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On the Non-relativistic Limit of Linear Wave Equations for Zero and Unity Spin Particles

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Abstract

In this work the non-relativistic limit of the linear wave equation for zero and unity spin bosons in the Duffin–Kemmer–Petiau representation is investigated by means of a unitary transformation, analogous to the Foldy–Wouthuysen canonical transformation for a relativistic electron. For non-interacting fields, it is shown that such transformation corresponds exactly to a Lorentz boost to the particle's rest frame. The interacting case is also analyzed, by considering a power series expansion of the transformed Hamiltonian, thus demonstrating that all features of particle dynamics can be recovered if corrections of order $1/m^2$ are properly taken into account, through a recursive iteration procedure.

1 Introduction

Recently, with the increasing technical complexity of string theories as the best candidates for the unification of the fundamental interactions, there is a renewed interest in quantum field theory of higher spins as a natural covariant formalism for accommodating the particle spectra in the Standard Model and quantum gravity gauge theories, as well as their supersymmetric counterparts. Thus, from the phenomenological standpoint, it is mandatory to investigate such theories in the low-energy regime, by examining their non-relativistic formal properties and taking into account the interaction with external electromagnetic and/or metric fields as a starting point.

In relativistic quantum mechanics, one must seek a relation between irreducible representations of the Poincaré group and wave equations. In Wigner's standard form, non-trivial wave equations can only be written for wave functions with a large number of components, simultaneously expressing constraints on redundant components and equations of motion for the physical ones. Studying general invariant equations, Gel'fand and Yaglom[1] expressed relativistic wave functions in terms of linear differential operators, simultaneously determining both these operators and the finite-dimensional representations of the homogeneous Lorentz group, according to which the components of the wave function transform. However, such a procedure is not applicable to non-relativistic wave equations whose solutions transform according to the homogeneous Galilei group. Following another approach, relying upon the Bargmann–Wigner method, J.M. Lévi-Leblond[2] constructed a basis in a ten-dimensional representation space of the homogeneous Galilei group for free massive particles of spin 1, by taking a complete set of independent linear combinations of symmetrical tensor products of two-component wave functions which describe non-relativistic particles of spin 1/2, and arriving at a system of equations involving linear operators.

In order to investigate the physical properties of particles of zero and unity spin in the presence of electromagnetic external sources, instead of starting from Galilean-covariant wave equations, we start from a Lorentz-covariant linear wave equation in the Hamiltonian form, and apply a canonical transformation, analogous to the Foldy–Wouthuysen (FW) transformation[3] for Dirac fermions, to a suitable reference frame where we can recognize the different couplings of charged bosons with the electromagnetic field. In this sense, the Duffin–Kemmer–Petiau representation[4] proves to be particularly useful, since all physical quantities are constructed from linear operators which satisfy convenient algebraic relations, in close similarity with the familiar Dirac operators.

This work is organized as follows. In the following section we present the linear wave equation which describes bosons of spin zero and unity and the basic identities of the DKP associated algebra, then rewriting this equation in the Hamiltonian form for non-interacting particles. In section 3 we discuss the quantum canonical transformation for the free boson Hamiltonian, by analogy with the ordinary FW transformation. Next, in section 4, we derive the non-relativistic limit of the Hamiltonian which describes charged bosons in interaction with an external electromagnetic field. In section 5 we make concluding remarks.

2 DKP Hamiltonian

Let us briefly review the DKP formalism for non-interacting bosons of spin zero and one. The relativistic wave equation in such a representation reads

$$(i\not{\partial} - m)\,\psi = 0\,, \tag{1}$$

where $\not{\partial} \equiv \beta_\mu \partial^\mu$ and ψ is a five(ten)-row column associated with the zero (unity) spin field. For instance,

$$\psi = \begin{pmatrix} m^{1/2}\phi \\ \vdots \\ m^{-1/2}\partial_1\phi \\ m^{-1/2}\partial_0\phi_0 \end{pmatrix}, \tag{2}$$

where ϕ obeys the Klein–Gordon equation for spin-0 particles and the Proca equation for spin-1 particles.

The β-matrices obey the following algebra:

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \beta_\mu g_{\nu\rho} + \beta_\rho g_{\nu\mu}\,. \tag{3}$$

Let us list some useful consequences:

$$\beta_0\beta_k\beta_0 = 0\,,\ k = 1,2,3\,, \tag{4}$$

$$\beta_0^3 = \beta_0\,, \tag{5}$$

$$\not{b}\beta_\nu\not{b} = \not{b}b_\nu\,, \tag{6}$$

$$(\vec{\beta}\cdot\vec{b})\beta_0(\vec{\beta}\cdot\vec{b}) = 0\,, \tag{7}$$

where $b_\mu = (b_0, \vec{b})$ is a generic four-vector.

Multiplying (1) by $\not{\partial}\beta_\nu$ and using (5),

$$(i\not{\partial}\beta_\nu\not{\partial} - m\not{\partial}\beta_\nu)\,\psi = (i\partial_\nu\not{\partial} - m\not{\partial}\beta_\nu)\,\psi = 0\,,$$

and then (1),

$$(m\partial_\nu - m\not{\partial}\beta_\nu)\,\psi = 0\,,$$

we obtain

$$\partial_\nu\psi = \not{\partial}\beta_\nu\psi\,. \tag{8}$$

Multiplying (1) by β_0 and taking the zero component of (7) times the imaginary unity, one obtains, upon adding the results,

$$\left\{i\left[\partial_0 + \partial^k\left(\beta_0\beta_k - \beta_k\beta_0\right)\right] - m\beta_0\right\}\psi = 0\,,$$

or

$$i\partial_0\psi = H\psi\,, \tag{9}$$

where

$$H = -i\vec{\alpha}\cdot\vec{\nabla} + \beta_0 m = \vec{\alpha}\cdot\vec{p} + \beta_0 m \tag{10}$$

is the DKP Hamiltonian, and $\vec{\alpha}$ is defined by its space components:

$$\alpha_k \equiv \beta_0\beta_k - \beta_k\beta_0\,,\ k = 1,2,3\,. \tag{11}$$

3 FW Transformation

As in the electron case, we now seek a unitary transformation

$$\psi' = e^{-iU}\psi\,, \tag{12}$$

$$H' = e^{iU}He^{-iU}\,, \tag{13}$$

which eliminates the term that mixes the space components of the four-momentum. In case H explicitly depends on time, equation (8) gives

$$i\partial_0(e^{-iU}\psi') = He^{-iU}\psi'\,,$$

so that

$$e^{-iU}\left(i\partial_0\psi'\right) = \left(He^{-iU} - i\partial_0 e^{-iU}\right)\psi'\,,$$

or

$$i\partial_0\psi' = H'\psi'\,, \tag{14}$$

where

$$H' = e^{iU}\left(H - i\partial_0\right)e^{-iU}\,. \tag{15}$$

Let us choose

$$U = -i\frac{\vec{\beta}.\vec{p}}{|\vec{p}|}\theta\,. \tag{16}$$

The β-algebra (2) implies the identity

$$2(\vec{\beta}.\vec{p})^3 = \sum_{ijk} p_i p_j p_k \left[\beta_i\beta_j\beta_k + \beta_k\beta_j\beta_i\right]$$
$$= -\sum_{ijk} p_i p_j p_k (\beta_i\delta_{jk} + \beta_k\delta_{ji}),$$

so that

$$(\vec{\beta}\cdot\vec{p})^3 = -|\vec{p}|^2(\vec{\beta}\cdot\vec{p}). \tag{17}$$

Note that, unlike in the electron case,

$$\left(\frac{\vec{\beta}\cdot\vec{p}}{|\vec{p}|}\right)^2 \neq \mathbf{1},$$

since $\vec{\beta}\cdot\vec{p}$ has no inverse.

Representing (16) in the form

$$[(\vec{\beta}\cdot\vec{p})^2 + |\vec{p}|^2](\vec{\beta}\cdot\vec{p}) = 0$$

and then, on the mass shell,

$$(\vec{\beta}\cdot\vec{p}) = \beta_0 p_0 - \not{p} = \beta_0 E + m,$$

we have

$$[(\vec{\beta}\cdot\vec{p})^2 + |\vec{p}|^2]\,(\beta_0 E + m)\,\psi = 0. \tag{18}$$

At the same time,

$$(\vec{\beta}\cdot\vec{p})^2\beta_0 = \sum_{ij} p_i p_j\,(\beta_i\beta_j\beta_0)$$
$$= -\sum_{ij} p_i p_j\,(\beta_0\beta_j\beta_i + \beta_0\delta_{ij}) = -\beta_0[(\vec{\beta}\cdot\vec{p})^2 + |p|^2].$$

Then (17) implies

$$(m - \beta_0 E)\,(\vec{\beta}\cdot\vec{p})^2\psi = -\,|\vec{p}|^2\,m\psi,$$

or

$$(m^2 - \beta_0^2 E^2)(\vec{\beta}\cdot\vec{p})^2\psi = -(m^2 + \beta_0 E m)\,|\vec{p}|^2\,\psi. \tag{19}$$

Since

$$m^2(m^2 - \beta_0^2 E^2)^{-1} = 1 - \frac{E^2}{|\vec{p}|^2}\beta_0^2,$$

one obtains from (4)

$$(\vec{\beta}\cdot\vec{p})^2 = -\,|\vec{p}|^2 + (Em)\,\beta_0 + E^2\beta_0^2. \tag{20}$$

Eq. (9) does not contain the complete information about the system due to the multiplication by the singular matrix β_0.

Multiplying (1) by $(1-\beta_0^2)$, one gets the additional constraint

$$\left[i\partial^k \beta_k \beta_0^2 - (1-\beta_0^2)m\right]\psi = 0\,,$$

or

$$(\vec{\beta}\cdot\vec{p})\beta_0^2 + (1-\beta_0^2)m = 0\,, \tag{21}$$

on the mass shell. Also, left-multiplying (19) by $(\vec{\beta}\cdot\vec{p})$, and using (16) and (20), one obtains

$$(\vec{\beta}\cdot\vec{p})\beta_0 = E(1-\beta_0^2)\,. \tag{22}$$

Now, multiplying (19) by $(\vec{\beta}\cdot\vec{p})^2$, and using (20) and (21), one gets

$$(\vec{\beta}\cdot\vec{p})^4 = -(\vec{\beta}\cdot\vec{p})^2\,|\vec{p}|^2\,. \tag{23}$$

Then

$$e^{iU} = e^{(\vec{\beta}\cdot\vec{p}/|\vec{p}|)\theta} = 1 + \frac{(\vec{\beta}\cdot\vec{p})^2}{|\vec{p}|^2}(1-\cos\theta) + \frac{(\vec{\beta}\cdot\vec{p})}{|\vec{p}|}\sin\theta\,, \tag{24}$$

where (16) and (22) have been used in the series expansion. Hence,

$$H' = (\vec{\alpha}.\vec{p})\left[\cos\theta - \frac{m}{|\vec{p}|}\sin\theta\right] + \beta_0\left(|\vec{p}|\sin\theta + m\cos\theta\right)\,.$$

Choosing

$$\sin\theta = \frac{|\vec{p}|}{E}\,,\ \cos\theta = \frac{m}{E}\,,$$

one arrives at

$$H' = \frac{\beta_0}{E}\left(\vec{p}^2 + m^2\right) = \beta_0 E\,. \tag{25}$$

From the matrix realization of the β-algebra[5] for β_0 and ψ, one obtains, according to (2),

$$\partial_0^2\phi' = -E^2\phi' = -m^2\phi'\,,$$

and, therefore, the transformed scalar field ϕ' satisfies the Schrödinger equation

$$i\partial_0\phi' = E\phi' \tag{26}$$

if and only if $E = m$, i.e., if the particle is in its rest frame.

4 DKP Interaction Hamiltonian

In order to have a better understanding of the particle content of the theory, let us examine the behaviour of charged bosons in the presence of an external electromagnetic field A_μ, transforming to a reference frame where particles carry low momenta. The electromagnetic interaction is introduced by means of the covariant derivative, so that

$$(i\not{D} - m)\psi = 0\,, \tag{27}$$

where the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu \tag{28}$$

satisfies the commutation relation

$$[D_\mu, D_\nu] = ieF_{\mu\nu}\,,\ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{29}$$

Multiplying (27) by $\not{D}\beta_\nu$, one obtains

$$[i\,(D^\rho D^\mu - ieF^{\rho\mu})\,(-\beta_\rho\beta_\nu\beta_\mu + \beta_\mu g_{\nu\rho} + \beta_\rho g_{\nu\mu}) - \not{D}\beta_\nu m]\,\psi = 0\,,$$

or

$$D_\nu\psi = \not{D}\beta_\nu\psi + \frac{e}{2m}F^{\rho\mu}(\beta_\rho\beta_\nu\beta_\mu - \beta_\rho g_{\nu\mu})\psi\,. \tag{30}$$

Then, from equations (27) and (30),

$$i\partial_0\psi = H\psi\,, \tag{31}$$

it follows that

$$H = H^{(0)} + H^{(1)}\,, \tag{32}$$

where

$$H^{(0)} = \vec{\alpha}\cdot\vec{\pi} + m\beta_0 - eA_0\,, \tag{33}$$

$$H^{(1)} = \frac{ie}{2m}F^{\rho\mu}\left(\beta_\rho\beta_0\beta_\mu - \beta_\rho g_{0\mu}\right)\,, \tag{34}$$

and

$$\vec{\pi} = \vec{p} - e\vec{A}\,. \tag{35}$$

From (14) and the Baker–Campbell–Hausdorf formula, one can write

$$H' = H + \frac{\partial U}{\partial t} + i\left[U, H + \frac{1}{2}\frac{\partial U}{\partial t}\right] - \frac{1}{2!}\left[U, \left[U, H + \frac{1}{3}\frac{\partial U}{\partial t}\right]\right] + \dots\,. \tag{36}$$

Since, in the nonrelativistic limit, $\theta \sim \sin\theta \sim |\vec{p}|/m$, one can choose, in the first approximation, by analogy with the free case,

$$U = -i\frac{\vec{\beta}\cdot\vec{\pi}}{m}\,. \tag{37}$$

From the commutation relations (A.1)–(A.7) and the vector relations (A.8) and (A.9), listed in the Appendix, one obtains

$$\left[U, H^{(1)}\right] = -\frac{e}{m^2}[\vec{\beta}\cdot\vec{\pi}, (\vec{\beta}\cdot\vec{E})\beta_0^2] + \frac{e}{m^2}[\vec{\beta}\cdot\vec{\pi}, \vec{\beta}\cdot\vec{E}] - \frac{e}{2m^2}[\vec{\beta}\cdot\vec{\pi}, F^{ij}\beta_i\beta_0\beta_j]\,. \tag{38}$$

In addition,

$$[\vec{\beta}\cdot\vec{\pi}, (\vec{\beta}\cdot\vec{E})\beta_0^2] = i\vec{S}\cdot[\vec{\pi}\times\vec{E}]\beta_0^2 + (\vec{\beta}\cdot\vec{E})[2(\vec{\beta}\cdot\vec{\pi})\beta_0^2 - \vec{\beta}\cdot\vec{\pi}]\,, \tag{39}$$

so that one arrives at

$$H' = m\beta_0 - eA_0 + \frac{\vec{\pi}^2}{2m}\left(\vec{\alpha}\cdot\frac{\vec{\pi}}{m} - \beta_0\right) + \frac{e}{2m}(\vec{S}\cdot\vec{H})\beta_0 + \frac{e}{2m}(\vec{\beta}\times\vec{\alpha}).\vec{H}$$
$$+ \frac{e}{2m^2}(\vec{S}\cdot(\vec{\pi}\times\vec{E}))\left(1+2\beta_0^2\right) + \frac{ie}{2m^2}[\vec{\beta}\cdot\vec{\pi},(\beta_0\vec{S}+\vec{\beta}\times\vec{\alpha})\cdot\vec{H}]$$
$$- \frac{ie}{m}(\vec{\beta}\cdot\vec{E})\beta_0^2 - \frac{ie}{m^2}(\vec{\beta}\cdot\vec{E})[2(\vec{\beta}\cdot\vec{\pi})\beta_0^2 - \vec{\beta}\cdot\vec{\pi}] + O\left(m^{-3}\right) \,, \qquad (40)$$

where relation (A.10) has been used. In the above expression, $\vec{S}$ stands for the spin operator of bosons,

$$S_{ij} = i\left(\beta_i\beta_0\beta_j - \beta_j\beta_0\beta_i\right) \,,\; i,j = 1,2,3 \,, \qquad (41)$$

with eigenvalues 0 or 1, and $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields, respectively.

Expression (40) is analogous to the Hamiltonian of the Pauli equation for spin-1/2 fermions, in the case of charged bosons of spin 0 and 1 in the presence of an external electromagnetic field. In expression (40) we can recognize each term individually. For example, the second term is related to the electrostatic potential and the third one corresponds to the kinetic term of the non-relativistic interaction Hamiltonian. In fact, following the same steps which led to equation (21) on the mass shell, and from the very definition of the matrices α_k, it results that the kinetic term in the transformed Hamiltonian can be rewritten as

$$\frac{\vec{\pi}^2}{2m}\left[\frac{\pi_0}{m}(2\beta_0^2 - 1)\right] \,,$$

which is indeed diagonal and non-singular in the matrix realization of the DKP β-algebra, as one would expect by analogy with the disentangling property of the FW transformation.

In this approach, the most interesting result is the appearance of the spin and orbital angular momentum couplings with the external magnetic field (the fourth and fifth terms, respectively), as well as the diagonal spin-orbital coupling (the sixth term), through the electric field; the last two terms may be interpreted as being analogous to the Darwin term for spin-1/2 fermions in the presence of an electric field; the remaining terms represent higher-order corrections to such effects, as well as the (non-diagonal) corrections to the rest-energy (the first term).

5 Concluding Remarks

In the preceding sections we have investigated the non-relativistic limit of the Lorentz-invariant wave equation which describes scalar and vector mesons in the so-called Duffin–Kemmer–Petiau representation. By constructing unitary operators involving the space components of the relativistic 4-momentum and those belonging to the associated DKP algebra, both for free particles and for charged bosons in an electromagnetic background, we performed a quantum canonical transformation to a reference frame where we succeeded in identifying the coupling terms with the electric and magnetic fields, in close similarity with the non-relativistic behaviour of interacting fermions described by the Pauli equation.

Our approach differs from the one by Lévi-Leblond[2] in the sense that he derived non-relativistic linear wave equations for particles of arbitrary spins which satisfy the Galilean invariance by construction, where the electromagnetic multipole moments are introduced on dimensional grounds. Nevertheless, in the case of massive particles of spin 1, he settled the corresponding wave equations by employing the Bargmann–Wigner construction, referring neither to the algebraic properties of the relevant physical quantities nor to any particular representation for them, which we have made explicitly in our treatment. Yet, in the scope of DKP theory, other authors have recently investigated the non-relativistic wave equation for spinless bosons via Galilean covariance, by introducing an extra degree of freedom in the free Lagrangian density[6], thus recovering the Schrödinger equation for a free particle. However, the introduction of electromagnetic potentials spoils the original structure of the associated Lie algebra on which the reasoning[6] is grounded. Another interesting issue related to the present work is a possible generalization of the above procedure to theories of higher spins, as well as to their non-Abelian counterparts[7].

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A Appendix

Below we present some useful commutation and vector relations derived from the algebra (2) of the β-matrices:

$$[U, \vec{\alpha}\cdot\vec{\pi}] = \frac{i}{m}\beta_0\vec{\pi}^2 \; ; \tag{A.1}$$

$$[U, \beta_0] = \frac{i}{m}\vec{\alpha}\cdot\vec{\pi} \; ; \tag{A.2}$$

$$[U, A_0] = -\frac{i}{m}\vec{\beta}\cdot\vec{\nabla}A_0 \; ; \tag{A.3}$$

$$[U, \partial U/\partial t] = \frac{ie}{m^2}\vec{S}\cdot\left(\vec{\pi}\times\partial\vec{A}/\partial t\right) ; \tag{A.4}$$

$$[U, [U, \vec{\alpha}\cdot\vec{\pi}]] = -\frac{\vec{\pi}^2}{m^2}\left(\vec{\alpha}\cdot\vec{\pi}\right) ; \tag{A.5}$$

$$[U, [U, \beta_0]] = -\frac{1}{m^2}\beta_0\vec{\pi}^2 \; ; \tag{A.6}$$

$$[U, [U, A_0]] = -\frac{1}{m^2}\vec{S}\cdot(\vec{\pi}\times\vec{\nabla}A_0) \; ; \tag{A.7}$$

$$F^{\rho\mu}\beta_\rho\beta_0\beta_\mu = -2(\vec{E}\cdot\vec{\beta})\beta_0^2 + \vec{E}\cdot\vec{\beta} + F^{ij}\beta_i\beta_0\beta_j \; ; \tag{A.8}$$

$$F^{\rho\mu}\beta_\rho g_{0\mu} = -\vec{E}\cdot\vec{\beta} ; \tag{A.9}$$

$$F^{ij}\beta_i\beta_0\beta_j = -i\beta_0\vec{S}\cdot\vec{H} - i(\vec{\beta}\times\vec{\alpha})\cdot\vec{H} \, . \tag{A.10}$$

References

[1] I.M. Gel'fand, R.A. Minlos and Z.Y. Shapiro, *Representations of the rotation and Lorentz Groups*, Pegamon Press, Oxford (1963).

[2] J.M. Lévy-Leblond, Commun. Math. Phys. **6** (1967) 286.

[3] L.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78** (1950) 29.

[4] G. Petiau, Acad. R. Belg. Cl. Sci. Mém. Collect 8 **16**, No. 2 (1936); N. Kemmer, Proc. Roy. Soc. **A 166** (1938) 127; R.J.Duffin, Phys. Rev. **54** (1938) 1114.

[5] A.I. Akhiezer and V.B. Berestetskii, *Quantum Electrodynamics*, Wiley Interscience Publishers, New York (1965).

[6] M de Montigny, F.C. Khanna, A.E. Santana, E.S. Santos and J.D.M. Vianna, J. Phys. A **33** (2000) 273.

[7] P.Yu. Moshin and J.L. Tomazelli, work in progress.