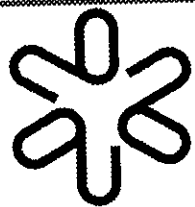


SYNNO 1429167 N688



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## COHOMOLOGIES OF THE POISSON SUPERALGEBRA OF THE GRASSMANN-VALUED FUNCTIONS ON 2-DIMENSIONAL SPACE

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Publicação IF – 1596/2004

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# Cohomologies of the Poisson superalgebra of the Grassmann-valued functions on 2-dimensional space.

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## Abstract

Cohomology spaces of the Poisson superalgebra realized on smooth Grassmann-valued functions with compact support on  $\mathbb{R}^2$  are investigated under suitable continuity restrictions on cochains. The first and second cohomology spaces with trivial coefficients and the zeroth, first and second cohomology spaces with coefficients in the adjoint representation of the Poisson superalgebra are found for the case of a nondegenerate constant Poisson superbracket.

The hope to construct the quantum mechanics on nontrivial manifolds is connected with geometrical or deformation quantization [1] - [4]. The functions on the phase space are associated with the operators, and the product and the commutator of the operators are described by associative  $*$ -product and  $*$ -commutator of the functions. These  $*$ -product and  $*$ -commutator are the deformations of usual product and of usual Poisson bracket.

To find the deformations of Poisson superalgebra, one should calculate the second cohomology of the Poisson superalgebra.

In [6], the lower cohomologies (up to second) were calculated for the Poisson algebra consisting of smooth complex-valued functions on  $\mathbb{R}^n$ . The pure Grassmanian case  $n = 0$  is considered in [7] and [8].

In [9], the lower cohomologies of the Poisson superalgebra in the trivial (up to third cohomology) and adjoint representation (up to second cohomology) for the case  $n > 2$  are calculated. It occurred that the case  $n = 2$  needs a separate consideration and an additional cohomologies arise in this case. The results of this consideration are presented in this publication. In [10] the central extensions of the algebras considered in this paper is considered.

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¶This work was supported by the RFRF (grants No. 02-01-00930 (I.T.) and No. 02-02-17067 (S.K.)), by INTAS (grant No. 00-00-262 (I.T.)) and by the grant LSS-1578.2003.2.

Let  $\mathbb{K}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . We denote by  $\mathcal{D}(\mathbb{R}^n)$  the space of smooth  $\mathbb{K}$ -valued functions with compact support on  $\mathbb{R}^n$ . This space is endowed by its standard topology: by definition, a sequence  $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$  converges to  $\varphi \in \mathcal{D}(\mathbb{R}^n)$  if  $\partial^\lambda \varphi_k$  converge uniformly to  $\partial^\lambda \varphi$  for every multi-index  $\lambda$ , and the supports of all  $\varphi_k$  are contained in a fixed compact set. We set

$$\mathbf{D}_{n_+}^{n_-} = \mathcal{D}(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad \mathbf{E}_{n_+}^{n_-} = C^\infty(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad \mathbf{D}_{n_+}^{m_-} = \mathcal{D}'(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-},$$

where  $\mathbb{G}^{n_-}$  is the Grassmann algebra with  $n_-$  generators and  $\mathcal{D}'(\mathbb{R}^{n_+})$  is the space of continuous linear functionals on  $\mathcal{D}(\mathbb{R}^{n_+})$ . The generators of the Grassmann algebra (resp., the coordinates of the space  $\mathbb{R}^{n_+}$ ) are denoted by  $\xi^\alpha$ ,  $\alpha = 1, \dots, n_-$  (resp.,  $x^i$ ,  $i = 1, \dots, n_+$ ). We shall also use collective variables  $z^A$  which are equal to  $x^A$  for  $A = 1, \dots, n_+$  and are equal to  $\xi^{A-n_+}$  for  $A = n_+ + 1, \dots, n_+ + n_-$ . The spaces  $\mathbf{D}_{n_+}^{n_-}$ ,  $\mathbf{E}_{n_+}^{n_-}$ , and  $\mathbf{D}_{n_+}^{m_-}$  possess a natural grading which is determined by that of the Grassmann algebra. The parity of an element  $f$  of these spaces is denoted by  $\varepsilon(f)$ . We also set  $\varepsilon_A = 0$  for  $A = 1, \dots, n_+$  and  $\varepsilon_A = 1$  for  $A = n_+ + 1, \dots, n_+ + n_-$ .

Let  $\partial/\partial z^A$  and  $\overleftarrow{\partial}/\partial z^A$  be the operators of the left and right differentiation. The Poisson bracket is defined by the relation

$$\{f, g\}(z) = f(z) \frac{\overleftarrow{\partial}}{\partial z^A} \omega^{AB} \frac{\partial}{\partial z^B} g(z) = -\sigma(f, g) \{g, f\}(z), \quad (1)$$

where  $\sigma(f, g) = (-1)^{\varepsilon(f)\varepsilon(g)}$  and the symplectic metric  $\omega^{AB} = (-1)^{\varepsilon_A \varepsilon_B} \omega^{BA}$  is a constant invertible matrix. For definiteness, we choose it in the form

$$\omega^{AB} = \begin{pmatrix} \omega^{ij} & 0 \\ 0 & \lambda_\alpha \delta^{\alpha\beta} \end{pmatrix}, \quad \lambda_\alpha = \pm 1, \quad i, j = 1, \dots, n_+, \quad \alpha, \beta = 1 + n_+, \dots, n_- + n_+$$

where  $\omega^{ij}$  is the canonical symplectic form (if  $\mathbb{K} = \mathbb{C}$ , then one can choose  $\lambda_\alpha = 1$ ). The nondegeneracy of the matrix  $\omega^{AB}$  implies, in particular, that  $n_+$  is even. The Poisson superbracket satisfies the Jacobi identity

$$\sigma(f, h) \{f, \{g, h\}\}(z) + \text{cycle}(f, g, h) = 0, \quad f, g, h \in \mathbf{E}_{n_+}^{n_-}. \quad (2)$$

By Poisson superalgebra, we mean the space  $\mathbf{D}_{n_+}^{n_-}$  with the Poisson bracket (1) on it. The relations (1) and (2) show that this bracket indeed determines a Lie superalgebra structure on  $\mathbf{D}_{n_+}^{n_-}$ .

The integral on  $\mathbf{D}_{n_+}^{n_-}$  is defined by the relation  $\int dz f(z) = \int_{\mathbb{R}^{n_+}} dx \int d\xi f(z)$ , where the integral on the Grassmann algebra is normed by the condition  $\int d\xi \xi^1 \dots \xi^{n_-} = 1$ . We identify  $\mathbb{G}^{n_-}$  with its dual space  $\mathbb{G}^{n_-}$  setting  $f(g) = \int d\xi f(\xi)g(\xi)$ ,  $f, g \in \mathbb{G}^{n_-}$ . Correspondingly,  $\mathbf{D}_{n_+}^{m_-}$ , i.e., the space of continuous linear functionals on  $\mathbf{D}_{n_+}^{n_-}$  is identified with the space  $\mathcal{D}'(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}$ . As a rule, the value  $m(f)$  of a functional  $m \in \mathbf{D}_{n_+}^{m_-}$  on a test function  $f \in \mathbf{D}_{n_+}^{n_-}$  will be written in the "integral" form:  $m(f) = \int dz m(z) f(z)$ .

Let  $L$  be a Lie superalgebra acting in a  $\mathbb{Z}_2$ -graded space  $V$  (the action of  $f \in L$  on  $v \in V$  will be denoted by  $f \cdot v$ ). The space  $C_p(L, V)$  of  $p$ -cochains consists of all multilinear super anti-symmetric mappings from  $L^p$  to  $V$ . The space  $C_p(L, V)$  possesses a natural  $\mathbb{Z}_2$ -grading: by definition,  $M_p \in C_p(L, V)$  has the definite parity  $\varepsilon(M_p)$  if

$$\varepsilon(M_p(f_1, \dots, f_p)) = \varepsilon(M_p) + \varepsilon(f_1) + \dots + \varepsilon(f_p)$$

for any  $f_j \in L$  with parities  $\epsilon(f_j)$ . The differential  $d_p^V$  is defined to be the linear operator from  $C_p(L, V)$  to  $C_{p+1}(L, V)$  such that

$$d_p^V M_p(f_1, \dots, f_{p+1}) = - \sum_{j=1}^{p+1} (-1)^{j+\epsilon(f_j)|\epsilon(f)|_{1,j-1} + \epsilon(f_j)\epsilon_{M_p}} f_j \cdot M_p(f_1, \dots, \check{f}_j, \dots, f_{p+1}) - \\ - \sum_{i < j} (-1)^{j+\epsilon(f_j)|\epsilon(f)|_{i+1,j-1}} M_p(f_1, \dots, f_{i-1}, \{f_i, f_j\}, f_{i+1}, \dots, \check{f}_j, \dots, f_{p+1}), \quad (3)$$

for any  $M_p \in C_p(L, V)$  and  $f_1, \dots, f_{p+1} \in L$  having definite parities. Here the sign  $\check{\phantom{x}}$  means that the argument is omitted and the notation

$$|\epsilon(f)|_{i,j} = \sum_{l=i}^j \epsilon(f_l)$$

has been used. The differential  $d^V$  is nilpotent (see [5]), i.e.,  $d_{p+1}^V d_p^V = 0$  for any  $p = 0, 1, \dots$ . The  $p$ -th cohomology space of the differential  $d_p^V$  will be denoted by  $H_p^V$ . The second cohomology space  $H_{\text{ad}}^2$  in the adjoint representation is closely related to the problem of finding formal deformations of the Lie bracket  $\{\cdot, \cdot\}$  of the form  $\{f, g\}_* = \{f, g\} + \hbar^2 \{f, g\}_1 + \dots$ . The condition that  $\{\cdot, \cdot\}$  is a 2-cocycle is equivalent to the Jacobi identity for  $\{\cdot, \cdot\}_*$  modulo the  $\hbar^4$ -order terms.

In the case of Poisson algebra, the problem of finding deformations can hardly be solved in such a general setting, and additional restrictions on cochains (apart from linearity and anti-symmetry) are usually imposed. In some papers on deformation quantization it is supposed that the kernels of  $n$ -th order deformations  $\{\cdot, \cdot\}_n$  are bidifferential operators. In the present paper, this requirement is replaced by the much weaker condition that cochains should be separately continuous multilinear mappings. It will be shown that this gives rise to additional cohomologies. We study the cohomologies of the Poisson algebra  $\mathbf{D}_{n_+}^{n_-}$  in the following representations:

1. The trivial representation:  $V = \mathbb{K}$ ,  $f \cdot a = 0$  for any  $f \in \mathbf{D}_{n_+}^{n_-}$  and  $a \in \mathbb{K}$ . The space  $C_p(\mathbf{D}_{n_+}^{n_-}, \mathbb{K})$  consists of separately continuous anti-symmetric multilinear forms on  $(\mathbf{D}_{n_+}^{n_-})^p$ . The cohomology spaces and the differentials will be denoted by  $H_{\text{tr}}^p$  and  $d_p^{\text{tr}}$  respectively.
2.  $V = \mathbf{D}_{n_+}^{m_-}$  and  $f \cdot m = \{f, m\}$  for any  $f \in \mathbf{D}_{n_+}^{n_-}$  and  $m \in \mathbf{D}_{n_+}^{m_-}$ . The space  $C_p(\mathbf{D}_{n_+}^{n_-}, \mathbf{D}_{n_+}^{m_-})$  consists of separately continuous anti-symmetric multilinear mappings from  $(\mathbf{D}_{n_+}^{n_-})^p$  to  $\mathbf{D}_{n_+}^{m_-}$ . The continuity of  $M \in C_p(\mathbf{D}_{n_+}^{n_-}, \mathbf{D}_{n_+}^{m_-})$  means that the  $(p+1)$ -form

$$(f_1, \dots, f_{p+1}) \rightarrow \int dz M(f_1, \dots, f_p)(z) f_{p+1}(z)$$

on  $(\mathbf{D}_{n_+}^{n_-})^{p+1}$  is separately continuous. The cohomology spaces will be denoted by  $H_{\mathbf{D}}^p$ .

3.  $V = \mathbf{E}_{n_+}^{n_-}$  and  $f \cdot m = \{f, m\}$  for every  $f \in \mathbf{D}_{n_+}^{n_-}$  and  $m \in \mathbf{E}_{n_+}^{n_-}$ . The space  $C_p(\mathbf{D}_{n_+}^{n_-}, \mathbf{E}_{n_+}^{n_-})$  is the subspace of  $C_p(\mathbf{D}_{n_+}^{n_-}, \mathbf{D}_{n_+}^{m_-})$  consisting of mappings taking values in  $\mathbf{E}_{n_+}^{n_-}$ . The cohomology spaces will be denoted by  $H_{\mathbf{E}}^p$ .

4. The adjoint representation:  $V = \mathbf{D}_{n_+}^{n_-}$  and  $f \cdot g = \{f, g\}$  for any  $f, g \in \mathbf{D}_{n_+}^{n_-}$ . The space  $C_p(\mathbf{D}_{n_+}^{n_-}, \mathbf{D}_{n_+}^{n_-})$  is the subspace of  $C_p(\mathbf{D}_{n_+}^{n_-}, \mathbf{D}_{n_+}^{n_-})$  consisting of mappings taking values in  $\mathbf{D}_{n_+}^{n_-}$ . The cohomology spaces and the differentials will be denoted by  $H_{\text{ad}}^p$  and  $d_p^{\text{ad}}$  respectively.

For the representations 2 and 3, we shall denote the differentials by the same symbol  $d_p^{\text{ad}}$  as in the adjoint representation. Note that in the case of the trivial representation,  $\mathbb{K}$  is considered as a graded space whose odd subspace is trivial. We shall call p-cocycles  $M_p^1, \dots, M_p^k$  independent if they give rise to linearly independent elements in  $H^p$ . For a multilinear form  $M_p$  taking values in  $\mathbf{D}_{n_+}^{n_-}$ ,  $\mathbf{E}_{n_+}^{n_-}$ , or  $\mathbf{D}_{n_+}^{n_-}$ , we write  $M_p(x|f_1, \dots, f_p)$  instead of more cumbersome  $[M_p(f_1, \dots, f_p)](x)$ .

Below we assume that  $n_+ = 2$ .

The results of the work are given by the following three theorems.

**Theorem 1.**

1.  $H_{\text{tr}}^1 \simeq \mathbb{K}$ ; the linear form  $M_1(f) = \bar{f} \stackrel{\text{def}}{=} \int dz f(z)$  is the nontrivial cocycle.
2. Let bilinear forms  $M_2^1$  and  $M_2^2$  be defined by the relations

$$M_2^1(f, g) = \bar{f}\bar{g}, \quad M_2^2(f, g) = \int dz z^A \left( \frac{\partial f(z)}{\partial z^A} g(z) - \sigma(f, g) \frac{\partial g(z)}{\partial z^A} f(z) \right), \quad f, g \in \mathbf{D}_2^{n_-}.$$

If  $n_-$  is even and  $n_- \neq 6$ , then  $H_{\text{tr}}^2 = 0$ ;

if  $n_- = 6$ , then  $H_{\text{tr}}^2 \simeq \mathbb{K}$  and the form  $M_2^2$  is a nontrivial cocycle;

if  $n_-$  is odd, then  $H_{\text{tr}}^2 \simeq \mathbb{K}$  and the form  $M_2^1$  is a nontrivial cocycle.

**Theorem 2.**

1.  $H_{\mathbf{D}}^0 \simeq H_{\mathbf{E}}^0 \simeq \mathbb{K}$ ; the function  $M_0(z) \equiv 1$  is a nontrivial cocycle.
2.  $H_{\mathbf{D}}^1 \simeq H_{\mathbf{E}}^1 \simeq \mathbb{K}^2$ ; independent nontrivial cocycles are given by

$$M_1^1(z|f) = \bar{f}, \quad M_1^2(z|f) = \left( 1 - \frac{1}{2} z^A \frac{\partial}{\partial z^A} \right) f(z).$$

3. Let the bilinear mappings  $M_2^1, M_2^2, M_2^3, M_2^4$  and  $M_2^5$  from  $(\mathbf{D}_2^{n_-})^2$  to  $\mathbf{E}_2^{n_-}$  be defined by the relations

$$M_2^1(z|f, g) = f(z) \left( \frac{\overleftarrow{\partial}}{\partial z^A} \omega^{AB} \frac{\partial}{\partial z^B} \right)^3 g(z),$$

$$M_2^2(z|f, g) = \bar{f}\bar{g},$$

$$M_2^3(z|f, g) = \bar{g}M_1^2(z|f) - \sigma(f, g)\bar{f}M_1^2(z|g),$$

$$M_2^4(z|f, g) = \int du u^A \left( \frac{\partial f(u)}{\partial u^A} g(u) - \sigma(f, g) \frac{\partial g(u)}{\partial u^A} f(u) \right),$$

$$M_2^5(z|f, g) = \Delta(x|\partial_2 f g) - \Delta(x|f \partial_2 g) - 2(-1)^{\varepsilon(f)} \partial_2 f(z) \Delta(x|g) + 2\Delta(x|f) \partial_2 g(z) \\ + \frac{1}{2} \overset{\circ}{f}(z) (z^A \partial_A g(z)) - \frac{1}{2} (-1)^{\varepsilon(f)} (z^A \partial_A f(z)) \overset{\circ}{g}(z),$$

where  $\Delta(x|f) \stackrel{def}{=} \int du \delta(x_1 - u_1) \theta(x_2 - u_2) f(u)$  and  $\overset{\circ}{f}(z) \stackrel{def}{=} \int d\xi_1 \dots d\xi_{n_-} f(z)$ .

If  $n_-$  is even and  $n_- \neq 6$  and  $n_- \neq 0$ , then  $H_{\mathbf{D}}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^2$  and the cochains  $M_2^1$  and  $M_2^3$  are independent nontrivial cocycles;

if  $n_- = 0$ , then  $H_{\mathbf{D}}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^3$  and the cochains  $M_2^1$ ,  $M_2^3$  and  $M_2^5$  are independent nontrivial cocycles;

if  $n_- = 6$ , then  $H_{\mathbf{D}}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^3$  and the cochains  $M_2^1$ ,  $M_2^3$ , and  $M_2^4$  are independent nontrivial cocycles;

if  $n_-$  is odd and  $n_- \neq 1$ , then  $H_{\mathbf{D}}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^3$  and the cochains  $M_2^1$ ,  $M_2^2$  and  $M_2^3$  are independent nontrivial cocycles;

if  $n_- = 1$ , then  $H_{\mathbf{D}}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^4$  and the cochains  $M_2^1$ ,  $M_2^2$ ,  $M_2^3$  and  $M_2^5$  are independent nontrivial cocycles.

### Theorem 3.

1.  $H_{\text{ad}}^0 = 0$ .
2. Let  $V_1$  be the one-dimensional subspace of  $C_1(\mathbf{D}_2^{n_-}, \mathbf{D}_2^{n_-})$  generated by the cocycle  $M_1^2$  defined in Theorem 2. Then there is a natural isomorphism  $V_1 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-}) \simeq H_{\text{ad}}^1$  taking  $(M_1, T) \in V_1 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-})$  to the cohomology class determined by the cocycle  $M_1(z|f) + \{t(z), f(z)\}$ , where  $t \in \mathbf{E}_2^{n_-}$  belongs to the equivalence class  $T$ .
3. Let  $V_2$  be the two-dimensional subspace of  $C_2(\mathbf{D}_2^{n_-}, \mathbf{D}_2^{n_-})$  generated by the cocycles  $M_2^1$  and  $M_2^3$  defined in Theorem 2. Then there is a natural isomorphism  $V_2 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-}) \simeq H_{\text{ad}}^2$  taking  $(M_2, T) \in V_2 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-})$  to the cohomology class determined by the cocycle

$$M_2(z|f, g) - \{t(z), f(z)\} \bar{g} + \sigma(f, g) \{t(z), g(z)\} \bar{f},$$

where  $t \in \mathbf{E}_2^{n_-}$  belongs to the equivalence class  $T$ .

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