to \mathcal{P}^{(2|a)} \to \mathcal{P}^{(3|a)} \ldots \mathcal{P}^{(a|a)}$, see the scheme below,

$$\begin{array}{ccccccccc}
\mathcal{P}^{(1|\aleph_\chi)} & \to & \mathcal{P}^{(2|\aleph_\chi)} & \to & \cdots & \to & \mathcal{P}^{(\aleph_\chi-1|\aleph_\chi)} & \to & \mathcal{P}^{(\aleph_\chi|\aleph_\chi)} \\
\mathcal{P}^{(1|\aleph_\chi-1)} & \to & \mathcal{P}^{(2|\aleph_\chi-1)} & \to & \cdots & \to & \mathcal{P}^{(\aleph_\chi-1|\aleph_\chi-1)} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
\mathcal{P}^{(1|2)} & \to & \mathcal{P}^{(2|2)} & & & & & & \\
\mathcal{P}^{(1|1)} & & & & & & & &
\end{array}$$

The consistency conditions for the constraints $\mathcal{P}^{(a|a)}$, $a = 1,...,\aleph_\chi$ do not create any new constraints. Note that in the canonical form the non-physical part of the total Hamiltonian does not depend on the coordinates $Q^{(a|a)}$.

Below we represent the proof of the above assertion.

First we consider the term $Q^{(1)}A\mathcal{P}^{(2\ldots)}$ from the Hamiltonian (2). Let the momenta $\mathcal{P}^{(1)}$ and the corresponding coordinates $Q^{(1)}$ are labeled by the Greek subscripts, whereas the momenta $\mathcal{P}^{(2\ldots)}$ and the corresponding coordinates $Q^{(2\ldots)}$ are labeled by the Latin subscripts,

$$Q = \left(Q^{(1)}_\nu, Q^{(2\ldots)}_b\right),\ \mathcal{P} = \left(\mathcal{P}^{(1)}_\nu, \mathcal{P}^{(2\ldots)}_b\right).$$

Suppose the defect of the rectangular matrix $A^{\nu b}$ is equal $\mathbf{a}$ (obviously $\left[Q^{(1)}\right] - \mathbf{a} \le \left[Q^{(2\ldots)}\right]$). Then there exists a nontrivial zero vectors $z_{(\tilde\alpha)}$, $\tilde\alpha = 1,...\mathbf{a}$, of the matrix A such that $z^\nu_{(\tilde\alpha)}A^{\nu b} = 0$. Let us construct a quadratic matrix

$$Z^\nu_\alpha = ||z^\nu_{(\tilde\alpha)} z^\nu_{(\bar\alpha)}||\,,\ \alpha = (\tilde\alpha,\bar\alpha)\,,\ [\bar\alpha] \le [b]\,,$$

where the vectors $z_{(\bar\alpha)}$ have to provide the nonsingularity of the complete matrix Z. Such vectors always exist. Then we perform the canonical transformation

$\left(Q^{(1)},\mathcal{P}^{(1)}\right)\rightarrow\left(Q'^{(1)},\mathcal{P}'^{(1)}\right)$, where $Q'^{(1)}_{\nu}Z^{\nu}_{\alpha}=Q^{(1)}_{\alpha}$. Such a canonical transformation can be performed with the generating function of the form

$$W=Q'^{(1)}_{\nu}Z^{\nu}_{\alpha}\mathcal{P}^{(1)}_{\alpha}. \tag{4}$$

We denote

$$Q'^{(1)}_{\alpha}=\left(Q'^{(1)}_{\tilde{\alpha}}=Q^{(1|1)}_{\tilde{\alpha}},Q'^{(1)}_{\bar{\alpha}}=\bar{Q}^{(1)}_{\bar{\alpha}}\right),\ \mathcal{P}'^{(1)}_{\alpha}=\left(\mathcal{P}^{(1|1)}_{\tilde{\alpha}},\bar{\mathcal{P}}^{(1)}_{\bar{\alpha}}\right).$$

Thus, now the primary FCC read $\mathcal{P}^{(1|1)},\bar{\mathcal{P}}^{(1)}$ and the corresponding conjugate coordinates are $Q^{(1|1)},\bar{Q}^{(1)}$. After the canonical transformation the total Hamiltonian $H^{(1)}=H_{\rm ph}+H_{\rm non-ph}$ becomes

$$\begin{aligned}H^{(1)}&=H_{\rm ph}+\bar{Q}^{(1)}A'\mathcal{P}^{(2\ldots)}+(Q^{(2\ldots)}B+\omega C)\mathcal{P}^{(2\ldots)}+\mathcal{P}^{(2\ldots)}D\mathcal{P}^{(2\ldots)}\\&+\mathcal{P}^{(2\ldots)}EU^{(2\ldots)}+U^{(2\ldots)}FU^{(2\ldots)}+\lambda_1\mathcal{P}^{(1|1)}+\bar{\lambda}\bar{\mathcal{P}}^{(1)}+\lambda_U U^{(1)},\end{aligned} \tag{5}$$

where

$$A'=(A')^{\bar{\nu}b}=Z^{\bar{\nu}}_{\alpha}A^{\alpha b},\ \operatorname{rank}A'=\max=[\bar{Q}^{(1)}];\ b=(\bar{\mu},\bar{b}),\ \det(A')^{\bar{\nu}\bar{\mu}}\neq 0,$$

and via $\lambda_1\mathcal{P}^{(1|1)}+\bar{\lambda}\bar{\mathcal{P}}^{(1)}$ we have denoted terms proportional to the primary FCC. At the same time the functions λ_1 and $\bar{\lambda}$ absorb the time derivative of the generating function (4). Note that the coordinates $Q^{(1|1)}$ do not enter the Hamiltonian $H^{(1)}$ (in fact, that was one of the aim of the above canonical transformation) and therefore, the consistency conditions for the constraints $\mathcal{P}^{(1|1)}$ do not create any new constraints,

$$\left\{\mathcal{P}^{(1|1)},H^{(1)}\right\}\equiv 0.$$

Consider the consistency conditions for the primary FCC $\bar{\mathcal{P}}^{(1)}$,

$$\left\{\bar{\mathcal{P}}^{(1)},H^{(1)}\right\}=-A'\mathcal{P}^{(2\ldots)}=0.$$

Since the rank of the matrix A' is maximal, the combinations $A'\mathcal{P}^{(2\ldots)}$ of the secondary FCC are independent. We can choose them as new momenta $\mathcal{P}'^{(2)}$ which are now second-stage FCC. To this end, we perform a canonical transformation $(Q^{(2\ldots)},\mathcal{P}^{(2\ldots)})\rightarrow(Q'^{(2\ldots)},\ \mathcal{P}'^{(2\ldots)})$ with the generating function

$$W=Q'^{(2)}A'\mathcal{P}^{(2\ldots)}+Q'^{(3\ldots)}A''\mathcal{P}^{(2\ldots)}. \tag{6}$$

Here the rectangular matrix A'' is chosen such that the quadratic matrix $\Lambda=||A'A''||$ is invertible, $\det\Lambda\neq 0$. Thus, the new variables are

$$\begin{aligned}&\mathcal{P}'^{(2\ldots)}=\left(\mathcal{P}'^{(2)},\mathcal{P}'^{(3\ldots)}\right),\ Q'^{(2\ldots)}=\left(Q'^{(2)},Q'^{(3\ldots)}\right),\\&\mathcal{P}'^{(2)}=A'\mathcal{P}^{(2\ldots)},\ \mathcal{P}'^{(3\ldots)}=\left(A''\mathcal{P}^{(2\ldots)}\right),\ Q'^{(2\ldots)}=Q^{(2\ldots)}\Lambda^{-1}.\end{aligned}$$

In the new variables the Hamiltonian (5) reads

$$\begin{aligned} H^{(1)} &= H_{\rm ph} + \bar{Q}^{(1)}\mathcal{P}'^{(2)} + (Q'^{(2\ldots)}B' + \omega C')\mathcal{P}'^{(2\ldots)} + \mathcal{P}'^{(2\ldots)}D'\mathcal{P}'^{(2\ldots)} \\ &+ \mathcal{P}'^{(2\ldots)}E'U^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)} . \end{aligned} \tag{7}$$

The matrices B', C', D', and E' differ from the ones B, C, D, and E due to the variable transformation and absorb the time derivative of the generating function (4). We note that the latter derivative does not modify the term $Q'^{(1)}\mathcal{P}'^{(2)}$.

Separating explicitly terms proportional to $\mathcal{P}'^{(2)}$ in the Eqs. (7) and omitting all the primes, we obtain

$$\begin{aligned} H^{(1)} &= H_{\rm ph} + \left(\bar{Q}^{(1)} + \Sigma_q S_q + \Sigma_p S_p\right)\mathcal{P}^{(2)} + (Q^{(2\ldots)}B + \omega C)\mathcal{P}^{(3\ldots)} \\ &+ \mathcal{P}^{(3\ldots)}D\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)} , \end{aligned} \tag{8}$$

where $\Sigma = (\Sigma_q, \Sigma_p)$ is the set of all the phase-space variables except $Q^{(1|1)}$, $\bar{Q}^{(1)}$ and $\mathcal{P}^{(1|1)}$, $\bar{\mathcal{P}}^{(1)}$, whereas S_q,S_p,B,C,D, E, and F are some matrices.

Now we perform a canonical transformation (we do not transform the variables $Q^{(1|1)}, \mathcal{P}^{(1|1)}$) with the generating function W,

$$W = \bar{\mathcal{P}}'^{(1)}\left(\bar{Q}^{(1)} + \Sigma_q S_q + \Sigma'_p S_p\right) + \Sigma'_p\Sigma_q .$$

Thus,

$$\bar{\mathcal{P}}'^{(1)} = \bar{\mathcal{P}}^{(1)} , \;\; \bar{Q}'^{(1)} = \bar{Q}^{(1)} + \Sigma_q S_q + \Sigma_p S_p + O(\bar{\mathcal{P}}^{(1)}), \; \Sigma' = \Sigma + O(\bar{\mathcal{P}}^{(1)}) .$$

In terms of the new variables, the Hamiltonian (8) takes the form (with primes omitted and redefined functions $\lambda_{\mathcal{P}}$ which absorb time derivative of the generating function)

$$\begin{aligned} H^{(1)} &= H_{\rm ph} + \bar{Q}^{(1)}\mathcal{P}^{(2)} + (Q^{(2\ldots)}B + \omega C)\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}D\mathcal{P}^{(3\ldots)} \\ &+ \mathcal{P}^{(3\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{9}$$

where B,C,D, E, and F are some matrices.

At this stage of the procedure, we consider the term $Q^{(2)}B\mathcal{P}^{(3\ldots)}$ from the Hamiltonian (9). Let the variables $\bar{Q}^{(1)}, \bar{\mathcal{P}}^{(1)}$; $Q^{(2)}, \mathcal{P}^{(2)}$ are numbered by the Greek subscripts, whereas the variables $Q^{(3\ldots)}, \mathcal{P}^{(3\ldots)}$ are labeled by the Latin subscripts (in the general case the number of these indices is different from the one of the first stage of the procedure). Suppose the defect of the rectangular matrix $B^{\nu k}$ is equal $\mathbf{b}$ (obviously $\left[Q^{(2)}\right] - \mathbf{b} \leq \left[Q^{(3\ldots)}\right]$). Then there exists $\mathbf{b}$ nontrivial zero vectors $v_{(\tilde{\alpha})}$, $\tilde{\alpha} = 1, \ldots, \mathbf{b}$ of the matrix B such that $v^{\nu}_{(\tilde{\alpha})}B^{\nu k} = 0$. Let us construct a quadratic matrix

$$V^{\nu\alpha} = ||v^{\nu}_{(\tilde{\alpha})}v^{\nu}_{(\bar{\alpha})}|| , \; \alpha = (\tilde{\alpha}, \bar{\alpha}) , \; [\bar{\alpha}] \leq [k] ,$$

where the vectors $v_{(\tilde{\alpha})}$ have to provide the nonsingularity of the complete matrix V. Such vectors always exist. Then we perform a canonical transformation

$$
\begin{aligned}
&\bar{Q}^{(1)}, \bar{\mathcal{P}}^{(1)}; Q^{(2)}, \mathcal{P}^{(2)} \to \bar{Q}'^{(1)}, \bar{\mathcal{P}}'^{(1)}; Q'^{(2)}, \mathcal{P}'^{(2)}, \\
&Q'^{(2)} = \left(Q'^{(2)}_{\tilde{\alpha}} = Q^{(2|2)}_{\tilde{\alpha}}, Q'^{(2)}_{\tilde{\alpha}} = \tilde{Q}^{(2)}_{\tilde{\alpha}}\right)
\end{aligned}
$$

with the generating function

$$
W = \bar{Q}'^{(1)} V \bar{\mathcal{P}}^{(1)} + Q'^{(2)} V \mathcal{P}^{(2)}.
$$

In the new variables the Hamiltonian (9) has the form (we omit all the primes and redefine λ)

$$
\begin{aligned}
H^{(1)} &= H_{\mathrm{ph}} + (\bar{Q}^{(1)} + Q^{(2)}\Delta)\mathcal{P}^{(2)} + \tilde{Q}^{(2)} B \mathcal{P}^{(3\ldots)} + (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(3\ldots)} \\
&+ \mathcal{P}^{(3\ldots)} D \mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)} E U^{(2\ldots)} + U^{(2\ldots)} F U^{(2\ldots)} + \lambda_1 \mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)},
\end{aligned} \tag{10}
$$

where $\Delta = \frac{\partial V}{\partial t} V^{-1}$, B, K, C, D, E, and F are some matrices, in particular,

$$
\operatorname{rank} B = \max = \left[\tilde{Q}^{(2)}\right] \leq \left[\mathcal{P}^{(3\ldots)}\right].
$$

Now we perform a canonical transformation $\bar{Q}^{(1)}, \bar{\mathcal{P}}^{(1)}, Q^{(2)}, \mathcal{P}^{(2)} \to \bar{Q}'^{(1)}, \bar{\mathcal{P}}'^{(1)}, Q'^{(2)}, \mathcal{P}'^{(2)}$ with the generating function

$$
W = \left(\bar{Q}^{(1)} + Q^{(2)}\Delta\right) \bar{\mathcal{P}}'^{(1)} + Q^{(2)} \mathcal{P}'^{(2)}.
$$

Thus we get:

$$
\bar{\mathcal{P}}'^{(1)} = \bar{\mathcal{P}}^{(1)}, \; \bar{Q}'^{(1)} = \bar{Q}^{(1)} + Q^{(2)}\Delta, \; \mathcal{P}'^{(2)} = \mathcal{P}^{(2)} - \Delta\bar{\mathcal{P}}^{(1)}, \; Q'^{(2)} = Q^{(2)}.
$$

In terms of new variables, the Hamiltonian (10) takes the form (omitting all the primes and redefining $\lambda_{\mathcal{P}}$)

$$
\begin{aligned}
H^{(1)} &= H_{\mathrm{ph}} + Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \tilde{Q}^{(2)} B \mathcal{P}^{(3\ldots)} \\
&+ (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)} D \mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)} E U^{(2\ldots)} \\
&+ U^{(2\ldots)} F U^{(2\ldots)} + \lambda_1 \mathcal{P}^{(1|1)} + \lambda_2 \mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}.
\end{aligned}
$$

The time derivative of the generating function is absorbed by the term $\tilde{\lambda}\tilde{\mathcal{P}}^{(1)}$.

Note that the variables $Q^{(2|2)}$ do not enter the Hamiltonian and therefore, the consistency conditions for the constraints $\mathcal{P}^{(2|2)}$ do not create any new constraints. In addition, we remark that at this stage of the procedure, the primary FCC are $\mathcal{P}^{(1|1)}, \mathcal{P}^{(1|2)}$ and $\tilde{\mathcal{P}}^{(1)}$.

Consider the consistency conditions for the second-stage FCC $\mathcal{P}^{(2)}$,

$$
\left\{\tilde{\mathcal{P}}^{(2)}, H^{(1)}\right\} = -B\mathcal{P}^{(3\ldots)} = 0.
$$

Since the rank of the matrix B is maximal, the combinations $B\mathcal{P}^{(3\dots)}$ are independent. We can choose them as new momenta $\mathcal{P}'^{(3)}$ which are now third-stage FCC. To this end, we perform a canonical transformation $(Q^{(3\dots)}, \mathcal{P}^{(3\dots)}) \to (Q'^{(3\dots)}, \mathcal{P}'^{(3\dots)})$ with the generating function

$$W = Q'^{(3)} B \mathcal{P}^{(3\dots)} + Q'^{(4\dots)} B' \mathcal{P}^{(3\dots)}\,,\; Q_k^{(3\dots)} = \left(Q_{\alpha'}^{(3)}, Q_{k'}^{(4\dots)}\right).$$

Here the rectangular matrix B' is chosen such that the quadratic matrix $\Lambda = ||BB'||$ is invertible, $\det\Lambda \neq 0$. Thus, we obtain

$$\mathcal{P}'^{(3\dots)} = \Lambda \mathcal{P} = \left(\mathcal{P}_{\alpha'}'^{(3)}, \mathcal{P}_{k'}'^{(4\dots)}\right),\; \mathcal{P}_{\alpha'}'^{(3)} = B^{\alpha' k} \mathcal{P}_k^{(3\dots)}\,,$$
$$Q'^{(3\dots)} = Q^{(3\dots)} \Lambda^{-1} = \left(Q_{\alpha'}'^{(3)}, Q_{k'}'^{(4\dots)}\right).$$

In the new variables the Hamiltonian (10) has the form (omitting primes)

$$H^{(1)} = H_{\rm ph} + Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \tilde{Q}^{(2)}\mathcal{P}^{(3)} + (Q^{(3\dots)}K + \omega C)\mathcal{P}^{(3\dots)} + \mathcal{P}^{(3\dots)}D\mathcal{P}^{(3\dots)}$$
$$+ \mathcal{P}^{(3\dots)}EU^{(2\dots)} + U^{(2\dots)}FU^{(2\dots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}\,,$$

where K,C,D, E, and F are some matrices.

Let us separate terms proportional to $\mathcal{P}^{(3)}$ in this expression. Thus, we obtain

$$\begin{aligned} H^{(1)} &= H_{\rm ph} + Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \mathcal{P}^{(3)}\left(\tilde{Q}^{(2)} + S_q\Xi_q + S_p\Xi_p\right) \\ &+ (Q^{(3\dots)}K + \omega C)\mathcal{P}^{(4\dots)} + \mathcal{P}^{(4\dots)}D\mathcal{P}^{(4\dots)} + \mathcal{P}^{(4\dots)}EU^{(2\dots)} \\ &+ U^{(2\dots)}FU^{(2\dots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}\,, \end{aligned} \tag{11}$$

where S_q, S_p, K, C, D, E,and F are some matrices and $\Xi = (\Xi_q, \Xi_p)$ is the set of all the phase-space variables except the ones $Q^{(1|1)}$, $\mathcal{P}^{(1|1)}$, $Q^{(1|2)}$, $\mathcal{P}^{(1|2)}$, $\tilde{Q}^{(1)}$, $\tilde{\mathcal{P}}^{(1)}$, $Q^{(2|2)}$, $\mathcal{P}^{(2|2)}$, $\tilde{Q}^{(2)}$, $\tilde{\mathcal{P}}^{(2)}$.

Now we perform a canonical transformation (we do not transform the variables $Q^{(1|1)}$, $\mathcal{P}^{(1|1)}$; $Q^{(1|2)}$, $\mathcal{P}^{(1|2)}$; $\tilde{Q}^{(1)}$, $\tilde{\mathcal{P}}^{(1)}$, $Q^{(2|2)}$, $\mathcal{P}^{(2|2)}$) with a generating function

$$W = \tilde{\mathcal{P}}'^{(2)}\left(\tilde{Q}^{(2)} + S_q\Xi_q + S_p\Xi'_p\right) + \Xi'_p\Xi_q\,.$$

Thus,

$$\tilde{\mathcal{P}}'^{(2)} = \tilde{\mathcal{P}}^{(2)}\,,\; \tilde{Q}'^{(2)} = \tilde{Q}^{(2)} + S_q\Xi_q + S_p\Xi_p + O(\tilde{\mathcal{P}}^{(2)})\,,\; \Xi' = \Xi + O(\tilde{\mathcal{P}}^{(2)})\,.$$

In terms of new variables, the Hamiltonian (11) takes the form (omitting all the primes)

$$\begin{aligned} H^{(1)} &= H_{\rm ph} + Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{\mathcal{P}}^{(2)}\left(\tilde{Q}^{(1)} + R_q\Sigma_q + R_p\Sigma_p\right) + \tilde{Q}^{(2)}\mathcal{P}^{(3)} \\ &+ (Q^{(3\dots)}K + \omega C)\mathcal{P}^{(4\dots)} + \mathcal{P}^{(4\dots)}D\mathcal{P}^{(4\dots)} + \mathcal{P}^{(4\dots)}EU^{(2\dots)} \\ &+ U^{(2\dots)}FU^{(2\dots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}\,, \end{aligned} \tag{12}$$

where $\Sigma = (\tilde{Q}^{(2)}, \tilde{\mathcal{P}}^{(2)}, \Xi) = (\Sigma_q, \Sigma_p)$ and R, K, C, D, E, and F are some matrices,

Let us perform a canonical transformation with a generating function

$$W = \tilde{\mathcal{P}}'^{(1)} \left(\tilde{Q}^{(1)} + R_q \Sigma_q + R_p \Sigma'_p \right) + \Sigma'_p \Sigma_q \,.$$

Thus,

$$\tilde{\mathcal{P}}'^{(1)} = \tilde{\mathcal{P}}^{(1)} \,,\ \tilde{Q}'^{(1)} = \tilde{Q}^{(1)} + R_q \Sigma_q + R_p \Sigma_p + O(\tilde{\mathcal{P}}^{(1)}) \,,\ \Sigma' = \Sigma + O(\tilde{\mathcal{P}}^{(1)}) \,.$$

In terms of new variables, the Hamiltonian (12) takes the form (omitting all the primes and redefining $\lambda_{\mathcal{P}}$)

$$\begin{aligned} H^{(1)} &= H_{\rm ph} + Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \tilde{Q}^{(2)}\mathcal{P}^{(3)} \\ &+ (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}D\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}EU^{(2\ldots)} \\ &+ U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)} \,, \end{aligned} \tag{13}$$

where K, C, D, E,and F are some matrices.

The further transformations of the Hamiltonian (13) can be done using the same kind of canonical transformations as were used above. In the end of the procedure we arrive to the form (??) for the non-physical part of the total Hamiltonian.

Let us stress some important facts related to the canonical transformation that was performed to reduce the total Hamiltonian to the form (??).

First of all, one ought to note that the finally transformed variables ω, Q, Ω, where $\Omega = (\mathcal{P}, U)$ (superspecial phase-space variables) still remain the special phase-space canonical variables ϑ and possess all the corresponding properties of such variables. Let us indicate below the final superspecial phase-space canonical variables by primes and the initial special phase-space variables without primes. One can see that

$$\begin{aligned} &\mathcal{P}' = T\mathcal{P} \,,\ \mathcal{P}^{(1)\prime} = T^{(1)}\mathcal{P}^{(1)} \,, \\ &U' = U + O(\mathcal{P}) \,,\ U^{(1)\prime} = U^{(1)} \,, \end{aligned}$$

such that $\mathcal{P}'$ are FCC, $\mathcal{P}^{(1)\prime}$ are primary FCC, U' are SCC, and $U^{(1)\prime}$ are primary SCC. The physical variables do not change on the constraint surface, $\omega \to \omega' = \omega + O(\mathcal{P})$. One ought to stress that the superspecial variables $\mathcal{P}^{(i|a)}$ coincide with the FCC $\chi^{(i|a)}$ from the orthogonal constraint basis introduced in [4]. In the general nonquadratic theory the relation is the following:

$$\chi^{(i|a)} = \mathcal{P}^{(i|a)} + O\left(\vartheta\Omega\right) \,. \tag{14}$$

One can also see that in the superspecial phase-space variables, the non-physical part of the Hamiltonian action can be written as:

$$S_{\rm non-ph} = \int \left[\mathcal{P}\hat{\Lambda}\mathcal{Q} + \sum_{i=1}^{\aleph_\chi} \mathcal{P}^{(i|i)}\dot{Q}^{(i|i)} + \mathcal{U}\hat{B}\mathcal{U} \right] dt \,, \tag{15}$$

where $\hat{\Lambda}$ and $\hat{B}$ are matrix first-order differential operators and

$$\mathcal{Q}=(\lambda_{\mathcal{P}}^{a},\,Q^{(i|a)})\,,\ i=1,...,a-1\,,\ a=1,...,\aleph_{\chi})\,,\ \mathcal{U}=(\lambda_{U},U)\,.$$

It is important to stress that $[\mathcal{Q}]=[\mathcal{P}]$, due to the fact that $[\lambda_{\mathcal{P}}]=\left[\mathcal{P}^{(1)}\right]$.

One can see that there exist local operators $\hat{\Lambda}^{-1}$ and $\hat{B}^{-1}$ such that $\hat{\Lambda}\hat{\Lambda}^{-1}=\hat{\Lambda}^{-1}\hat{\Lambda}=1$, $\hat{B}\hat{B}^{-1}=\hat{B}^{-1}\hat{B}=1$. This assertion can be derived from the fact that by the construction of the special phase-space variables, the equations of motion that follow from the Hamiltonian action have the unique solution $\mathcal{P}=0$ and $\mathcal{U}=0$. Thus, the equations

$$\frac{\delta S_{\mathrm{H}}}{\delta\mathcal{Q}}=0\Longrightarrow\hat{\Lambda}^{T}\mathcal{P}=0\,,\ \frac{\delta S_{\mathrm{H}}}{\delta\mathcal{U}}=0\Longrightarrow\hat{B}\mathcal{U}=0 \tag{16}$$

must have only the solution $\mathcal{P}=0$ and $\mathcal{U}=0$. Let us represent $\hat{\Lambda}$ as

$$\hat{\Lambda}=\Lambda\left(\frac{d}{dt}\right)=a\frac{d}{dt}+b\,,$$

where a and b are some constant matrices, and consider solutions of the form $\mathcal{P}(t)=e^{-Et}\mathcal{P}(0)$, where E is a complex number. Thus, we obtain $\Lambda^{T}(E)\mathcal{P}(0)=0$. The existence of the unique solution $\mathcal{P}(0)=0$ implies

$$\forall E:\ \det\Lambda(E)\neq0\,. \tag{17}$$

On the other side $\det\Lambda(E)$ is a polynomial of E. Due to (17) such a polynomial has no roots. That means that $\det\Lambda(E)=\mathrm{const}=c$. In turn, that implies that

$$\Lambda^{-1}(E)=\frac{1}{c}\Delta(E)\,,$$

where $\Delta(E)$ are the corresponding minors of the matrix $\Lambda(E)$. The latter minors are finite order polynomials of E. Thus, the operator

$$\hat{\Lambda}^{-1}=\frac{1}{c}\Delta\left(\frac{d}{dt}\right)$$

is a local operator. In the same manner, we can prove the existence of the local operator $\hat{B}^{-1}$ (to this aim it is convenient to reduce the Hamiltonian $H_{\mathrm{SCC}}^{(1)}$ to the canonical form as well, see below).

3 Functional arbitrariness in solutions of equations of motion

In theories with FCC equations of motion do not determine an unique trajectory for any given initial data. Below we are going to study this problem for quadratic gauge theories under consideration using the superspecial phase-space variables.

The equations of motion that follow from the action (1) and (2), with account taken of (3), have the form:

$$\dot{\omega} = \{\omega, H_{\mathrm{ph}}\}\,,\ \Omega = 0\,, \tag{18}$$

and

$$\dot{Q}^{(i|a)} = \{Q^{(i|a)}, H_{\mathrm{non-ph}}\} \Longrightarrow \left\{ \begin{array}{l} \dot{Q}^{(1|a)} = \lambda^a_{\mathcal{P}}\,, \\ \dot{Q}^{(2|a)} = Q^{(1|a)}\,, \\ \ldots. \\ \dot{Q}^{(a|a)} = Q^{(a-1|a)}\,. \end{array} \right. \tag{19}$$

One can see that the equations (18) for the physical variables ω and for Ω have a unique solution whenever initial data for these variables are given. There exists a functional arbitrariness in solutions of the equations of motion (19) for the variables Q since these equations contain arbitrary functions of time $\lambda_{\mathcal{P}}\,(t)$. One ought to note that the number of the variables Q is equal to the number of all FCC and, in the general case, that number is larger than the number of the arbitrary functions $\lambda_{\mathcal{P}}\,(t)$. However, as will be seen below, due to the specific structure of the equations, the "influence" of these arbitrary functions on solutions for Q is very strong. That fact is extremely important for the physical interpretation of the variables Q and for the general physical interpretation of theories with FCC. The extent to which the variables Q are affected by the arbitrary functions $\lambda_{\mathcal{P}}\,(t)$ is described by the following proposition:

The equations of motion (19) for the variables Q are completely controllable[1] by the functions $\lambda_{\mathcal{P}}\,(t)$. In the case under consideration, that means that by a proper choice of the functions $\lambda^a_{\mathcal{P}}\,(t)$ the equations (19) have a solution with the properties

$$\begin{aligned} &Q^{(i|a)}\Big|_{t=0} = 0\,,\quad Q^{(i|a)}\Big|_{t=\tau} = \Delta^{(i|a)},\ i = 1, ..., a\,, \\ &\frac{d^s\lambda^a_{\mathcal{P}}}{d^s t}\bigg|_{t=0} = 0\,,\quad \frac{d^s\lambda^a_{\mathcal{P}}}{d^s t}\bigg|_{t=\delta} = \delta^a_{(s)}\,,\ s = 0, 1, ..., K\,, \end{aligned} \tag{20}$$

where τ, $\Delta^{(i|a)}$, $\Delta^{(i|a)}$, $\delta^a_{(s)}$, and the integer K are arbitrary.

Due to simple structure of the equations of motion in superspecial phase-space variables, the proof of the above assertion can be done in a constrictive manner. Namely, we represent explicitly such a solution. It has the form

$$Q^{(i|a)} = \frac{d^{a-i}X^a}{dt^{a-i}}\,,\ i = 1, ...a\,,$$

if we chose

$$\lambda^a_{\mathcal{P}} = \frac{d^a X^a}{dt^a}\,,$$

[1] For exact definition of the controlabilty see e.g. the book [?, ?].

where $X^a(t)$ are arbitrary smooth functions, obeying the following boundary conditions

$$\left.\frac{d^s X^a}{dt^s}\right|_{t=0} = 0\,,\; s = 0, ..., K + a\,,$$
$$\left.\frac{d^s X^a}{dt^s}\right|_{t=\tau} = \begin{cases} \left.Q^{(a-s|a)}\right|_{t=\tau} = \Delta^{(a-s|a)}\,,\; s = 0, ..., a-1\,, \\ \left.\frac{d^{s-a}\lambda^a_{\mathcal{P}}}{dt^{s-a}}\right|_{t=\tau} = \delta^a_{(s-a)}\,,\; s = a, ..., K + a\,. \end{cases}$$

For example, the functions $X^a(t)$ can be chosen to be:

$$X^a(t) = f(t)\left[\sum_{s=0}^{a-1}\frac{1}{s!}\Delta^{(a-s|a)}(t-\delta)^s + \sum_{s=a}^{K+a}\frac{1}{s!}\delta^a_{(s-a+1)}(t-\delta)^s\right]$$

where $f(t)$ is an arbitrary smooth function that is equal to zero and to one in the neighborhoods of the points $t = 0$ and $t = \tau$, respectively. An example of such a function is given below

$$f(t) = \begin{cases} 0\,, & t \leq \varepsilon\,, \\ \frac{1}{1+e^u}\,,\; u = \frac{1}{t-\varepsilon} + \frac{1}{t-(\tau-\varepsilon)}\,, & \varepsilon \leq t \leq \tau - \varepsilon\,, \\ 1\,, & t \geq \tau - \varepsilon\,, \end{cases}$$
$$\lim_{t\to\varepsilon+0} f^{[s]}(t) = 0\,,\quad \lim_{t\to\tau-\varepsilon-0} f^{[s]}(t) = \delta_{0,s}\,,\quad s \geq 0\,. \tag{21}$$

The proposition that was proved is crucial for the understanding of the structure of theories with FCC (gauge theories) and for their physical interpretation. The most remarkable fact is the following: The functional arbitrariness in equations of motion of theories with FCC (gauge theories) is due to the undetermined Lagrange multipliers to the primary FCC. However, this arbitrariness affects essentially more variables. In the special variables all the variables Q are controllable by the undetermined Lagrange multipliers. The number of Q is equal to the number of all the FCC, and is greater than the number of the Lagrange multipliers.

4 Symmetries

We recall that a finite transformation $q(t) \to q'(t)$ is called a symmetry of an action S if

$$L(q,\dot{q}) \to L'(q,\dot{q}) = L(q,\dot{q}) + \frac{dF}{dt}\,, \tag{22}$$

where F is a local function. The finite symmetry transformations can be discrete, continuous global, gauge ones, and trivial. Continuous global symmetry transformations are parametrized by a set of time-independent parameters. Continuous symmetry transformations are gauge transformations if they are parametrized by some arbitrary functions of time, gauge parameters (in the case of a field theory the gauge parameters depend on all space-time variables).

We below consider only infinitesimal symmetry transformations $q \to q+\delta q$. Any such a transformation implies a conservation low (Nöether theorem):

$$\frac{dG}{dt} = -\delta q^a \frac{\delta S}{\delta q^a} \Longrightarrow G = \text{const. on extremals,} \tag{23}$$
$$G = P - F\,,\ P = \frac{\partial L}{\partial \dot{q}^a}\delta q^a,\ \delta L = \frac{dF}{dt}\,.$$

The local function G is referred to as the conserved charge related to the symmetry δq of the action S. The quantities δq, S, and G are related by the equation (23). In what follows, we call this equation the symmetry equation. The symmetry equation for the action S_{H} has the form

$$\delta\vartheta \frac{\delta S_{\mathrm{H}}}{\delta \vartheta} + \frac{dG}{dt} = 0\,. \tag{24}$$

4.1 Trivial symmetries

For any action there exist trivial symmetry transformations,

$$\delta_{\mathrm{tr}} q^a = \hat{U}^{ab}\frac{\delta S}{\delta q^b}\,, \tag{25}$$

where $\hat{U}$ is an antisymmetric local operator, that is $\left(\hat{U}^T\right)^{ab} = -\hat{U}^{ab}$. The trivial symmetry transformations do not affect genuine trajectories.

Using the simple action structure in superspecial phase-space variables, we can prove the following assertion: For theories with FCC, symmetries of the Hamiltonian action that vanish on the extremals are trivial symmetries.

To prove this assertion let us consider the Hamiltonian action $S_{\mathrm{H}}(\vartheta)$, $\vartheta = (\vartheta, \lambda)$ of a theory with FCC in the superspecial phase-space variables ϑ. The equations of motion that follow from this action have the form

$$\begin{aligned}
&\frac{\delta S_{\mathrm{H}}}{\delta \mathcal{U}} = 0 \Longrightarrow \mathcal{U} = 0\\
&\frac{\delta S_{\mathrm{H}}}{\delta \mathcal{Q}} = 0 \Longrightarrow \mathcal{P} = 0\,,\\
&\frac{\delta S_{\mathrm{H}}}{\delta \mathcal{P}} = 0 \Longrightarrow \mathcal{Q} = -\hat{\Lambda}^{-1}T\,,\ T = \left(T^{(i|a)} = \delta_{ia}\dot{Q}^{(a|a)}\right)\,.
\end{aligned} \tag{26}$$

Note, that the exact solution of the equations (26) has the form $\mathcal{U} = \mathcal{P} = 0$, $\mathcal{Q} = \psi\left(Q^{(i|i)}\right)$, where ψ is a local functions of the indicated arguments. Therefore, the variables $\mathcal{U}, \mathcal{P}$, and $\mathcal{Q}$ are auxiliary ones[2]. Excluding these variables from

[2]Suppose an action $S[q,y]$ contains two groups of coordinates q and y such that the coordinates y can be expressed as local functions $y = \bar{y}\left(q^{[l]}, l < \infty\right)$ of q and their time derivatives by the help of the equations $\delta S/\delta y = 0$. We call y the auxiliary coordinates. The action $S[q,y]$ and the reduced action $S[q] = S[q,\bar{y}]$ lead to the same equations for the coordinates q, see [5, 6]. The actions $S[q,y]$ and $S[q]$ are called the dynamicaly equivalent actions. One can prove that there exists one to one correspondence (isomorphism) between the symmetry classes of of the extended action. We call symmetries equivalent if they differ by a trivial transformation.

the action S_{H}, we obtain a dynamically equivalent action $\bar{S}_{\mathrm{H}}\left[\omega, Q^{(i|i)}\right]$. Taking into account that $\mathcal{U}=\mathcal{P}=0 \Longrightarrow \Omega=0$, and the relation (??), we obtain

$$\bar{S}_{\mathrm{H}}\left[\omega, Q^{(i|i)}\right]=\left.S_{\mathrm{H}}\right|_{\mathcal{U}=\mathcal{P}=0, Q=\psi}=S_{\mathrm{ph}}[\omega] . \tag{27}$$

Let a transformation $\delta\vartheta$ vanishing on the extremals is a symmetry of the action S_{H} . Consider the reduced transformation $\bar{\delta}\omega$, $\bar{\delta}Q^{(i|i)}$,

$$\left(\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}\right)=\left.\delta\vartheta\right|_{\mathcal{U}=\mathcal{P}=0, Q=\psi} .$$

Obviously, the reduced transformation vanishes on extremals of the reduced action $\bar{S}_{\mathrm{H}}$ and is a symmetry transformation of the action $\bar{S}_{\mathrm{H}}$. That implies

$$\bar{\delta}\omega=\hat{m}\frac{\delta S_{\mathrm{ph}}}{\delta\omega}, \quad \bar{\delta}Q^{(i|i)}=\left(\hat{n}\frac{\delta S_{\mathrm{ph}}}{\delta\omega}\right)^{(i|i)},$$

where $\hat{m}$ and $\hat{n}$ are some local operators. The transformation $\bar{\delta}\omega$ is obviously the symmetry transformation of the nonsingular action S_{ph}, and in addition, this symmetry transformation vanishes on its extremals. One can prove that such a symmetry is always trivial one and, therefore, $\hat{m}$ is antisymmetric. Thus, the complete transformation $\bar{\delta}\omega$, $\bar{\delta}Q^{(i|i)}$ can be represented in the form

$$\begin{pmatrix} \bar{\delta}\omega \\ \bar{\delta}Q^{(|)} \end{pmatrix}=\hat{M}\begin{pmatrix} \frac{\delta\bar{S}_{\mathrm{H}}}{\delta\omega} \\ \frac{\delta\bar{S}_{\mathrm{H}}}{\delta Q^{(|)}} \end{pmatrix}, \quad \hat{M}=\begin{pmatrix} \hat{m} & -\hat{n}^{T} \\ \hat{n} & 0 \end{pmatrix} .$$

Obviously, here $\hat{M}$ is an antisymmetric matrix.

Finally, the transformation $\bar{\delta}\omega$, $\bar{\delta}Q^{(i|i)}$ is a trivial symmetry of the action $\bar{S}_{\mathrm{H}}$. That implies that the extended transformation $\delta\vartheta$ is a trivial symmetry of the action S_{H}.

4.2 Gauge symmetries

Below, we are going to prove the following assertions:

In theories with FCC there exist nontrivial symmetries $\delta\vartheta$ of the Hamiltonian action S_{H} that are gauge transformations. These symmetries are parametrized by the gauge parameters ν. The latter parameters are arbitrary functions of time t, moreover, they can be arbitrary local functions of $\vartheta=(\vartheta, \lambda)$.

The corresponding conserved charge (the gauge charge) is a local function $G=G\left(\mathcal{P}, \nu^{[l]}\right)$, which vanishes on the extremals. The gauge charge has the following decomposition with respect to the FCC:

$$G=\sum_{i=1}^{\aleph_{\chi}}\nu_{(a)}\mathcal{P}^{(a|a)}+\sum_{i=1}^{\aleph_{\chi}-1}\sum_{a=i+1}^{\aleph_{\chi}}C_{i|a}\mathcal{P}^{(i|a)} . \tag{28}$$

Here $C_{i|a}\left(\nu^{[l]}\right)$ are some local functions, which can be determined from the symmetry equation in an algebraic way, and $\nu=\left(\nu_{(a)}\right)$, $\nu_{(a)}=\left(\nu_{(a)}^{\mu_{a}}\right)$, $a=$

$1,...,\aleph_\chi$. Here $\nu_{(a)}$ are gauge parameters related to the FCC in a chain with the number a. The number of the gauge parameters $\nu_{(a)}$ is equal to the number of the FCC in the chain a. The index μ_a labels constraints (and gauge parameters) inside the chain.

The total number of the gauge parameters is equal to the number of the primary FCC. The total number of the independent gauge parameters together with their time derivatives that enter essentially in the gauge charge is equal to the number of all the FCC,

The gauge charge is the generating function for the variations $\delta\vartheta$ of the phase-space variables,

$$\delta\vartheta = \{\vartheta, G\}\ . \tag{29}$$

The variations $\delta\lambda_{\mathcal{U}}$ are zero and $\delta\lambda^{a}_{\mathcal{P}} = \nu^{[a]}_{(a)}$.

To prove the above assertion, we consider the symmetry equation (23) for the case under consideration. Taking into account the action structure (1,3) and the anticipated form of the gauge charge (28) and of the variations $\delta\vartheta$, we may rewrite this equation as follows:

$$\hat{H}G - \sum_{a=1}^{\aleph_\chi} \lambda^{a}_{\mathcal{P}} \left\{\mathcal{P}^{(1|a|)}, G\right\} = U^{(1)}\delta\lambda_U + \sum_{a=1}^{\aleph_\chi} \delta\lambda^{a}_{\mathcal{P}}\mathcal{P}^{(1|a)}\,, \tag{30}$$

where

$$\hat{H}G = \{G, H\} + \left(\frac{\partial}{\partial t} + \nu^{[m+1]}\frac{\partial}{\partial \nu^{[m]}}\right)G\,,$$

$$H = H_{\rm ph}(\omega) + \sum_{a=1}^{\aleph_\chi}\sum_{i=1}^{a-1} Q^{(i|a)}\mathcal{P}^{(i+1|a)} + U^{(2...)}FU^{(2...)}\,. \tag{31}$$

The following commutation relations hold true

$$\left\{\mathcal{P}^{(i|a)}, H\right\} = -\mathcal{P}^{(i+1|a)}\,,\ i = 1, ..., \aleph_\chi - 1\,,\ a = i+1, ..., \aleph_\chi\,,$$

$$\left\{\mathcal{P}^{(i|i)}, H\right\} = 0\,,\ i = 1, ..., \aleph_\chi\,;\ \{\mathcal{P}, \Omega\} = 0\,.$$

The equation (30) implies the following equations for the functions $C_{i|a}$ and the variations $\delta\lambda$

$$\sum_{i=1}^{\aleph_\chi - 1}\sum_{a=i+1}^{\aleph_\chi}\left[-\mathcal{P}^{(i+1|a)} + \mathcal{P}^{(i|a)}\left(\frac{\partial}{\partial t} + \nu^{[m+1]}\frac{\partial}{\partial \nu^{[m]}}\right)\right]C_{i|a}$$

$$+\sum_{a=1}^{\aleph_\chi}\mathcal{P}^{(a|a)}\dot{\nu}_{(a)} = U^{(1)}\delta\lambda_U + \sum_{a=1}^{\aleph_\chi}\mathcal{P}^{(1|a)}\delta\lambda^{a}_{\mathcal{P}}\,. \tag{32}$$

Considering Eq. (32) on the constraint surface $\mathcal{P}^{(i|a)} = 0$, $i = 1, ..., \aleph_\chi - 1$, $a = i, ..., \aleph_\chi$, $U^{(1)} = 0$, we can choose $C_{\aleph_\chi - 1|\aleph_\chi} = \dot{\nu}_{(\aleph_\chi)}$. Substituting this

$C_{\aleph_\chi-1|\aleph_\chi}$ into Eq. (32), we get

$$\sum_{i=1}^{\aleph_\chi-2}\sum_{a=i+1}^{\aleph_\chi}\left[-\mathcal{P}^{(i+1|a)}+\mathcal{P}^{(i|a)}\left(\frac{\partial}{\partial t}+\nu^{[m+1]}\frac{\partial}{\partial\nu^{[m]}}\right)\right]C_{i|a}$$
$$+\mathcal{P}^{(\aleph_\chi-1|\aleph_\chi)}\nu^{[2]}_{\aleph_\chi}+\sum_{a=1}^{\aleph_\chi-1}\mathcal{P}^{(a|a)}\dot{\nu}_{(a)}=U^{(1)}\delta\lambda_U+\sum_{a=1}^{\aleph_\chi}\mathcal{P}^{(1|a)}\delta\lambda^a_{\mathcal{P}}\,. \qquad (33)$$

Considering the equation (33) on the constraint surface $\mathcal{P}^{(i|a)}=0,\ i=1,...,\aleph_\chi-2,\ a=i,...,\aleph_\chi,\ U^{(1)}=0$, we choose $C_{\aleph_\chi-2|\aleph_\chi-1}=\dot{\nu}_{(\aleph_\chi-1)}$, $C_{\aleph_\chi-2|\aleph_\chi}=\ddot{\nu}_{(\aleph_\chi)}$. One can see that in the same manner, we determine all the $C_{i|a}$ to be

$$C_{i|a}=\nu^{[a-i]}_{(a)},\ i=1,...,\aleph_\chi-1,\ a=i+1,...,\aleph_\chi\,. \qquad (34)$$

Thus, in the case under consideration, the gauge charge has the following form

$$G=\sum_{i=1}^{\aleph_\chi}\sum_{a=i}^{\aleph_\chi}\nu^{[a-i]}_{(a)}\mathcal{P}^{(i|a)}\,. \qquad (35)$$

The form of the variations $\delta\vartheta$ follows from (29),

$$\delta Q^{(i|a)}=\nu^{[a-i]}_{(a)},\ \delta\omega=\delta\Omega=0\,. \qquad (36)$$

Thereafter all the $C_{i|a}$ are known, the variations $\delta\lambda$ can be determined from Eq. (32),

$$\delta\lambda_U=0\,,\ \delta\lambda^a_{\mathcal{P}}=\nu^{[a]}_{(a)}\,. \qquad (37)$$

Thus, the assertion is justified.

5 Structure of arbitrary symmetry

Analyzing the symmetry equation, we are going to prove that:

Any symmetry $\delta\vartheta$ and G of the action $S_{\rm H}$ can be presented as the sum of three type of symmetries

$$\begin{pmatrix}\delta\vartheta\\ G\end{pmatrix}=\begin{pmatrix}\delta_c\vartheta\\ G_c\end{pmatrix}+\begin{pmatrix}\delta_g\vartheta\\ G_g\end{pmatrix}+\begin{pmatrix}\delta_{\rm tr}\vartheta\\ G_{\rm tr}\end{pmatrix}, \qquad (38)$$

such that:

The set $\delta_c\vartheta$ and G_c is a global symmetry, canonical for the phase-space variables ϑ. All the variations $\delta_c\vartheta$ and the corresponding conserved charge G_c are either identically zero or do not vanish on the extremals.

The set $\delta_g\vartheta$ and G_g is a particular gauge transformation given by Eqs. (35), (36), and (37) with specific fixed gauge parameters (i.e. specific fixed forms of the functions $\nu=\bar{\nu}\left(t,\eta^{[l]},\lambda^{[l]}\right)$) that are either identically zero or do not vanish on the extremals. In the latter case, the corresponding conserved charge G_g vanishes on the extremals, whereas the variations $\delta_g\vartheta$ do not.

The set $\delta_{\rm tr}\vartheta$ and $G_{\rm tr}$ is a trivial symmetry. All the variations $\delta_{\rm tr}\vartheta$ and the corresponding conserved charge $G_{\rm tr}$ vanish on the extremals. The charge $G_{\rm tr}$ depends on the extremals as $G_{\rm tr} = O\left(I^2\right)$.

Below we present a constructive way to find the components of the decomposition (38).

5.0.1 Constructing the global canonical part of a symmetry

Supposing that $\delta\vartheta$ and G is a symmetry, and taking into account the structure of the total Hamiltonian in the case under consideration, we can write the symmetry equation (??) as follows

$$\delta\vartheta E^{-1}\mathcal{I} - U^{(1)}\delta\lambda_U - \lambda_U\delta U^{(1)} - \mathcal{P}^{(1)}\delta\lambda_{\mathcal{P}} + \frac{dG}{dt} = 0, \tag{39}$$

$$I = (\Omega, J) = O\left(\frac{\delta S_{\rm H}}{\delta\vartheta}\right), \; J = (\mathcal{I},\ \lambda_U), \; \mathcal{I} = \dot{\vartheta} - \{\vartheta, H\} - \left\{\vartheta, \mathcal{P}^{(1)}\right\}\lambda_{\mathcal{P}},$$

Let us denote via $\delta_J\vartheta$, G'_J the corresponding zero-order terms in the decomposition of the quantities $\delta\vartheta$, G with respect to the extremals J. Then

$$\begin{pmatrix} \delta\vartheta \\ G \end{pmatrix} = \begin{pmatrix} \delta_J\vartheta(\eta, \lambda_{\mathcal{P}}^{[l]}) + O(J) \\ G'_J(\eta, \lambda_{\mathcal{P}}^{[l]}) + B_m(\omega, \lambda_{\mathcal{P}}^{[l]})J^{[m]} + O(J^2) \end{pmatrix}. \tag{40}$$

Then we rewrite the equation (39) retaining only the terms of zero and first order with respect to the extremals J. Thus, we obtain

$$\begin{aligned} &\delta_J\vartheta E^{-1}\mathcal{I} - \mathcal{P}^{(1)}\delta\lambda_{\mathcal{P}} = -\hat{H}G'_J + \left\{\mathcal{P}^{(1)}, G'_J\right\}\lambda_{\mathcal{P}} + \lambda_U\{U^{(1)}, G'_J\} \\ &+ \{\vartheta, G'_J\}E^{-1}\mathcal{I} - J^{[m]}\hat{H}B_m + \lambda_{\mathcal{P}}\left\{\mathcal{P}^{(1)}, B_m\right\}J^{[m]} - B_m J^{[m+1]} + O(\Omega I). \end{aligned} \tag{41}$$

Here, the contributions from the terms $U^{(1)}\delta\lambda_U$ and $\delta\lambda_U U^{(1)}$ are accumulated in the term $O(\Omega I)$, and the operator $\hat{H}$ is defined by

$$\hat{H}F = \{F, H\} + \left(\frac{\partial}{\partial t} + \lambda^{[m+1]}\frac{\partial}{\partial\lambda^{[m]}}\right)F. \tag{42}$$

Analyzing terms with the extremals $J^{[m]}$ (beginning with the highest derivative) in Eq. (41), we can see that all $B_m = 0$. Considering then terms proportional to $\mathcal{I}$ in Eq. (41), we get for the variation $\delta_J\vartheta$ the expression

$$\delta_J\vartheta = \{\vartheta, G'_J\} + O(\Omega).$$

Then, with account taken of the relations

$$\delta\Omega = O(I) \Longrightarrow \delta_J\Omega = O(\Omega) = \{\Omega, G'_J\} + O(\Omega),$$

we can see that

$$\{\Omega, G'_J\} = O(\Omega)\,. \tag{43}$$

One can check that $\{\mathcal{P}, G'_J\}$ is the first-class function, which means that

$$\{\mathcal{P}, G'_J\} = O(\mathcal{P}) + O(\Omega^2)\,. \tag{44}$$

Considering the remaining terms in Eq. (41), we get the equation

$$\mathcal{P}^{(1)}\delta_J\lambda_{\mathcal{P}} = \hat{H}G'_J + \lambda_{\mathcal{P}}\left\{\mathcal{P}^{(1)}, G'_J\right\} + O(\Omega^2)\,, \tag{45}$$

which relates $\delta_J\lambda_{\mathcal{P}}$ and G'_J.

This equation (45) allows us to study the function G'_J in more detail. To this aim, we rewrite this equation (taking into account (42) and (44)) as

$$\{G'_J, H\} + \left(\frac{\partial}{\partial t} + \lambda_{\mathcal{P}}^{[m+1]}\frac{\partial}{\partial\lambda_{\mathcal{P}}^{[m]}}\right)G'_J = O(\mathcal{P}) + O(\Omega^2)\,.$$

Analyzing terms with the Lagrange multipliers $\lambda_{\chi}^{[m]}$ (beginning with the highest derivative) in this equation, we can see that these multipliers may enter only the terms that vanish on the constraint surface. For example, considering terms with the highest derivative $\lambda_{\mathcal{P}}^{[M+1]}$ in the latter equation, we get

$$\frac{\partial G'_J}{\partial\lambda_{\mathcal{P}}^{[M]}} = O(\mathcal{P}) + O(\Omega^2) \Longrightarrow G'_J = G'_J(\cdots\lambda_{\mathcal{P}}^{[M-1]}) + O(\mathcal{P}) + O(\Omega^2).$$

In the same manner we finally obtain: $G'_J = G_J(\vartheta) + O(\mathcal{P}) + O(\Omega^2)$. With account taken of Eqs. (43) and (44) this implies:

$$\{U, G_J\} = O(\Omega)\,,\ \{\mathcal{P}, G_J\} = O(\mathcal{P}) + O(\Omega^2)\,.$$

Thus, the above consideration allows us to represent a refined version of the representation (40)

$$\begin{pmatrix} \delta\vartheta \\ \delta\lambda_U \\ \delta\lambda_{\mathcal{P}} \\ G \end{pmatrix} = \begin{pmatrix} \{\vartheta, G_J + B_{\mathcal{P}}\mathcal{P}\} + O(I) \\ O(I) \\ \delta_J\lambda_{\mathcal{P}}(\vartheta, \lambda_{\mathcal{P}}^{[l]}) + O(J) \\ G_J + B_{\mathcal{P}}\mathcal{P} + O(I^2) \end{pmatrix}. \tag{46}$$

where $B_{\mathcal{P}} = B_{\mathcal{P}}(\vartheta, \lambda_{\mathcal{P}}^{[l]})$, and the function $G_J(\vartheta)$ obeys the relations

$$\{G_J, U\} = O(\Omega)\,,\ \{G_J, \mathcal{P}\} = O(\mathcal{P}) + O(\Omega^2)\,,\ \{G_J, H\} = O(\mathcal{P}) + O(\Omega^2)\,. \tag{47}$$

We select from the function G_J a part G_I that does not vanish on the constraint surface,

$$G_J(\vartheta) = g\,(\omega) + g_1\,(\omega, Q)\,Q + O\,(\Omega)\,. \tag{48}$$

Due to the relation (47), the function $g_1(\omega, Q)$ in (48) is zero, and, moreover, $O(\Omega) = O(\mathcal{P}) + O(\Omega^2)$. We define $G_I(\vartheta)$ as

$$G_I(\vartheta) = g(\omega) \,. \tag{49}$$

Then

$$G_J(\vartheta) = G_I(\vartheta) + G_1(\vartheta),\; G_1(\vartheta) = O(\mathcal{P}) + O(\Omega^2)\,.$$

Therefore, in virtue of (47),

$$\begin{aligned} &G = G_I(\vartheta) + O(\mathcal{P}) + O(I^2)\,, \\ &\hat{H}G_I = 0,\; \{U, G_I\} = \{\mathcal{P}, G_I\} = 0\,. \end{aligned} \tag{50}$$

Now we define the variations $\delta_I\vartheta$ as

$$\begin{aligned} &\delta_I\vartheta = \{\vartheta, G_I\} \Longrightarrow \delta_I\omega = \{\omega, G_I\}\,,\; \delta_I Q = \delta_I\mathcal{P} = \delta_I U = 0\,, \\ &\delta_I\lambda_U = \delta_I\lambda_{\mathcal{P}} = 0\,. \end{aligned} \tag{51}$$

The set $\delta_I\vartheta$, G_I is an exact symmetry of the action S_{H}. In what follows this symmetry is denoted as

$$\delta_I\vartheta = \delta_c\vartheta = \{\vartheta, G_c\},\;\; \delta_I\lambda = \delta_c\lambda = 0\,,\; G_I = G_c = g(\omega)\,.$$

5.0.2 Constructing the gauge and trivial parts of a symmetry

At this step we represent a symmetry $\delta\vartheta$, G as

$$\delta\vartheta = \delta_c\vartheta + \delta_r\vartheta,\; G = G_c + G_r\,. \tag{52}$$

Since $\delta_c\vartheta$, G_c is a symmetry, it is obvious that $\delta_r\vartheta$, G_r is a symmetry as well. By the help of Eqs. (47), we can verify that the following relations hold true

$$\begin{aligned} G_r &= \sum_{i=1}^{\aleph_{\mathcal{P}}}\sum_{a=i}^{\aleph_{\mathcal{P}}} K_{i|a}\left(\omega, Q, \lambda_{\mathcal{P}}^{[l]}\right)\mathcal{P}^{(i|a)} + O\left(I^2\right)\,, \\ \delta_r\eta &= \sum_{i=1}^{\aleph_{\mathcal{P}}}\sum_{a=i}^{\aleph_{\mathcal{P}}}\left\{\eta, \mathcal{P}^{(i|a)}\right\} K_{i|a}\left(\omega, Q, \lambda_{\mathcal{P}}^{[l]}\right) + O\left(I\right)\,, \end{aligned} \tag{53}$$

where K are some LF.

In turn, let us represent the symmetry $\delta_r\vartheta$, G_r in the following form

$$\delta_r\vartheta = \delta_{\bar{\nu}}\vartheta + \delta_{\mathrm{tr}}\vartheta\,,\; G_r = G_{\bar{\nu}} + G_{\mathrm{tr}}\,, \tag{54}$$

where the set $\delta_{\bar{\nu}}\vartheta$, $G_{\bar{\nu}}$ is the gauge transformation given by Eqs. (??), (29), and (??) with specific fixed values of the gauge parameters,

$$\nu_i = \bar{\nu}_i\left(t, \eta, \lambda^{[l]}\right) = K_{i|i}\left(\omega, Q, \lambda_{\mathcal{P}}^{[l]}\right), \tag{55}$$

that either are identically zero or do not vanish on the constraint surface. This implies:

$$G_{\bar{\nu}} = O\left(\mathcal{P}\right) + O\left(I^2\right)\ ,\ \delta_{\bar{\nu}}\vartheta = \{\vartheta, G_{\bar{\nu}}\}\ . \tag{56}$$

One ought to stress that by the construction, the functions $K_{i|i}$ (and therefore the gauge transformations) are identically zero whenever they vanish on the constraint surface.

It follows the from Eqs. (53) that $\delta_{\mathrm{tr}}\vartheta$, G_{tr} is a symmetry with the charge of the form

$$G_{\mathrm{tr}} = G'_{\mathrm{tr}} + O(I^2)\,,\ G'_{\mathrm{tr}} = \sum_{i=1}^{\aleph_{\mathcal{P}}-1}\sum_{a=i+1}^{\aleph_{\mathcal{P}}} K_{i|a}\left(\omega, Q, \lambda_{\mathcal{P}}^{[l]}\right)\mathcal{P}^{(i|a)}\,.$$

Below we will see that $\delta_{\mathrm{tr}}\vartheta$, G_{tr} is a trivial symmetry. Let us write for the symmetry $\delta_{\mathrm{tr}}\vartheta$, G_{tr} the decomposition of the form (40), taking into account that $B_m = O(\Omega)$,

$$\begin{pmatrix} \delta\vartheta_{\mathrm{tr}} \\ G_{\mathrm{tr}} \end{pmatrix} = \begin{pmatrix} \delta_{\mathrm{tr}J}\vartheta(\omega, Q, \lambda_{\mathcal{P}}^{[l]}) + O(J) \\ G'_{\mathrm{tr}J}(\vartheta, \lambda_{\mathcal{P}}^{[l]}) + O(J^2),\ G'_{\mathrm{tr}J} = G'_{\mathrm{tr}} + O(\Omega^2) \end{pmatrix}. \tag{57}$$

All the relations that take place for the quantities $\delta_J\vartheta$, G_J hold true for the quantities $\delta_{\mathrm{tr}J}\vartheta$, $G'_{\mathrm{tr}J}$ as well. In particular, the charge G'_{tr} obeys the equation

$$\mathcal{P}^{(1)}\delta'_{\mathrm{tr}}\lambda_{\chi} = \hat{H}G'_{\mathrm{tr}} + \lambda_{\mathcal{P}}\left\{\mathcal{P}^{(1)}, G'_{\mathrm{tr}}\right\} + O(\Omega^2),\ \delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} = \delta_{\mathrm{tr}}\lambda_{\mathcal{P}}|_{I=0} = \delta_{\mathrm{tr}J}\lambda_{\mathcal{P}} + O(\Omega), \tag{58}$$

which is similar to the one (45). The equation (58) implies the following equation for the LF $K_{i|a}$, $a = i+1, ..., \aleph_{\mathcal{P}}$:

$$\sum_{i=1}^{\aleph_{\mathcal{P}}-1}\sum_{a=i+1}^{\aleph_{\mathcal{P}}}\left(\mathcal{P}^{(i|a)}\hat{H}K_{i|a} + K_{i|a}\mathcal{P}^{(i+1|a)}\right) = \mathcal{P}^{(1)}\delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} + O(\Omega^2)\,.$$

Considering this equation on the constraint surface $\Omega^{(\ldots\aleph_{\mathcal{P}}-1)} = 0$, we obtain

$$K_{\aleph_{\mathcal{P}}-1|\aleph_{\mathcal{P}}}\mathcal{P}^{(\aleph_{\mathcal{P}}|\aleph_{\mathcal{P}})} = O(\Omega^2) \Longrightarrow K_{\aleph_{\mathcal{P}}-1|\aleph_{\mathcal{P}}} = 0\,.$$

Substituting the expression for $K_{\aleph_{\mathcal{P}}-1|\aleph_{\mathcal{P}}}$ into Eq. (58), and considering the resulting equation on the constraint surface $\Omega^{(\ldots\aleph_{\mathcal{P}}-2)} = 0$, we obtain $K_{\aleph_{\mathcal{P}}-2|\aleph_{\mathcal{P}}} = 0$, and so on. Thus, we can see that all $K_{i|a} = 0$, $a = i+1, ..., \aleph_{\mathcal{P}}$, and therefore

$$G_{\mathrm{tr}} = O(I^2)\,. \tag{59}$$

Then it follows from Eq. (58)

$$\mathcal{P}^{(1)}\delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} = O(\Omega^2) \Longrightarrow \delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} = O(\Omega) \Longrightarrow \delta_{\mathrm{tr}}\lambda_{\mathcal{P}} = O(I)\,.$$

By the construction, the transformation $\delta_{\mathrm{tr}J}$ is completely similar to the one δ_J. Therefore, the relation (47) holds true for this transformation and implies:

$$\delta_{\mathrm{tr}J}\vartheta = \{\vartheta, G'_{\mathrm{tr}}\} + O\left(\Omega\right) = O(\Omega)\,,\ \ \delta_{\mathrm{tr}J}\lambda_U = O(\Omega)\,.$$

Therefore

$$\delta_{\mathrm{tr}}\vartheta = O(I)\,. \tag{60}$$

The relations (59) and (60) prove that the symmetry $\delta_{\mathrm{tr}}\vartheta$, G_{tr} is trivial.

We note in addition, that the reduction of symmetry variations $\delta\omega$ on the extremals are global canonical symmetries of the physical action with the conserved charge that is the reduction of the complete conserved charge on the extremals (the Eqs. (??) take place).

Any global canonical symmetry $\delta_{\mathrm{ph}}\omega$, $g(\omega)$ (here $g(\omega)$ is the corresponding conserved charge) of the physical action is nontrivial global symmetry $\delta_c\vartheta$, G_c of the Hamiltonian action S_{H}.

6 Physical functions

First of all we recall general understanding which physics may be described in terms of gauge theories [2]. Let the time evolution of a classical system is given by genuine trajectories $\kappa(t)$ in the configuration space, the latter are solutions of the equations of motion of the theory. On the other side the state of the classical system at any given time instant t is characterized by the set $\kappa^{[l]}(t)\,,\ l\geq 0$ at this time instant, i.e. by a point in the jet space. The trajectory in the configuration space creates a trajectory in the jet space. The latter trajectory can be called the trajectory of system states. We call two trajectories in the configuration space intersecting if the corresponding trajectories in the jet space intersect at a time instant. Using such a terminology and results of the Sect. III, one can say that intersecting trajectories do exist in gauge theories . On the other side, we believe that for classical systems one can introduce the notion of the system physical state at each time instant, such that there exist a causal evolution of the physical states in time. Namely, once a physical state is given at a certain time, at all other times the physical states are determined in a unique way. All the physical quantities are single-valued functions of the physical state at a given time instant. The physical state is completely determined as soon as all possible physical quantities at this time are given. Thus, on the first glance, there is a disagreement between the causal evolution of the physical states and the absence of the causal evolution of trajectories in the jet space for gauge theories. To eliminate this discrepancy and to be able describe consistently classical systems by the help of gauge theories, one can resort the following natural interpretation:

a) Physical states of a classical system and, therefore all local physical quantities are determined uniquely by points of genuine trajectories in the jet space.

b) All the functions that are used to describe physical quantities must coincide at equal-time points of genuine intersecting trajectories in the jet space.

The item (b) ensures independence of the physical quantities of the arbitrariness inherent in solutions of a gauge theory and reconciles the item (a) with the causal development of the physical states in time. The item (b) imposes limitations on the possible form of these functions. The local functions that obey the item (b) are called physical functions. Suppose the local functions $\mathcal{A}_{\mathrm{ph}}\left(\kappa^{[l]}\right)$ be physical. This implies that for two arbitrary genuine intersecting trajectories κ and κ' the equality

$$\mathcal{A}_{\text{ph}}\left(\kappa^{[l]}\right)=\mathcal{A}_{\text{ph}}\left(\kappa'^{[l]}\right) \tag{61}$$

holds true at any time instant.

Let us consider physical local functions in the Hamiltonian formulation and in special phase-space variables ϑ. Taking into account the equations of motion (??) and $\Omega=0$, we may conclude that any physical local functions of the form $\mathcal{A}_{\text{ph}}\left(\vartheta^{[l]}\right)$ can be represented as follows

$$\mathcal{A}_{\text{ph}}\left(\vartheta^{[l]}\right)=a_{\text{ph}}\left(\omega,Q,\lambda_{\mathcal{P}}^{[l]}\right)+O\left(\frac{\delta S}{\delta\vartheta}\right). \tag{62}$$

Now it easy to establish restrictions on the functions a_{ph} that follow from the condition (61) of physicality. To this end, we recall that there exist two genuine intersecting at $t=0$ trajectories ϑ and ϑ' such that at the time instant t they differ by the values of the variables Q and $\lambda_{\mathcal{P}}^{[l]}$ only, having the same ω. Namely,

$$\begin{aligned}
&\vartheta\left(t\right)\Longrightarrow\left(Q,\,\lambda_{\mathcal{P}}^{[l]}\right),\\
&\vartheta'\left(t\right)\Longrightarrow\left(Q+\delta Q,\,\lambda_{\mathcal{P}}^{[l]}+\delta\lambda_{\mathcal{P}}^{[l]}\right),
\end{aligned} \tag{63}$$

where all the quantities $Q,\ \lambda_{\mathcal{P}}^{[l]},\delta Q$, and $\delta\lambda_{\mathcal{P}}^{[l]}$ are arbitrary given. The existence of such intersecting trajectories follows from the above consideration. The relation (61) taken for such two intersecting trajectories implies for the function a_{ph},

$$a_{\text{ph}}\left(\omega\left(t\right),Q,\,\lambda_{\mathcal{P}}^{[l]}\right)=a_{\text{ph}}\left(\omega\left(t\right),Q+\delta Q,\,\lambda_{\mathcal{P}}^{[l]}+\delta\lambda_{\mathcal{P}}^{[l]}\right). \tag{64}$$

Because of the arbitrariness of the quantities $Q,\ \lambda_{\mathcal{P}}^{[l]},\delta Q$, and $\delta\lambda_{\mathcal{P}}^{[l]}$, we obtain from the equation (64)

$$\frac{\partial a_{\text{ph}}}{\partial Q}=\frac{\partial a_{\text{ph}}}{\partial\lambda^{[l]}}=0\Longrightarrow a_{\text{ph}}=a_{\text{ph}}\left(\omega\right). \tag{65}$$

Therefore, physical local functions of the form $\mathcal{A}_{\text{ph}}\left(\vartheta^{[l]}\right)$ can be represented as follows

$$\mathcal{A}_{\text{ph}}\left(\vartheta^{[l]}\right)=a_{\text{ph}}\left(\omega\right)+O\left(\frac{\delta S_{\text{H}}}{\delta\vartheta}\right). \tag{66}$$

In terms of the initial phase-space variables $\eta=(\eta,\lambda)$, $\eta=(q,p)$, any physical local functions of the form $A_{\text{ph}}\left(\eta^{[l]}\right)$ has the structure

$$A_{\text{ph}}\left(\eta^{[l]}\right)=a_{\text{ph}}\left(\eta\right)+O\left(\frac{\delta S_{\text{H}}}{\delta\eta}\right). \tag{67}$$

It follows from (66) that bearing in mind that the set of constraints $\mathcal{P}$ is equivalent to all FCC $\chi\left(\eta\right)$ in the initial phase-space variables, the set of constraints

Ω is equivalent to all the initial constraints $\Phi(\eta)$, one can write the physicality conditions for the functions $a_{\mathrm{ph}}(\eta)$:

$$\frac{\partial a_{\mathrm{ph}}}{\partial Q} = O(\Omega) \Longleftrightarrow \{a_{\mathrm{ph}}, \chi\} = O(\Phi) . \tag{68}$$

We are going to call conditions (67) and (68) the physicality condition in the Hamiltonian sense. Exactly in such a sense one has to understand the usual assertion that physical functions must commute with first-class constraints on extremals. In fact, that is these condition of physicality which usually is called the Dirac conjecture.

On the other hand, a consideration in the Lagrangian formulation implies that physical functions must be gauge invariant on the extremals, see e.g. [2]. Such a condition of physicality we call the physicality condition in the Lagrangian sense. Below, we are going to demonstrate equivalence of these two conditions.

Let a local functions $A = A\left(\eta^{[l]}\right)$ be physical in the Hamiltonian sense. Consider its gauge variation $\delta\mathcal{A}$. Such a variation has the form with account taken of (67)

$$\delta A = \delta a(\eta) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta\eta}\right) . \tag{69}$$

Here we have used the fact that gauge variations of extremals are proportional to extremals. Let us consider δa taking into account (29) and (28). Then one easily see that

$$\delta a = \{a, G\} = O(\{a, \chi\}) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta\eta}\right) .$$

Taking into account (68), we obtain that gauge variation of a physical functions are proportional to extremals,

$$\delta A = O\left(\frac{\delta S_{\mathrm{H}}}{\delta\eta}\right) . \tag{70}$$

Let now a local functions $A = A\left(\eta^{[l]}\right)$ be physical in the Lagrangian sense, i.e. it obeys Eq. (70). One can always to represent the function in the form

$$A = f(\eta, \lambda_{\mathcal{P}}^{[l]}) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta\eta}\right) .$$

The condition (70) implies:

$$\{f, G\} + \sum_{m=0}^{m_{\max}} \frac{\partial f}{\partial\lambda_{\mathcal{P}}^{[m]}} \delta\lambda_{\mathcal{P}}^{[m]} = O\left(\frac{\delta S_{\mathrm{H}}}{\delta\eta}\right) . \tag{71}$$

Let us consider terms with highest time-derivatives of the gauge parameters in the left-hand side of (71). Taking into account that $\delta\lambda_{\mathcal{P}}^{a} = \nu_a^{[a]}$, see (37), and the fact that G contains only the time derivatives $\nu_a^{[l]}$, $l < a$), such terms have the form:

$$\sum_a^{\aleph_\chi} \frac{\partial f}{\partial \lambda_{\mathcal{P}}^{a[m_{\max}]}} \nu_a^{[a+m_{\max}]} .$$

These terms have to be proportional to the extremals, which implies

$$\frac{\partial f}{\partial \lambda_{\mathcal{P}}^{a[m_{\max}]}} = O\left(\frac{\delta S_{\mathrm{H}}}{\delta \eta}\right) .$$

Similarly, we can verify that the function f does not contain any λ on the extremals, i.e.

$$f(\eta, \lambda_\chi^{[l]}) = a\,(\eta) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta \eta}\right) .$$

Therefore,

$$\mathcal{A} = a(\eta) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta \eta}\right) . \qquad (72)$$

Considering the equation (70) for the function (??), we obtain

$$\{a, G\} = \sum_{i=1}^{\aleph_\chi} \sum_{b=i}^{\aleph_\chi} \left\{a, \mathcal{P}^{(i|b)}\right\} \nu_b^{[b-i]} = O\left(\frac{\delta S_{\mathrm{H}}}{\delta \eta}\right) ,$$

which implies (due to the independence of $\nu_b^{[b-i]}$)

$$\left\{a, \mathcal{P}^{(i|b)}\right\} = O\left(\frac{\delta S_{\mathrm{H}}}{\delta \eta}\right) = O(\Phi) .$$

This completes the proof of the equivalence of the two definitions of physical functions.

7 Conclusion

Below we summarize the main conclusions.

Any continuos symmetry transformation can be represented as a sum of three kind of symmetries, global, gauge, and trivial one. If the global part of a symmetry and the corresponding canonical charge are not identically zeros, they do not vanish on extremals. One ought to say that this separation is not non-unique. In particular, the determination of the canonical charge from the corresponding equation and thus the determination of the canonical part of a symmetry transformation is then ambiguous. However, one can see the ambiguity in the canonical part of a symmetry transformation is always a sum of a gauge and a trivial transformation. The gauge part of a symmetry does not vanish on extremals, but the gauge charge vanishes on the extremals. We stress that the gauge charge necessary contains a part that vanish linearly on the FCC, and the rest part of the gauge charge vanishes quadratically on extremals. The trivial part of any symmetry vanish on extremals and the corresponding charge vanishes quadratically on extremals.

The reduction of global symmetry transformations on extremals are global canonical symmetries of the physical action with the conserved charge that is the reduction of the complete conserved charge on the extremals.

Any global symmetry of the physical action is a global symmetry of the complete Hamiltonian action.

The gauge transformations, taken on extremals, transform only the nonphysical variables Q and $\lambda_{\mathcal{P}}$.

Now one can see that do not exist other gauge transformations that cannot be represented in the form (28). That follows from the structure of arbitrary symmetry transformation represented above. Namely, as was demonstrated, any symmetry transformation with the charge that vanishes on extremals is a sum of the particular gauge transformation and of a trivial transformation.

The gauge charge contains time derivatives of the gauge parameters whenever there exist secondary FCC.

In the same manner as was done in Sect. II, one can demonstrate that there exist a choice of superspecial phase-space variables that includes already the variables U and which simplifies significantly the Hamiltonian $H^{(1)}_{\mathrm{SCC}}$. Namely, at such a choice the variables U have the following structure $U = (V;\, u)$, where both V and u are sets of pairs of conjugate coordinates and momenta. The variables from these sets are divided into groups according to stages of the Dirac procedure and organized in chains (labeled by the index a). The variables V consist of coordinates Θ and conjugate momenta Π. Namely:

$$
\begin{aligned}
&V = (\Theta^{(i|2a)}_{\mu_a},\, \Theta^{(i|2a+1)}_{\nu_a,s};\Pi^{(i|2a)}_{\mu_a},\, \Pi^{(i|2a+1)}_{\nu_a,s})\,,\ \ 1 \le a \le \aleph_\varphi/2\,,\ i = 1,\dots,a\,,\ s = 1,2,\\
&u = \left(u^{(1)}_{\zeta,s}\,,\, u^{(2a+1)}_{\nu_a,s}\right).
\end{aligned}
$$

The variables $u^{(1)}$ are primary constraints (constraints of the first stage); the variables $u^{(2a+1)}$ are $2a{+}1-$stage constraints; the variables $\Pi^{(i|2a)}$ and $\Pi^{(i|2a+1)}$ are i-stage constraints; the variables $\Theta^{(i|2a)}$ are $2a-(i-1)$-stage constraints; the variables $\Theta^{(i|2a+1)}$ are $2a+1-(i-1)$-stage constraints.

The variables are divided in chains with even and odd numbers. Variables in the chains with even numbers $2a$ are labeled by the index μ_a, in the chains with odd numbers 1, $2a+1$ are labeled by the index ζ, ν_a and by the index s. The number of the indices μ_a and ζ, ν_a can be equal to zero.

In the variables V, u the Hamiltonian $H^{(1)}_{\mathrm{SCC}}$ takes the following form:

$$
\begin{aligned}
&H^{(1)}_{\mathrm{scc}} = h_{\mathrm{odd}} + h_{\mathrm{even}} + \lambda^{(1)}u^{(1)}\,,\\
&h_{\mathrm{even}} = \sum_{a=1}\left(\sum_{i=1}^{a-1}\Theta^{(i|2a)}\Pi^{(i+1|2a)} + \sigma_{2a}(\Theta^{(i|2a)})^2 + \lambda^{(2a)}\Pi^{(1|2a)}\right),\\
&h_{\mathrm{odd}} = \sum_{a=1}\left(\sum_{i=1}^{a-1}\Theta^{(i|2a+1)}\Pi^{(i+1|2a+1)} + \sigma_{2a+1}\Theta^{(a|2a+1)}u^{(2a+1)} + \lambda^{(2a+1)}\Pi^{(1|2a+1)}\right),
\end{aligned}
$$

where $\sigma \neq 0$ are some numbers. Certainly, there is a summation over the indices μ, ν, and ζ, in particular $\sigma_{2a}(\Theta^{(i|2a)})^2 = \sum_{\mu_a} \sigma_{2a,\mu_a}(\Theta_{\mu_a}^{(i|2a)})^2$.

In the refined superspecial phase-space variables, the consistency conditions that starts with the primary SCC determine all the corresponding Lagrange multipliers $\lambda^{(1)}, \lambda^{(2a)}$, and $\lambda^{(2a+1)}$ to be zero, see the scheme of constraint chains below

$$
\begin{array}{llllllllllllll}
u_s^{(1)} & \rightarrow & \lambda_s^{(1)} & & & & & & & & & & & \\
\Pi^{(1|2)} & \rightarrow & \Theta^{(1|2)} & \rightarrow & \lambda^{(2)} & & & & & & & & & \\
\Pi_s^{(1|3)} & \rightarrow & \Pi_s^{(2|3)} & \rightarrow & u_s^{(3)} & \rightarrow & \Theta_s^{(2|3)} & \rightarrow & \Theta_s^{(1|3)} & \rightarrow & \lambda_s^{(3)} & & & \\
\Pi^{(1|4)} & \rightarrow & \Pi^{(2|4)} & \rightarrow & \Theta^{(2|4)} & \rightarrow & \Theta^{(1|4)} & \rightarrow & \lambda^{(4)} & & & & & \\
\vdots & & & & & & & & & & & & & \\
\Pi^{(1|2a)} & \rightarrow & \ldots & \rightarrow & \Pi^{(a|2a)} & \rightarrow & \Theta^{(a|2a)} & \rightarrow & \ldots & \rightarrow & \Theta^{(1|2a)} & \rightarrow & \lambda^{(2a)} & \\
\Pi_s^{(1|2a+1)} & \rightarrow & \ldots & \rightarrow & \Pi_s^{(a|2a+1)} & \rightarrow & u_s^{(2a+1)} & \rightarrow & \Theta_s^{(a|2a+1)} & \rightarrow & \ldots & \rightarrow & \Theta_s^{(1|2a+1)} & - \\
\vdots & & & & & & & & & & & & &
\end{array}
$$

Finally, one ought to mention that we have demonstrated (using some natural suppositions) the equivalence of two definitions of physicality, one definition of physical functions in the Lagrangian formulation as those that are gauge invariant on extremals and another one in the Hamiltonian formulation that demands physical functions to commute with FCC (Dirac conjecture).

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References

[1] P.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York 1964)

[2] D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin 1990)

[3] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton 1992)

[4] D. M. Gitman, and I.V. Tyutin, Gravitation & Cosmology, **8**, No.1-2 (2002) 138; *Constraint reorganization consistent with Dirac procedure,* Michael Marinov Memorial Volume: *Multiple Facets of Quantization and Supersymmetry*, ed. M. Olshanetsky and A. Vainstein (World Publishing, Singapore 2002) pp.184-204

[5] M. Henneaux, Phys. Lett., **B238** (1990) 299; I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. A**11** (1996) 1352; D. M. Gitman, and I.V. Tyutin, Nucl. Phys. B630 (3) (2002) 509

[6] V.A. Borochov and I.V. Tyutin, Physics of Atomic Nuclei, **61** (1998) 1603; ibid **62** (1999) 1070