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An Embedding of the BV Quantization into an N=1 Local Superfield Formalism

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Abstract

An $N=1$ local superfield formulation of Lagrangian quantization in non-Abelian hypergauges is proposed on the basis of an extension of general reducible gauge theories to special superfield models with a Grassmann parameter θ. We solve the problem of describing the quantum action and the gauge algebra of an L-stage-reducible superfield model in terms of a BRST charge for a formal dynamical system with first-class constraints of $(L+1)$-stage reducibility. Starting from θ-local functions of the quantum and gauge-fixing actions, with an essential use of Darboux coordinates on the antisymplectic manifold, we construct a superfield generating functionals of Green's functions, including the effective action. We present two superfield forms of BRST transformations, considered as θ-shifts along vector fields defined by Hamiltonian-like systems constructed in terms of the quantum and gauge-fixing actions and an arbitrary θ-local boson function, as well as via corresponding fermion functionals, in terms of Poisson brackets with opposite Grassmann parities. The gauge independence of the S-matrix is proved. The Ward identities are derived. A connection of the suggested $N=1$ local superfield quantization is established with the BV method and the multilevel Batalin–Tyutin formalism.

1 Introduction

The construction of superfield counterparts of the Lagrangian [1] and Hamiltonian [2, 3] quantization schemes for gauge theories on the basis of BRST symmetry [4] has been covered in a number of papers [5, 6, 7]. These works are based on nontrivial (represented by the operator $D=\partial_\theta+\theta\partial_t$, $[D,D]_+=2\partial_t$) and trivial relations between the even t and odd θ components of supertime $\chi=(t,\theta)$, introduced in [8]. In [5, 6, 7], the geometric interpretation [9] of BRST transformations is realized by special translations in superspace, which originally provided a basis for a superspace description [10] of quantum theories of Yang–Mills type.

The superfield Lagrangian partition function of [5] is derived from a Hamiltonian partition function through functional integration over so-called Pfaffian ghosts and momenta. On the other hand, the quantization rules [6, 7] present a superfield modification of the BV method by including non-Abelian hypergauges [11]. The corresponding hypergauge functions are introduced into a gauge-fixing action which obeys (following the ideas of [12]) the same generating equation that holds for the quantum action [6, 7], except that the first-order operator V in this equation is replaced by the first-order operator U. The operators V, U are crucial ingredients of [6, 7] from the viewpoint of a superspace interpretation of BRST transformations.

The formalism [6, 7] provides a comparatively detailed analysis of superfield quantization (BRST invariance, S-matix gauge-independence). This analysis is based on the structure of solutions to the generating equations [6, 7]; however, a detailed relation between these solutions and a gauge model is not indicated. To achieve a better understanding of the quantum properties based on solutions of the superfield generating equations, it is natural to equip the formalism [6, 7] with an *explicit superfield*

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description of gauge algebra structure functions that determine a given model. So far, this problem has remained unsolved. For instance, the definition of a classical action of superfields, $\mathcal{A}^i(\theta) = A^i + \lambda^i\theta$, on a superspace with coordinates (x^μ, θ), $\mu = 0, \ldots, D-1$, as an integral of a nontrivial odd density, $\mathcal{L}(\mathcal{A}(x,\theta), \partial_\mu\mathcal{A}(x,\theta), \ldots; x, \theta) \equiv \mathcal{L}(x,\theta)$, is a problem for every given model. Here, by trivial densities $\mathcal{L}(x,\theta)$ we understand those of the form

$$\int d^D x\, d\theta\, \mathcal{L}(x,\theta) = \int d\theta\, \theta\, S_0\left(\mathcal{A}(\theta)\right) = S_0(A),$$

where $S_0(A)$ is a usual classical action.

In this paper, we propose an $N=1$ local superfield formalism of Lagrangian quantization, in which the quantities of an initial classical theory are realized in the framework of a *θ-local superfield model* (LSM). The idea of LSM is to represent the objects of a gauge theory (classical action, generators of gauge transformations, etc.) in terms of θ-local functions, trivially related[1] to the spacetime coordinates. Using an analogy with classical mechanics (or classical field theory), we reproduce the dynamics and gauge invariance (in particular, BRST transformations) of the initial theory (the one with $\theta = 0$) in terms of θ-local equations, called *Lagrangian* and *Hamiltonian systems* (LS, HS) with a dynamical θ.[2]

On the basis of the suggested formalism, we solve the following problems:

1. We develop a *dual description*[3] of an arbitrary reducible LSM of Ref. [15] in the case of irreducible gauge theories (with bosonic classical fields), in terms of a BRST charge related to a formal dynamical system with first-class constraints of a higher stage of reducibility.

2. An HS constructed from θ-local quantities, i.e., a quantum action, a gauge-fixing action, and an arbitrary bosonic function, encodes, through a θ-local antibracket, both anticanonical and BRST transformations in terms of a universal set of equations underlying the gauge-independence of the S-matrix. This set of equations is generated, in terms of an even superfield Poisson bracket, by a linear combination of fermionic functionals corresponding to the above θ-local quantities, e.g., the quantum and gauge-fixing actions and the bosonic function.

3. For the first time in the framework of superfield approach, we introduce a *superfield effective action* (also in the case of non-Abelian hypergauges).

In addition to DeWitt's condensed notation [17], we partially use the conventions of Refs. [6, 7]. For the indices of quantities and geometric objects used for the description of a general and restricted LSM, we reserve mainly the capital letters I, P_{MIN}, $\mathcal{A}_s$, CL and the small letters i, $p_{\min}$, α_s, cl, for instance, $\mathcal{M}_{\mathrm{CL}}$, $\Pi T^*\mathcal{M}_{\min}$, with the corresponding coordinates $\mathcal{A}^I(\theta)$ and $\Gamma^{p_{\min}}_{\min}(\theta) = (\Phi^{A_{\min}}_{\min}, \Phi^*_{A_{\min}})(\theta)$. We distinguish between two types of superfield derivatives: the right (left) variational derivative $\delta_{(l)}F/\delta\Phi^A(\theta)$, and the right (left) derivative $\partial_{(l)}\mathcal{F}(\theta)/\partial\Phi^A(\theta)$ for a fixed θ. Derivatives with respect to super(anti)fields and their components are understood as right (left), for instance, $\delta/\delta\lambda^A$, or $\delta/\delta\Phi^*_A(\theta)$, while the corresponding left (right) derivatives are labelled by the subscript $l(r)$. For right-hand derivatives with respect to $\mathcal{A}^I(\theta)$, with a fixed θ, we use the notation $\mathcal{F}_{,I}(\theta) \equiv \partial\mathcal{F}(\theta)/\partial\mathcal{A}^I(\theta)$. The $\delta(\theta)$-function and integration over θ are given, respectively, by $\delta(\theta) = \theta$ and left-hand differentiation over θ.

The rank of an even θ-local supermatrix $K(\theta)$ with Z_2-grading ε is characterized by a pair of numbers $\overline{m} = (m_+, m_-)$, where m_+ (m_-) is the rank of the Bose–Bose (Fermi–Fermi) block of the θ-independent part of the supermatrix $K(\theta)$: $\mathrm{rank}\|K(\theta)\| = \mathrm{rank}\|K(0)\|$. With respect to the same Grassmann parity ε, we understand the dimension of a smooth supersurface, also characterized by a pair of numbers, in the sense of the definition [18] of a supermanifold, so that the above pair coincides with the corresponding numbers of the Bose and Fermi components of $z^i(0)$, being the θ-independent parts of local coordinates $z^i(\theta)$ parameterizing this supersurface. On these pairs, we consider the operations of component addition, $\overline{m} + \overline{n} = (m_+ + n_+, m_- + n_-)$, and comparison,

$$\overline{m} = \overline{n} \Leftrightarrow m_\pm = n_\pm,\ \overline{m} > \overline{n} \Leftrightarrow (m_+ > n_+,\ m_- \geq n_-) \text{ or } (m_+ \geq n_-,\ m_+ > n_-).$$

[1]By *trivial* relation to spacetime coordinates, we imply, in contrast to Hamiltonian formalism, that derivatives with respect to the even t and odd θ component of supertime are taken independently.

[2]By *dynamical* θ, we imply that this coordinate enters an LS or HS not as a parameter, but rather as part of a differential operator ∂_θ that describes the θ-evolution of a system.

[3]By *dual* description, we understand such a treatment of a gauge model that interrelates the Lagrangian and Hamiltonian formalisms (the latter is understood in the sense of *formal* dynamical systems).

2 Lagrangian Formulation

In this section, we propose a Lagrangian formulation of an LSM as an extension of a usual model of classical fields A^i, $i = 1,...,n = n_+ + n_-$, to a θ-local theory, defined on the odd tangent bundle $T_{\rm odd}\mathcal{M}_{\rm CL} \equiv \Pi T\mathcal{M}_{\rm CL} = \{\mathcal{A}^I, \partial_\theta\mathcal{A}^I\}(\theta)$, $I = 1,\ldots,N = N_+ + N_-$[4], $(n_+, n_-) \leq (N_+, N_-)$. The superfields $(\mathcal{A}^I, \partial_\theta\mathcal{A}^I)(\theta)$ are defined on a superspace $\mathcal{M} = \widetilde{\mathcal{M}} \times \widetilde{P}$ parameterized by $\left(z^M, \theta\right)$, where the spacetime coordinates $z^M \subset i \subset I$ include Lorentz vectors and spinors of the superspace $\widetilde{\mathcal{M}}$. We shall investigate the superfield equations of motion, introduce the notions of reducible *general* and *special* superfield gauge theories and apply Noether's first theorem to θ-translations.

The basic objects of the Lagrangian formulation of an LSM are a *Lagrangian action* $S_{\rm L}$: $\Pi T\mathcal{M}_{\rm CL}\times\{\theta\} \to \Lambda_1(\theta;\mathbb{R})$, being a $C^\infty(\Pi T\mathcal{M}_{\rm CL})$-function taking values in a real Grassmann algebra, $\Lambda_1(\theta;\mathbb{R})$, and a (nonequivalent) functional $Z[\mathcal{A}]$, whose θ-density is defined with accuracy up to an arbitrary function $f((\mathcal{A}, \partial_\theta\mathcal{A})(\theta), \theta) \in \ker\{\partial_\theta\}$, $\vec{\varepsilon}(f) = \vec{0}$,

$$Z[\mathcal{A}] = \partial_\theta S_{\rm L}(\theta),\ \vec{\varepsilon}(Z) = \vec{\varepsilon}(\theta) = (1,0,1),\ \vec{\varepsilon}(S_{\rm L}) = \vec{0}. \tag{1}$$

The values $\vec{\varepsilon} = (\varepsilon_P, \varepsilon_{\bar{J}}, \varepsilon)$, $\varepsilon = \varepsilon_P + \varepsilon_{\bar{J}}$, of Z_2-grading, with the auxiliary components $\varepsilon_{\bar{J}}$, ε_P related to the respective coordinates $\left(z^M, \theta\right)$ of a superspace $\mathcal{M}$, are defined on superfields $\mathcal{A}^I(\theta)$ by the relation $\vec{\varepsilon}(\mathcal{A}^I) = ((\varepsilon_P)_I, (\varepsilon_{\bar{J}})_I, \varepsilon_I)$. Note that $\mathcal{M}$ may be realized as the quotient of a symmetry supergroup $J = \bar{J} \times P$, $P = \exp(i\mu p_\theta)$, for the functional $Z[\mathcal{A}]$, where μ and p_θ are, respectively, a nilpotent parameter and a generator of θ-shifts, whereas $\bar{J}$ is chosen as the spacetime SUSY group. The quantities $\varepsilon_{\bar{J}}, \varepsilon_P$ are the respective Grassmann parities of the coordinates of representation spaces of the supergroups $\bar{J}$, P. The introduced objects allow one to achieve a correct incorporation of the spin-statistic relation into operator quantization.

Among the objects $S_{\rm L}(\theta)$ and $Z[\mathcal{A}]$, invariant under the action of a J-superfield representation T restricted to $\bar{J}$, $T|_{\bar{J}}$, it is only $S_{\rm L}(\theta)$ that transforms nontrivially with respect to the total representation T under $\mathcal{A}^I(\theta) \to \mathcal{A}'^I(\theta) = (T|_{\bar{J}}\mathcal{A})^I(\theta - \mu)$,

$$\delta S_{\rm L}(\theta) = S_{\rm L}\left(\mathcal{A}'(\theta), \partial_\theta\mathcal{A}'(\theta), \theta\right) - S_{\rm L}(\theta) = -\mu\left[\frac{\partial}{\partial\theta} + P_0(\theta)(\partial_\theta U)(\theta)\right] S_{\rm L}(\theta). \tag{2}$$

Here, we have introduced the nilpotent operator $(\partial_\theta U)(\theta) = \partial_\theta\mathcal{A}^I(\theta)\partial_l/\partial\mathcal{A}^I(\theta) = [\partial_\theta, U(\theta)]_-$, $U(\theta) = P_1\mathcal{A}^I(\theta)\partial_l/\partial\mathcal{A}^I(\theta)$.

Assuming the existence of a critical superfield configuration for $Z[\mathcal{A}]$, one presents the dynamics of an LSM in terms of superfield Euler–Lagrange equations:

$$\frac{\delta_l Z[\mathcal{A}]}{\delta\mathcal{A}^I(\theta)} = \left[\frac{\partial_l}{\partial\mathcal{A}^I(\theta)} - (-1)^{\varepsilon_I}\partial_\theta\frac{\partial_l}{\partial(\partial_\theta\mathcal{A}^I(\theta))}\right] S_{\rm L}(\theta) \equiv \mathcal{L}^l_I(\theta)S_{\rm L}(\theta) = 0, \tag{3}$$

equivalent, in view of $\partial^2_\theta\mathcal{A}^I(\theta) \equiv 0$, to an LS characterized by $2N$ formally second-order differential equations in θ,

$$\partial^2_\theta\mathcal{A}^J(\theta)\frac{\partial^2_l S_{\rm L}(\theta)}{\partial(\partial_\theta\mathcal{A}^I(\theta))\partial(\partial_\theta\mathcal{A}^J(\theta))} \equiv \partial^2_\theta\mathcal{A}^J(\theta)(S''_{\rm L})_{IJ}(\theta) = 0,$$
$$\Theta_I(\theta) \equiv \frac{\partial_l S_{\rm L}(\theta)}{\partial\mathcal{A}^I(\theta)} - (-1)^{\varepsilon_I}\left[\frac{\partial}{\partial\theta}\frac{\partial_l S_{\rm L}(\theta)}{\partial(\partial_\theta\mathcal{A}^I(\theta))} + (\partial_\theta U)(\theta)\frac{\partial_l S_{\rm L}(\theta)}{\partial(\partial_\theta\mathcal{A}^I(\theta))}\right] = 0. \tag{4}$$

The *Lagrangian constraints* $\Theta_I(\theta) = \Theta_I(\mathcal{A}(\theta), \partial_\theta\mathcal{A}(\theta), \theta)$ restrict the setting of the Cauchy problem for the LS and may be functionally dependent, as first-order equations in θ.

Provided that there exists (at least locally) a supersurface $\Sigma \subset \mathcal{M}_{\rm CL}$ such that

$$\Theta_I(\theta)|_\Sigma = 0,\ \dim\Sigma = \overline{M},\ \mathrm{rank}\left\|\mathcal{L}^l_J(\theta_1)\left[\mathcal{L}^l_I(\theta_1)S_{\rm L}(\theta_1)(-1)^{\varepsilon_I}\right]\right\|_\Sigma = \overline{N} - \overline{M}, \tag{5}$$

there exist $M = M_+ + M_-$ independent identities:

$$\int d\theta\frac{\delta Z[\mathcal{A}]}{\delta\mathcal{A}^I(\theta)}\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0) = 0,\ \hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0) = \sum_{k\geq 0}\left((\partial_\theta)^k\delta(\theta - \theta_0)\right)\hat{\mathcal{R}}_k{}^I_{\mathcal{A}_0}\left(\mathcal{A}(\theta), \partial_\theta\mathcal{A}(\theta), \theta\right). \tag{6}$$

[4]Π denotes the exchange operation of the coordinates of a tangent fiber bundle $T\mathcal{M}_{\rm CL}$ over a configuration $\mathcal{A}^I$ into the coordinates of the opposite Grassmann parity [16], and N_+, N_- are the numbers of bosonic and fermionic fields, among which there may be superfields corresponding to the ghosts of the minimal sector in the BV quantization scheme (in condensed notation [17] used in this paper).

The generators $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)$ of *general gauge transformations,*

$$\delta_g \mathcal{A}^I(\theta) = \int d\theta_0 \hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)\xi^{\mathcal{A}_0}(\theta_0),\ \vec{\varepsilon}(\xi^{\mathcal{A}_0}) = \vec{\varepsilon}_{\mathcal{A}_0},\ \mathcal{A}_0 = 1,\ldots,\ M_0 = M_{0+} + M_{0-},$$

that leave $Z[\mathcal{A}]$ invariant, are functionally dependent under the assumption of locality and $\bar{J}$-covariance, provided that

$$\operatorname{rank}\left\|\sum_{k\geq 0}\hat{\mathcal{R}}_k{}^I_{\mathcal{A}_0}(\theta)\left(\partial_\theta\right)^k\right\|_\Sigma = \overline{M} < \overline{M}_0.$$

The dependence of $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)$ implies the existence (on solutions of the LS) of proper zero-eigenvalue eigenvectors, $\hat{\mathcal{Z}}^{\mathcal{A}_0}_{\mathcal{A}_1}(\mathcal{A}(\theta_0),\partial_{\theta_0}\mathcal{A}(\theta_0),\theta_0;\theta_1)$, with a structure analogous to $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)$ in (6), which exhaust the zero-modes of the generators, and are dependent in case

$$\operatorname{rank}\left\|\sum_{k}\hat{\mathcal{Z}}_k{}^{\mathcal{A}_0}_{\mathcal{A}_1}(\theta_0)\left(\partial_{\theta_0}\right)^k\right\|_\Sigma = \overline{M}_0 - \overline{M} < \overline{M}_1.$$

As a result, the relations of dependence for eigenvectors that define a general L_g-stage reducible LSM are given by

$$\begin{aligned}
&\int d\theta' \hat{\mathcal{Z}}^{\mathcal{A}_{s-2}}_{\mathcal{A}_{s-1}}(\theta_{s-2};\theta')\hat{\mathcal{Z}}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\theta';\theta_s) = \int d\theta' \Theta_J(\theta')\mathcal{L}^{\mathcal{A}_{s-2}J}_{\mathcal{A}_s}\left((\mathcal{A},\partial_\theta\mathcal{A})(\theta_{s-2}),\theta_{s-2},\theta';\theta_s\right),\\
&\overline{M}_{s-1} > \sum_{k=0}^{s-1}(-1)^k\overline{M}_{s-k-2} = \operatorname{rank}\left\|\sum_{k\geq 0}\hat{\mathcal{Z}}^{\mathcal{A}_{s-2}}_{\mathcal{A}_{s-1}}(\theta_{s-2})\left(\partial_{\theta_{s-2}}\right)^k\right\|_\Sigma,\\
&\overline{M}_{L_g} = \sum_{k=0}^{L_g}(-1)^k\overline{M}_{L_g-k-1} = \operatorname{rank}\left\|\sum_{k\geq 0}\hat{\mathcal{Z}}^{\mathcal{A}_{L_g-1}}_{\mathcal{A}_{L_g}}(\theta_{L_g-1})\left(\partial_{\theta_{L_g-1}}\right)^k\right\|_\Sigma,\\
&\vec{\varepsilon}(\hat{\mathcal{Z}}^{\mathcal{A}_s}_{\mathcal{A}_{s+1}}) = \vec{\varepsilon}_{\mathcal{A}_s} + \vec{\varepsilon}_{\mathcal{A}_{s+1}} + (1,0,1),\ \hat{\mathcal{Z}}^{\mathcal{A}_{-1}}_{\mathcal{A}_0}(\theta_{-1};\theta_0) \equiv \hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta_{-1};\theta_0),\\
&\mathcal{L}^{\mathcal{A}_{-1}J}_{\mathcal{A}_1}(\theta_{-1},\theta';\theta_1) \equiv \mathcal{K}^{IJ}_{\mathcal{A}_1}(\theta_{-1},\theta';\theta_1) = -(-1)^{(\varepsilon_I+1)(\varepsilon_J+1)}\mathcal{K}^{JI}_{\mathcal{A}_1}(\theta',\theta_{-1};\theta_1).
\end{aligned}\tag{7}$$

for $s = 1,\ldots,L_g$, $\mathcal{A}_s = 1,\ldots,\ M_s = M_{s+} + M_{s-}$, $\overline{M} \equiv \overline{M}_{-1}$. For $L_g = 0$, the LSM is an irreducible *general gauge theory.*

In case an LSM admits the form $S_{\rm L}(\theta) = T\left(\partial_\theta\mathcal{A}(\theta)\right) - S\left(\mathcal{A}(\theta),\theta\right)$, the functions $\Theta_I(\theta)$ are given on the extended configuration space $\mathcal{M}_{\rm CL}\times\{\theta\}$ by the relations

$$\Theta_I(\theta) = -S_{,I}\left(\mathcal{A}(\theta),\theta\right)(-1)^{\varepsilon_I} = 0, \tag{8}$$

being the usual extremals of the functional $S_0(A) = S\left(\mathcal{A}(0),0\right)$, corresponding to $\theta = 0$. In case $\theta = 0$, condition (5) and identities (6) take the usual form

$$\operatorname{rank}\left\|S_{,IJ}\left(\mathcal{A}(\theta),\theta\right)\right\|_\Sigma = \overline{N} - \overline{M},\quad S_{,I}\left(\mathcal{A}(\theta),\theta\right)\mathcal{R}_0{}^I_{\mathcal{A}_0}\left(\mathcal{A}(\theta),\theta\right) = 0, \tag{9}$$

with linearly-dependent (for $\overline{M}_0 > \overline{M}$) generators of *special gauge transformations,*

$$\delta\mathcal{A}^I(\theta) = \mathcal{R}_0{}^I_{\mathcal{A}_0}\left(\mathcal{A}(\theta),\theta\right)\xi_0^{\mathcal{A}_0}(\theta),$$

with leave invariant only $S(\theta)$, in contrast to $T(\theta)$. The dependence of generators $\mathcal{R}_0{}^I_{\mathcal{A}_0}(\theta)$, as well as of their zero-eigenvalue eigenvectors $\mathcal{Z}^{\mathcal{A}_0}_{\mathcal{A}_1}(\mathcal{A}(\theta),\theta)$, and so on, can be expressed also by special relations of reducibility for $s = 1,\ldots,L_g$, namely,

$$\begin{aligned}
&\mathcal{Z}^{\mathcal{A}_{s-2}}_{\mathcal{A}_{s-1}}(\mathcal{A}(\theta),\theta)\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta),\theta) = S_{,J}(\theta)\mathcal{L}^{\mathcal{A}_{s-2}J}_{\mathcal{A}_s}(\mathcal{A}(\theta),\theta),\ \vec{\varepsilon}(\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}) = \vec{\varepsilon}_{\mathcal{A}_{s-1}} + \vec{\varepsilon}_{\mathcal{A}_s},\\
&\mathcal{Z}^{\mathcal{A}_{-1}}_{\mathcal{A}_0}(\theta) \equiv \mathcal{R}_0{}^I_{\mathcal{A}_0}(\theta),\ \mathcal{L}^{\mathcal{A}_{-1}J}_{\mathcal{A}_1}(\theta) \equiv \mathcal{K}^{IJ}_{\mathcal{A}_1}(\theta) = -(-1)^{\varepsilon_I\varepsilon_J}\mathcal{K}^{JI}_{\mathcal{A}_1}(\theta).
\end{aligned}\tag{10}$$

For $\overline{M}_{L_g} = \sum_{k=0}^{L_g}(-1)^k\overline{M}_{L_g-k-1} = \operatorname{rank}\left\|\mathcal{Z}^{\mathcal{A}_{L_g-1}}_{\mathcal{A}_{L_g}}\right\|_\Sigma$, relations (8)–(10) determine a *special gauge theory* of L_g-stage reducibility. The gauge algebra of such a theory is θ-locally embedded into the gauge algebra of a general gauge theory with the functional $Z[\mathcal{A}] = \partial_\theta(T(\theta) - S(\theta))$, which implies the relation between the eigenvectors

$$\hat{\mathcal{Z}}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta_{s-1}), \theta_{s-1}; \theta_s) = -\delta(\theta_{s-1} - \theta_s)\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta_{s-1}), \theta_{s-1}) \tag{11}$$

and the fact that the structure functions of the gauge algebra of a special gauge theory may depend on $\partial_\theta \mathcal{A}^I(\theta)$ only parametrically. Note that an extended (as compared to $\{P_a(\theta)\}$, $a = 1, 2$) system of projectors onto $C^\infty(\Pi T\mathcal{M}_{\mathrm{CL}}) \times \{\theta\}$, $\{P_0(\theta), \theta\partial/\partial\theta, U(\theta)\}$, selects from (10) two kinds of gauge algebras: one with structure equations and functions $S(\mathcal{A}(\theta))$, $\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta))$ not depending on θ in an explicit form; another with the standard relations for the gauge algebra of a reducible model with quantities $S_0(A)$, $\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(A)$, in case $\theta = 0$, $(\varepsilon_P)_I = (\varepsilon_P)_{\mathcal{A}_s} = 0$, $s = 1, ..., L_g$, and under the assumption of completeness of the reduced generators $\mathcal{R}^i_{\alpha_0}(\mathcal{A}(\theta))$ and eigenvectors $\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A}(\theta))$.

An extension of a usual field theory to a θ-local LSM permits one to apply Noether's first theorem [19] to the invariance of the density $d\theta S_{\mathrm{L}}(\theta)$ with respect to global θ-translations as symmetry transformations of the superfields $\mathcal{A}^I(\theta)$ and coordinates (z^M, θ), $(\mathcal{A}^I, z^M, \theta) \to (\mathcal{A}^I, z^M, \theta + \mu)$. By direct verification, one establishes that the function

$$S_E((\mathcal{A}, \partial_\theta\mathcal{A})(\theta), \theta) \equiv \frac{\partial S_{\mathrm{L}}(\theta)}{\partial(\partial^r_\theta \mathcal{A}^I(\theta))}\partial^r_\theta\mathcal{A}^I(\theta) - S_{\mathrm{L}}(\theta) \tag{12}$$

is an LS integral of motion, i.e., a conserved quantity under the θ-evolution, in case there holds the equation

$$\left.\frac{\partial}{\partial\theta}S_{\mathrm{L}}(\theta) + 2(\partial_\theta U)(\theta)S_{\mathrm{L}}(\theta)\right|_{\mathcal{L}^l_I S_{\mathrm{L}}=0} = 0. \tag{13}$$

In contrast to its analogue in a t-local field theory, the energy $E(t)$, the function $S_E(\theta)$ is an LS integral also in the case of an explicit dependence on θ. This fact takes place in case $S_{\mathrm{L}}(\theta)$ admits the structure

$$S_{\mathrm{L}}((\mathcal{A}, \partial_\theta\mathcal{A})(\theta), \theta) = S^0_{\mathrm{L}}(\mathcal{A}, \partial_\theta\mathcal{A})(\theta) - 2\theta(\partial_\theta U)(\theta)S^0_{\mathrm{L}}(\theta),\ \vec{\varepsilon}(S^0_{\mathrm{L}}) = \vec{0}. \tag{14}$$

3 Hamiltonian Formulation

This section is devoted to a Hamiltonian formulation of an LSM, on the odd cotangent bundle $T^*_{\mathrm{odd}}\mathcal{M}_{\mathrm{CL}} \equiv \Pi T^*\mathcal{M}_{\mathrm{CL}} = \{\mathcal{A}^I, \mathcal{A}^*_I\}(\theta)$. Here, we shall establish a connection to the Lagrangian formalism and investigate the existence of a Noether integral, related to θ-translations, that leads to the fulfillment of a θ-local master equation.

Independently, an LSM can be formulated without an $\mathcal{M}_{\mathrm{CL}}$-extension, in terms of a *Hamiltonian action*, being a $C^\infty(\Pi T^*\mathcal{M}_{\mathrm{CL}})$-function, $S_{\mathrm{H}} : \Pi T^*\mathcal{M}_{\mathrm{CL}} \times \{\theta\} \to \Lambda_1(\theta; \mathbb{R})$, depending on superantifields $\mathcal{A}^*_I(\theta) = (A^*_I - \theta J_I)$, included in the local coordinates of $\Pi T^*\mathcal{M}_{\mathrm{CL}}$: $\Gamma^P_{\mathrm{CL}}(\theta) = (\mathcal{A}^I, \mathcal{A}^*_I)(\theta)$, $\vec{\varepsilon}(\mathcal{A}^*_I) = \vec{\varepsilon}(\mathcal{A}^I) + (1, 0, 1)$. The equivalence of the Lagrangian and Hamiltonian formulations is implied by the nondegeneracy of the supermatrix $\|(S''_{\mathrm{L}})_{IJ}(\theta)\|$ given by (4), in the framework of a Legendre transformation of $S_{\mathrm{L}}(\theta)$ with respect to $\partial^r_\theta\mathcal{A}^I(\theta)$,

$$S_{\mathrm{H}}(\Gamma_{\mathrm{CL}}(\theta), \theta) = \mathcal{A}^*_I(\theta)\partial^r_\theta\mathcal{A}^I(\theta) - S_{\mathrm{L}}(\theta),\ \mathcal{A}^*_I(\theta) = \frac{\partial S_{\mathrm{L}}(\theta)}{\partial(\partial^r_\theta\mathcal{A}^I(\theta))}, \tag{15}$$

where $S_{\mathrm{H}}(\Gamma_{\mathrm{CL}}(\theta), \theta)$ coincides with $S_E(\theta)$ in terms of the $\Pi T^*\mathcal{M}_{\mathrm{CL}}$-coordinates.

The dynamics of an LSM is given by a *generalized Hamiltonian system* of $3N$ first-order equations in θ, equivalent to the LS equations in (4), and expressed through a θ-local antibracket $(\cdot\,,\cdot)_\theta$, namely,

$$\begin{aligned}&\partial^r_\theta\Gamma^P_{\mathrm{CL}}(\theta) = \left(\Gamma^P_{\mathrm{CL}}(\theta), S_{\mathrm{H}}(\theta)\right)_\theta,\ \Theta^{\mathrm{H}}_I(\Gamma_{\mathrm{CL}}(\theta), \theta) = \Theta_I(\mathcal{A}(\theta), \partial_\theta\mathcal{A}(\Gamma_{\mathrm{CL}}(\theta), \theta), \theta) = 0,\\ &(\mathcal{F}_1, \mathcal{F}_2)_\theta \equiv \frac{\partial\mathcal{F}_1}{\partial\mathcal{A}^I(\theta)}\frac{\partial\mathcal{F}_2}{\partial\mathcal{A}^*_I(\theta)} - \frac{\partial_r\mathcal{F}_1}{\partial\mathcal{A}^*_I(\theta)}\frac{\partial_l\mathcal{F}_2}{\partial\mathcal{A}^I(\theta)},\end{aligned} \tag{16}$$

with *Hamiltonian constraints* $\Theta^{\mathrm{H}}_I(\Gamma_{\mathrm{CL}}(\theta), \theta)$. The latter coincide with half of the equations of the HS proper, due to transformations (15) and their consequences:

$$\Theta^{\mathrm{H}}_I(\Gamma_{\mathrm{CL}}(\theta), \theta) = -\partial^r_\theta\mathcal{A}^*_I(\theta) - S_{\mathrm{H},I}(\theta)(-1)^{\varepsilon_I}. \tag{17}$$

Formula (17) establishes the equivalence of an HS with a generalized HS, and hence with an LS, in the corresponding, formal in view of the degeneracy conditions (5), setting ($\theta = 0$, $k = \mathrm{CL}$) of the Cauchy problem for integral curves $\hat{\mathcal{A}}^I(\theta)$, $\hat{\Gamma}^P_k(\theta)$,

$$\left(\hat{\mathcal{A}}^I, \partial^r_\theta \hat{\mathcal{A}}^I\right)(0) = \left(\overline{\mathcal{A}}^I, \overline{\partial^r_\theta \mathcal{A}}^I\right), \; \hat{\Gamma}^P_k(0) = \left(\overline{\mathcal{A}}^I, \overline{\mathcal{A}}^*_I\right) : \; \overline{\mathcal{A}}^*_I = P_0\left[\frac{\partial S_{\mathrm{L}}(\theta)}{\partial(\partial^r_\theta \mathcal{A}^I(\theta))}\right]\left(\overline{\mathcal{A}}^I, \overline{\partial^r_\theta \mathcal{A}}^I\right) \qquad (18)$$

(we ignore the continuous part of the indices I). The equivalence between an HS and a generalized HS holds due to the coincidence (mutual inclusion) of the corresponding sets of solutions. Indeed, the solutions of a generalized HS are included into those of an HS by construction, while the reverse is valid due to (17).

The HS is defined through a variational problem for a functional identical with $Z[\mathcal{A}]$,

$$Z_{\mathrm{H}}[\Gamma_k] = \int d\theta\left[\frac{1}{2}\Gamma^P_k(\theta)\omega^k_{PQ}(\theta)\partial^r_\theta \Gamma^Q_k(\theta) - S_{\mathrm{H}}(\Gamma_k(\theta),\theta)\right],$$
$$\omega^{PQ}_k(\theta) \equiv \left(\Gamma^P_k(\theta), \Gamma^Q_k(\theta)\right)_\theta, \; \omega^{PD}_k(\theta)\omega^k_{DQ}(\theta) = \delta^P{}_Q. \qquad (19)$$

Definitions (8)–(10) remain the same for special gauge theories, while definitions (6), (7), in the case of general gauge theories of L_g-stage reducibility, are transformed by the rule

$$\hat{\mathcal{Z}}_{\mathrm{H}}{}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}\left(\Gamma_k(\theta_{s-1}), \theta_{s-1}; \theta_s\right) = \hat{\mathcal{Z}}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}\left(\mathcal{A}(\theta_{s-1}), \partial_{\theta_{s-1}}\mathcal{A}(\Gamma_k(\theta_{s-1}), \theta_{s-1}), \theta_{s-1}; \theta_s\right), \; s = 0, ..., L_g. \quad (20)$$

From eqs. (13), as well as from transformations (15) and their consequence $\frac{\partial}{\partial\theta}(S_{\mathrm{L}} + S_{\mathrm{H}})(\theta) = 0$, there follows the invariance of $S_{\mathrm{H}}(\theta)$ under θ-shifts along arbitrary solutions $\hat{\Gamma}^P_k(\theta)$, or, equivalently, along an $(\varepsilon_P, \varepsilon)$-odd vector field $\mathbf{Q}(\theta) = \mathrm{ad}S_{\mathrm{H}}(\theta) \equiv (S_{\mathrm{H}}(\theta), \cdot\,)_\theta$. Therefore,

$$\delta_\mu S_{\mathrm{H}}(\theta)|_{\hat{\Gamma}_k(\theta)} = \mu\left[\frac{\partial}{\partial\theta}S_{\mathrm{H}}(\theta) - ((S_{\mathrm{H}}(\theta), S_{\mathrm{H}}(\theta))_\theta\right] = 0, \; \delta_\mu S_{\mathrm{H}}(\theta) = \mu\partial_\theta S_{\mathrm{H}}(\theta) \qquad (21)$$

holds true, provided that $S_{\mathrm{H}}(\theta)$ can be presented, according to (13), in the form

$$S_{\mathrm{H}}(\Gamma_k(\theta), \theta) = S^0_{\mathrm{H}}(\Gamma_k(\theta)) + \theta\left((S^0_{\mathrm{H}}(\Gamma_k(\theta)), S^0_{\mathrm{H}}(\Gamma_k(\theta))\right)_\theta, \qquad (22)$$

where $(\partial_\theta U)(\theta)S_{\mathrm{L}}(\theta) = 1/2\,(S_{\mathrm{H}}(\theta), S_{\mathrm{H}}(\theta))_\theta$ and $S^0_{\mathrm{H}}(\Gamma_k(\theta))$ is the Legendre transform of $S^0_{\mathrm{L}}(\theta)$, defined by (14).

If $S_{\mathrm{H}}(\theta)$, or $S_{\mathrm{L}}(\theta)$, does not depend on θ explicitly, then eq. (21), or (13), implies the fulfilment of the equation $(S_{\mathrm{H}}(\theta), S_{\mathrm{H}}(\theta))_\theta = 0$, or $(\partial_\theta U)(\theta)S_{\mathrm{L}}(\theta)|_{\hat{\mathcal{A}}(\theta)} = 0$, which has no counterpart in a t-local field theory, and imposes the known condition [1] that $S_{\mathrm{H}}(\theta)$, or $S_{\mathrm{L}}(\theta)$, be proper, although for an LSM at the classical level. In this case, a θ-superfield integrability[5] of the HS in (16) is guaranteed by the standard properties of the antibracket, including the Jacobi identity:

$$(\partial^r_\theta)^2\Gamma^P_k(\theta) = \frac{1}{2}\left(\Gamma^P_k(\theta), (S_{\mathrm{H}}(\Gamma_k(\theta)); S_{\mathrm{H}}(\Gamma_k(\theta)))_\theta\right)_\theta = 0. \qquad (23)$$

This fact ensures the validity on $C^\infty(\Pi T^*\mathcal{M}_{\mathrm{CL}} \times \{\theta\})$ of the θ-translation formula

$$\delta_\mu \mathcal{F}(\theta)|_{\hat{\Gamma}_k(\theta)} = \mu\left[\frac{\partial}{\partial\theta} - \mathrm{ad}S_{\mathrm{H}}(\theta)\right]\mathcal{F}(\theta) \equiv \mu\check{s}^l(\theta)\mathcal{F}(\theta), \qquad (24)$$

as well as the nilpotency of a BRST-like generator of θ-shifts along $\mathbf{Q}(\theta)$, $\check{s}^l(\theta)$.

Depending on the realization of additional properties of a gauge theory (see Section 4), we shall henceforth assume the fulfillment of the equation

$$\Delta^k(\theta)S_{\mathrm{H}}(\theta) = 0, \; \Delta^k(\theta) \equiv \frac{1}{2}(-1)^{\varepsilon(\Gamma^Q)}\omega^k_{QP}(\theta)\left(\Gamma^P_k(\theta), \left(\Gamma^Q_k(\theta), \cdot\,\right)_\theta\right)_\theta. \qquad (25)$$

Eq. (25) is equivalent to a vanishing divergence of the vector field $\mathbf{Q}(\theta)$, namely,

$$\mathrm{div}\,(\partial^r_\theta \Gamma_k(\theta)|_{\hat{\Gamma}_k(\theta)}) = \frac{\partial_r}{\partial\Gamma^P_k(\theta)}\left(\partial^r_\theta \Gamma^P_k(\theta)|_{\hat{\Gamma}_k(\theta)}\right) = 2\Delta^k(\theta)S_{\mathrm{H}}(\theta) = 0. \qquad (26)$$

This condition holds trivially for the symplectic analogue of formula (26). The validity of the *Hamiltonian master equation* $(S_{\mathrm{H}}(\theta), S_{\mathrm{H}}(\theta))_\theta = 0$ for $\frac{\partial}{\partial\theta}S_{\mathrm{H}}(\theta) = 0$ justifies the interpretation of the equivalent equation in (13), for $\frac{\partial}{\partial\theta}S_{\mathrm{L}}(\theta) = 0$, $(\partial_\theta U)(\theta)S_{\mathrm{L}}(\theta)|_{\mathcal{L}^l_I S_{\mathrm{L}}=0} = 0$, as a *Lagrangian master equation*.

[5]The notion of θ-superfield integrability is introduced by analogy with the treatment of Ref. [14].

4 Local Superfield Quantization

4.1 Superfield Quantum Action in Initial Coordinates

In this subsection, we shall transform the reducibility relations of a specially *restricted* LSM into a sequence of new gauge transformations for the ghost superfields of the minimal sector. Together with the gauge transformations of the classical superfields $\mathcal{A}^i(\theta)$, extracted from $\mathcal{A}^I(\theta)$, the new gauge transformations are translated into a Hamiltonian system related to the restricted HS. A requirement of superfield integrability for the resulting HS produces a deformation of the θ-local Hamiltonian in powers of the ghosts and superantifields of the minimal sector, and leads to a quantum action, and, independently, to a gauge-fixing action (see Subsection 4.3), subject to different θ-local master equations.

Given the standard distribution of ghost number [1] for $\Gamma^P_{\mathrm{CL}}(\theta)$, $\mathrm{gh}(\mathcal{A}^*_I) = -1 - \mathrm{gh}(\mathcal{A}^I) = -1$, the choice $\mathrm{gh}(\theta, \partial_\theta) = (-1, 1)$ implying the absence of ghosts among $\mathcal{A}^I$, and, in particular, the relations $(\varepsilon_P)_I = 0$, the quantization rules, firstly, consists in restricting an LSM (in both Lagrangian and Hamiltonian formulations) by the equations

$$\left(\mathrm{gh}, \frac{\partial}{\partial\theta}\right) S_{\mathrm{H(L)}}(\theta) = (0, 0). \tag{27}$$

Given the existence of a potential term in $S_{\mathrm{H(L)}}(\theta)$, $S(\mathcal{A}(\theta), 0) = S(\mathcal{A}(\theta))$ and the absence in $S_{\mathrm{H(L)}}(\theta)$ of a dimensional constant with a nonzero ghost number, solutions of eqs. (27) select from an LSM a standard field theory model with a classical action $S_0(A)$ in which the fields A^i are extended to $\mathcal{A}^i(\theta)$. Then an extended HS in (16) is transformed into a θ-integrable system defined in $\Pi T^*\mathcal{M}_{\mathrm{cl}} = \{\Gamma^p_{\mathrm{cl}}(\theta)\} = \{(\mathcal{A}^i, \mathcal{A}^*_i)(\theta)\}$, with $\Theta^{\mathrm{H}}_i(\mathcal{A}(\theta)) = \Theta_i(\mathcal{A}(\theta))$,

$$\partial^r_\theta \Gamma^p_{\mathrm{cl}}(\theta) = (\Gamma^p_{\mathrm{cl}}(\theta), S_0(\mathcal{A}(\theta)))_\theta,\ \Theta^{\mathrm{H}}_i(\mathcal{A}(\theta)) = -(-1)^{\varepsilon_i} S_{0,i}(\mathcal{A}(\theta)). \tag{28}$$

The restricted special gauge transformations $\delta\mathcal{A}^i(\theta) = \mathcal{R}^i_{0\alpha_0}(\mathcal{A}(\theta))\,\xi^{\alpha_0}_0(\theta)$, $\vec{\varepsilon}(\xi^{\alpha_0}_0(\theta)) = \vec{\varepsilon}_{\alpha_0}$, with the condition $(\varepsilon_P)_{\alpha_0} = 0$, are embedded by the substitution $\xi^{\alpha_0}_0(\theta) = d\tilde{\xi}^{\alpha_0}_0(\theta) = \mathcal{C}^{\alpha_0}(\theta)d\theta$, $\alpha_0 = 1, ...,$ $m_0 = m_{0-} + m_{0+}$, into a Hamiltonian system with $2n$ equations for unknown $\Gamma^p_{\mathrm{cl}}(\theta)$, with the Hamiltonian $S^0_1(\Gamma_{\mathrm{cl}}, C_0)(\theta) = (\mathcal{A}^*_i \mathcal{R}_0{}^i_{\alpha_0}(\mathcal{A})\mathcal{C}^{\alpha_0})(\theta)$. A union of this system with the HS in (28), extended to $2(n + m_0)$ equations, has the form

$$\partial^r_\theta \Gamma^{p_{[0]}}_{[0]}(\theta) = \left(\Gamma^{p_{[0]}}_{[0]}(\theta), S^0_{[1]}(\theta)\right)_\theta,\ S^0_{[1]}(\theta) = (S_0 + S^0_1)(\theta),\ \Gamma^{p_{[0]}}_{[0]} \equiv (\Gamma^p_{\mathrm{cl}}, \Gamma^{p_0}_0),\ \Gamma^{p_0}_0 \equiv (\mathcal{C}^{\alpha_0}, \mathcal{C}^*_{\alpha_0}). \tag{29}$$

By virtue of (10), the function $S^0_1(\theta)$ is invariant, modulo $S_{0,i}(\theta)$, under special gauge transformations of ghost superfields $\mathcal{C}^{\alpha_0}(\theta)$, with arbitrary functions $\xi^{\alpha_1}_1(\theta)$, $(\varepsilon_P)_{\alpha_1} = 0$, on the superspace $\mathcal{M}$:

$$\delta\mathcal{C}^{\alpha_0}(\theta) = \mathcal{Z}^{\alpha_0}_{\alpha_1}(\mathcal{A}(\theta))\xi^{\alpha_1}_1(\theta),\ (\vec{\varepsilon}, \mathrm{gh})\xi^{\alpha_1}_1(\theta) = (\vec{\varepsilon}_{\alpha_1} + (1,0,1), 1). \tag{30}$$

Making the substitution $\xi^{\alpha_1}_1(\theta) = d\tilde{\xi}^{\alpha_1}_1(\theta) = \mathcal{C}^{\alpha_1}(\theta)d\theta$, $\alpha_1 = 1, ..., m_1$, and an enlargement of m_0 first-order equations in θ, with respect to the unknowns $\mathcal{C}^{\alpha_0}(\theta)$ in transformations (30), to an HS of $2m_0$ equations with the Hamiltonian $S^1_1(\mathcal{A}, C^*_0, C_1)(\theta) = (\mathcal{C}^*_{\alpha_0}\mathcal{Z}^{\alpha_0}_{\alpha_1}(\mathcal{A})\mathcal{C}^{\alpha_1})(\theta)$, we obtain a system of the form (29), written for $\partial^r_\theta \Gamma^{p_0}_0(\theta)$. The enlargement of the union of the latter HS with eqs. (29) is formally identical to the system (29) under the replacement

$$(\Gamma^{p_{[0]}}_{[0]}, S^0_{[1]}) \to (\Gamma^{p_{[1]}}_{[1]}, S^1_{[1]}):\ \left\{\Gamma^{p_{[1]}}_{[1]} = (\Gamma^{p_{[0]}}_{[0]}, \Gamma^{p_1}_1),\ \Gamma^{p_1}_1 = (\mathcal{C}^{\alpha_1}, \mathcal{C}^*_{\alpha_1}),\ S^1_{[1]} = S^0_{[1]} + S^1_1\right\}.$$

The iteration sequence related to a reformulation of the special gauge transformations of ghosts $\mathcal{C}^{\alpha_0}, ..., \mathcal{C}^{\alpha_{s-2}}$, obtained from (possibly) enhanced[6] relations (10), leads, for an L-stage-reducible restricted LSM at the s-th step with $0 < s \leq L$ and $\Gamma^p_{\mathrm{cl}} \equiv \Gamma^{p_{-1}}_{-1}$, to invariance transformations for $S^{s-1}_1(\theta)$, modulo $S_{0,i}(\theta)$, namely,

[6]From $\mathrm{gh}(\mathcal{A}^I) = 0$ in eqs. (27), with $(\varepsilon_P)_{\mathcal{A}_s} = (\varepsilon_P)_I = 0$, $s = 0, ..., L_g$, it follows that the values of $\overline{m}$, $\overline{m}_s$ may be both larger and smaller than the corresponding values $\overline{M}$, $\overline{M}_s$, in contrast to the values of $\overline{n}$, $\overline{N}$. Indeed, for a restricted LSM, the presence of additional gauge symmetries is possible; therefore, we suppose that (possibly) enhanced sets of restricted functions $\mathcal{R}_0{}^i_{\alpha_0}(\theta)$, $\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\theta)$ exhaust, correspondingly, on the surface $S_{0,i}(\theta) = 0$, the zero-modes of both the Hessian $S_{0,ij}(\theta)$ and $\mathcal{Z}^{\alpha_{s-2}}_{\alpha_{s-1}}(\theta)$. As a consequence, this implies that the final stage of reducibility for a restricted model L is different from L_g.

$$\delta\mathcal{C}^{\alpha_{s-1}}(\theta) = \mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A}(\theta))\xi^{\alpha_s}_s(\theta),\ (\vec{\varepsilon},\mathrm{gh})\xi^{\alpha_s}_s(\theta) = (\vec{\varepsilon}_{\alpha_s} + s(1,0,1),s),\ (\varepsilon_P)_{\alpha_s} = 0,$$
$$S^{s-1}_1(\theta) = (\mathcal{C}^*_{\alpha_{s-2}}\mathcal{Z}^{\alpha_{s-2}}_{\alpha_{s-1}}(\mathcal{A})\mathcal{C}^{\alpha_{s-1}})(\theta),\ \left(\mathrm{gh}, \frac{\partial}{\partial\theta}\right) S^{s-1}_1(\theta) = (0,0). \qquad (31)$$

The substitution $\xi^{\alpha_s}_s(\theta) = d\tilde{\xi}^{\alpha_s}_s(\theta) = \mathcal{C}^{\alpha_s}(\theta)d\theta$, $\alpha_s = 1,\ldots, m_s = m_{s-} + m_{s+}$, transforms special gauge transformations (31) into m_{s-1} equations with respect to unknown $\mathcal{C}^{\alpha_{s-1}}(\theta)$, extended by the introduction of superantifields $\mathcal{C}^*_{\alpha_{s-1}}(\theta)$ to an HS:

$$\partial^r_\theta\Gamma^{p_{s-1}}_{s-1}(\theta) = \left(\Gamma^{p_{s-1}}_{s-1}(\theta), S^s_1(\theta)\right)_\theta,\ S^s_1(\theta) = (\mathcal{C}^*_{\alpha_{s-1}}\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A})\mathcal{C}^{\alpha_s})(\theta),\ \Gamma^{p_{s-1}}_{s-1} = (\mathcal{C}^{\alpha_{s-1}},\mathcal{C}^*_{\alpha_{s-1}}). \qquad (32)$$

Having combined the system (32) with an HS of the same form, although with $\partial^r_\theta\Gamma^{p_{[s-1]}}_{[s-1]}(\theta)$ and the Hamiltonian $S^{s-1}_{[1]}(\theta) = (S_0+\sum^{s-1}_{r=0} S^r_1)(\theta)$, and having expressed the result for $2\,(n + \sum^s_{r=0} m_r)$ equations with $S^s_{[1]}(\theta) = (S^{s-1}_{[1]} + S^s_1)(\theta)$, we obtain, by induction, the following HS:

$$\partial^r_\theta\Gamma^{p_{[L]}}_{[L]}(\theta) = \left(\Gamma^{p_{[L]}}_{[L]}(\theta), S^L_{[1]}(\theta)\right)_\theta,\ S^L_{[1]}(\theta) = S_0(\mathcal{A}(\theta)) + \sum^L_{s=0}(\mathcal{C}^*_{\alpha_{s-1}}\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A})\mathcal{C}^{\alpha_s})(\theta)\,. \qquad (33)$$

The function $S^L_{[1]}(\theta)$, subject to the condition of a proper θ-local solution of the classical master equation [1], with the antibracket extended in $\Pi T^*\mathcal{M}_k=\{\Gamma^{p_{[L]}}_{[L]}(\theta) \equiv \Gamma^{p_k}_k(\theta)=(\Phi^{A_k},\Phi^*_{A_k})(\theta),\ A_k = n + \sum^L_{r=0} m_r,\ k=\min\}$, is a solution with accuracy up to $O(C^{\alpha_s})$, modulo $S_{0,i}$. The integrability of the HS in (33) is guaranteed by a double deformation of $S^L_{[1]}(\theta)$: first in powers of $\Phi^*_{A_k}(\theta)$ and then in powers of $\mathcal{C}^{\alpha_s}(\theta)$, in the framework of the existence theorem [20] for the classical master equation in the minimal sector:

$$((S_{\mathrm{H};k}(\Gamma_k(\theta)), S_{\mathrm{H};k}(\Gamma_k(\theta)))_\theta = 0,\ \left(\vec{\varepsilon},\mathrm{gh},\frac{\partial}{\partial\theta}\right) S_{\mathrm{H};k}(\Gamma_k(\theta)) = \left(\vec{0},0,0\right),\ k = \min. \qquad (34)$$

The proposed superfield algorithm for constructing the function $S_{\mathrm{H;min}}(\theta)$ may be considered as a superfield version of the Koszul–Tate complex resolution [21]. We remind that the enlargement of $S_{\mathrm{H;min}}(\theta)$ to $S_{\mathrm{H};k}(\Gamma_k(\theta))$, $S_{\mathrm{H};k}(\theta) = S_{\mathrm{H;min}}(\theta) + \sum^L_{s=0}\sum^s_{s'=0}(\mathcal{C}^*_{s'\alpha_s}\mathcal{B}^{\alpha_s}_{s'})(\theta)$, being a proper solution [1] in $\Pi T^*\mathcal{M}_k = \{\Gamma^{p_k}_k(\theta)\}$,

$$\Gamma^{p_k}_k(\theta) = (\Gamma^{p_{\min}}_{\min},\mathcal{C}^{\alpha_s}_{s'},\mathcal{B}^{\alpha_s}_{s'},\mathcal{C}^*_{s'\alpha_s},\mathcal{B}^*_{s'\alpha_s})(\theta),\ s' = 0,\ldots,s,\ s = 0,\ldots,L,$$
$$(\vec{\varepsilon},\mathrm{gh})\mathcal{C}^{\alpha_s}_{s'}(\theta) = (\vec{\varepsilon}_{\alpha_s} + (s+1)(1,0,1), 2s' - s - 1) = (\vec{\varepsilon},\mathrm{gh})\mathcal{B}^{\alpha_s}_{s'}(\theta) + ((1,0,1),-1)$$

(henceforth we assume $k = \mathrm{ext}$ and take into account that $(\vec{\varepsilon},\mathrm{gh})\Phi^*_{A_k}(\theta) = -((1,0,1),1) - (\vec{\varepsilon},\mathrm{gh})\Phi^{A_k}(\theta))$, with the pyramids of ghosts and Nakanishi–Lautrup superfields, and with a deformation in the Planck constant $\hbar$, determines the quantum action $S^\Psi_{\mathrm{H}}(\Gamma(\theta),\hbar)$, e.g., in case of an Abelian hypergauge defined as an anticanonical phase transformation:

$$\Gamma^{p_k}_k(\theta) \to \Gamma'^{p_k}_k(\theta) = \left(\Phi^{A_k}(\theta), \Phi^*_{A_k}(\theta) - \frac{\partial\Psi(\Phi(\theta))}{\partial\Phi^{A_k}(\theta)}\right):\ S^\Psi_{\mathrm{H}}(\Gamma(\theta),\hbar) = e^{\mathrm{ad}\Psi}S_{\mathrm{H};k}(\Gamma_k(\theta),\hbar). \qquad (35)$$

The functions $(S^\Psi_{\mathrm{H}}, S_{\mathrm{H};k})(\theta,\hbar)$ obey eqs. (25), (34) in case the $\hbar$-deformation of $S_{\mathrm{H;min}}(\theta)$ is their solution. It is known that this choice of equations ensures the integrability of a non-equivalent HS constructed from S^Ψ_{H}, $S_{\mathrm{H};k}$, as well as the anticanonical [preserving the volume element $dV_k(\theta) = \prod_{p_k} d\Gamma^{p_k}_k(\theta)$] nature of this change of variables, corresponding to a θ-shift by a constant parameter μ along the corresponding HS solutions. In its turn, the quantum master equation

$$\Delta^k(\theta)\exp\left[\frac{i}{\hbar}E(\theta,\hbar)\right] = 0,\ E \in \{S^\Psi_{\mathrm{H}}, S_{\mathrm{H};k}\} \qquad (36)$$

determines a non-integrable HS, with the respective anticanonical change of variables preserving $d\hat{V}_k(\theta) = \exp\left[(i/\hbar)\,E(\theta,\hbar)\right]dV_k(\theta)$. It is the latter nonintegrable HS with the Hamiltonian $S^\Psi_{\mathrm{H}}(\theta,\hbar)$ that is crucial, for $\theta = 0$, in the BV formalism. This HS determines on $\Pi T^*\mathcal{M}_k$ a θ-local, but not nilpotent, generator of BRST transformations, $\breve{s}^{l(\Psi)}(\theta)$, which is associated with its θ-nonintegrable consequence:

$$\partial^r_\theta\left(\Phi^{A_k},\Phi^*_{A_k}\right)(\theta) = \left(\left(\Phi^{A_k}(\theta), S^\Psi_{\mathrm{H}}(\theta,\hbar)\right)_\theta, 0\right),\ \breve{s}^{l(\Psi)}(\theta) = \frac{\partial}{\partial\theta} + \frac{\partial_r S^\Psi_{\mathrm{H}}(\theta,\hbar)}{\partial\Phi^*_{A_k}(\theta)}\frac{\partial_l}{\partial\Phi^{A_k}(\theta)}. \qquad (37)$$

4.2 Duality between the BV and BFV Superfield Quantities

In this subsection, we shall construct a dual description of an LSM. Namely, an embedding of a restricted LSM gauge algebra, described by the action $S_{\mathrm{H;min}}(\theta)$ and by eq. (34), into the gauge algebra of a general gauge theory in Lagrangian formalism, see eqs. (6)–(11), can be effectively realized by means of dual functional counterparts, with the opposite $(\varepsilon_P, \varepsilon)$-parity, of the action and antibracket, following, in part, the approach of Refs. [13, 15]. To this end, let us consider the functional

$$Z_k[\Gamma_k] = -\partial_\theta S_{\mathrm{H};k}(\theta)\,,\ (\vec{\varepsilon}, \mathrm{gh})Z_k = ((1,0,1),1)$$

on the supermanifold $\Pi T(\Pi T^*\mathcal{M}_k) = \{(\Gamma_k^{p_k}, \partial_\theta\Gamma_k^{p_k})(\theta), k = \min\}$ with natural $(\varepsilon_P, \varepsilon)$-even, symplectic, and $(\varepsilon_P, \varepsilon)$-odd Poisson structures. These structures define an $(\varepsilon_P, \varepsilon)$-even functional $\{\,\cdot\,,\,\cdot\,\}$ with canonical pairs $\{(\Phi_k^{A_k}, \partial_\theta\Phi^*_{A_k}), (\partial_\theta\Phi_k^{A_k}, \Phi^*_{A_k})\}(\theta)$, and $(\varepsilon_P, \varepsilon)$-odd θ-local, $(\,\cdot\,,\,\cdot\,)_\theta^{(\Gamma_k, \partial_\theta\Gamma_k)}$, Poisson brackets. The latter act on the superalgebra $C^\infty(\Pi T(\Pi T^*\mathcal{M}_k) \times \theta)$ and provide the lifting of the antibracket $(\,\cdot\,,\,\cdot\,)_\theta$ defined on $\Pi T^*\mathcal{M}_k$. For arbitrary functionals $F_{\mathrm{t}}[\Gamma_k] = \partial_\theta\mathcal{F}_{\mathrm{t}}((\Gamma_k, \partial_\theta\Gamma_k)(\theta), \theta)$, $\mathrm{t} = 1, 2$, we have the following correspondence between the Poisson brackets of opposite Grassmann grading:

$$\begin{aligned}
&\{F_1, F_2\} = \int d\theta \left[\frac{\delta F_1}{\delta\Phi^{A_k}(\theta)}\frac{\delta F_2}{\delta\Phi^*_{A_k}(\theta)} - \frac{\delta_r F_1}{\delta\Phi^*_{A_k}(\theta)}\frac{\delta_l F_2}{\delta\Phi^{A_k}(\theta)}\right] = \int d\theta (\mathcal{F}_1(\theta), \mathcal{F}_2(\theta))_\theta^{(\Gamma_k, \partial_\theta\Gamma_k)},\\
&(\mathcal{F}_1(\theta), \mathcal{F}_2(\theta))_\theta^{(\Gamma_k, \partial_\theta\Gamma_k)} \equiv \left[(\mathcal{L}_{A_k}\mathcal{F}_1)\,\mathcal{L}^{*A_k}\mathcal{F}_2 - (\mathcal{L}_r^{*A_k}\mathcal{F}_1)\mathcal{L}^l_{A_k}\mathcal{F}_2\right](\theta),
\end{aligned} \tag{38}$$

where the Euler–Lagrange superfield derivative, e.g., with respect to $\Phi^*_{A_k}(\theta)$, for a fixed θ, has the form $\mathcal{L}^{*A_k}(\theta) = \partial/\partial\Phi^*_{A_k}(\theta) - (-1)^{\varepsilon_{A_k}+1}\partial_\theta \cdot \partial/\partial\left(\partial_\theta\Phi^*_{A_k}(\theta)\right)$.

By construction, the functional Z_k is nilpotent:

$$\{Z_k, Z_k\} = \int d\theta (S_{\mathrm{H};k}(\theta), S_{\mathrm{H};k}(\theta))_\theta = 0,\ k = \min, \tag{39}$$

and, due to the absence of the additional time coordinate, is formally related to the BRST charge of a dynamical system with first-class constraints [2]. Indeed, after identifying the fields $(\Gamma_k, \partial_\theta\Gamma_k)(0)$ with the phase-space coordinates of the minimal sector, canonical with respect to the $(\varepsilon_P, \varepsilon)$-even brackets in the framework of the BFV method [2] for first-class constrained systems of $(L+1)$-stage reducibility,

$$\begin{aligned}
&(q^i, p_i) = (\mathcal{A}^i, \partial_\theta\mathcal{A}^*_i)(0),\ (C^{A_s}, \mathcal{P}_{A_s}) = \left((\partial_\theta^r C^{\alpha_{s-1}}, C^{\alpha_s}), (C^*_{\alpha_{s-1}}, \partial_\theta C^*_{\alpha_s})\right)(0),\\
&A_s = (\alpha_{s-1}, \alpha_s),\ s = 0, ..., L,\ (C^{A_{L+1}}, \mathcal{P}_{A_{L+1}}) = \left(\partial_\theta^r C^{\alpha_L}, C^*_{\alpha_L}\right)(0),
\end{aligned} \tag{40}$$

the functional Z_k takes the form

$$Z_k[\Gamma_k] = T_{A_0}(q,p)C^{A_0} + \sum_{s=1}^{L+1}\mathcal{P}_{A_{s-1}}Z^{A_{s-1}}_{A_s}(q)C^{A_s} + O(C^2). \tag{41}$$

With allowance for the gauge algebra structure functions of the original L-stage-reducible restricted LSM in the enhanced eqs. (10), the constraints $T_{A_0}(q,p)$ and the set of $(L+1)$-stage-reducible eigenvectors $Z^{A_{s-1}}_{A_s}(q)$ are defined by the relations (the symbol T below stands for transposition)

$$\begin{aligned}
&T_{A_0}(q,p) = \left(S_{0,i}(q), -p_i\mathcal{R}_{0\alpha_0}^{i}(q)\right),\ Z^{A_{s-1}}_{A_s}(q) = \mathrm{diag}\left(\mathcal{Z}^{\alpha_{s-2}}_{\alpha_{s-1}}, \mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}\right)(q),\\
&s = 1, ..., L,\ \left(Z^{A_L}_{A_{L+1}}\right)^T(q) = \left(\mathcal{Z}^{\alpha_{L-1}}_{\alpha_L}, 0\right)^T(q),
\end{aligned} \tag{42}$$

$$\begin{aligned}
&Z^{A_{s-2}}_{A_{s-1}}Z^{A_{s-1}}_{A_s} = T_{B_0}L^{A_{s-2}B_0}_{A_s}(q,p),\ s = 1, ...,\ L+1,\ Z^{A_{-1}}_{A_0} \equiv T_{A_0},\ L^{A_{s-2}\beta_0}_{A_s} = 0,\\
&L^{A_{s-2}j}_{A_s} = \mathrm{diag}\left(\mathcal{L}^{\alpha_{s-3}j}_{\alpha_{s-1}},\ \mathcal{L}^{\alpha_{s-2}j}_{\alpha_s}\right),\ \mathcal{L}^{\alpha_{-2}j}_{\alpha_0} = \mathcal{L}^{\alpha_{L-1}j}_{\alpha_{L+1}} = 0,\ \mathcal{L}^{\alpha_{-1}j}_{\alpha_1}(q,p) = (-1)^{\varepsilon_j+1}p_i\mathcal{K}^{ji}_{\alpha_1}(q).
\end{aligned} \tag{43}$$

Formulae (38)–(43) generalize, to the case of arbitrary reducible theories, the results of Ref. [15] concerning a dual description (for $\varepsilon_i = \varepsilon_{\alpha_0} = L = 0$) of the quantum action and classical master equation in terms of a nilpotent BRST charge.

Note that the variables $(C^*_{s'\alpha_s}, B^*_{s'\alpha_s}, B^{\alpha_s}_{s'})(\theta)$ are identical, by the rule (40), to the respective ghost momenta $\mathcal{P}_{s'A_s}$, Lagrangian multipliers $\lambda_{s'A_s}$, and their conjugate momenta $\pi^{A_s}_{s'}$ in [2]. Then a comparison of the superfields $\mathcal{C}^{\alpha_s}_{s'}(\theta)$, $s'=0,...,s$, selected from the non-minimal configuration space of an L-stage-reducible LSM, with the coordinates $C^{A_s}_{s'}$ selected from the non-minimal phase space of the corresponding $(L+1)$-stage-reducible dynamical system [2] shows the only possible embedding of $\Pi T(\Pi T^{*}\mathcal{M}_{\rm ext})$ into the phase space of the BFV method. Indeed, for the coordinates $C^{A_{L+1}}_0$, $\mathrm{gh}(C^{A_{L+1}}_0)=-L-2$, there exists no pre-image among $(\mathcal{C}^{\alpha_s}_{s'},\partial_\theta\mathcal{C}^{\alpha_s}_{s'})(0)$, because the ghost number spectrum for the latter variables is bounded from below:

$$\min \mathrm{gh}(\mathcal{C}^{\alpha_s}_{s'},\partial_\theta\mathcal{C}^{\alpha_s}_{s'})=\mathrm{gh}(\mathcal{C}^{\alpha_L}_0)=-L-1.$$

As a consequence, the nilpotent functional $Z_k[\Gamma_k]=-\partial_\theta S_{\mathrm{H};k}(\theta)$, $k=\mathrm{ext}$, is embedded into the total BRST charge constructed by the prescription of Ref. [2].

It should be noted that the systems constructed with respect to the Hamiltonians $S^{\Psi}_{\mathrm{H}}(\Gamma(\theta),\hbar)$ and $S_{\mathrm{H};k}(\theta)$, $k=\min,\,\mathrm{ext}$, are equivalently described by dual fermion functionals $Z_k[\Gamma_k]$ and $Z^{\Psi}[\Gamma]=-\partial_\theta S^{\Psi}_{\mathrm{H}}(\Gamma(\theta),\hbar)$, in terms of even Poisson brackets, for instance,

$$\partial^r_\theta\Gamma^p(\theta)=\left(\Gamma^p(\theta),S^{\Psi}_{\mathrm{H}}(\Gamma(\theta),\hbar)\right)_\theta=-\left\{\Gamma^p(\theta),Z^{\Psi}[\Gamma]\right\}. \tag{44}$$

Thereby, BRST transformations in the Lagrangian formalism with Abelian hypergauges can be encoded by a formal BRST charge, $Z^{\Psi}[\Gamma]$, related to $Z_k[\Gamma_k]$, $k=\mathrm{ext}$, by means of a phase canonical transformation with the $(\varepsilon_P,\varepsilon)$-even phase $F^{\Psi}[\Phi]{=}\partial_\theta\Psi(\Phi(\theta))$,

$$Z^{\Psi}[\Gamma]=e^{\overline{\mathrm{ad}}\,F^{\Psi}}Z_k[\Gamma_k]\,,\;\overline{\mathrm{ad}}\,F^{\Psi}\equiv\left\{F^{\Psi},\cdot\right\}. \tag{45}$$

On the assumption that an additional gauge invariance does not appear in deriving the restricted LSM model from the initial general gauge theory, i.e., $\overline{m}_s\le\overline{M}_s$, and, therefore, $L\le L_g$, cf. footnote 6, the problem of including the restricted LSM gauge algebra into the initial gauge algebra, defined by (1), (6), (7), is solved with the help of a nilpotent functional defined on $\Pi T(\Pi T^{*}\mathcal{M}_k){=}\{(\Gamma^{P_k}_k,\partial_\theta\Gamma^{P_k}_k)(\theta)$, $\Gamma^{P_k}_k(\theta)=\left(\Gamma^{P_{\rm CL}}_{\rm CL},\mathcal{C}^{\mathcal{A}_s},\mathcal{C}^{*}_{\mathcal{A}_s}\right)(\theta)$, $s=0,1,...,L_g$, $k=\mathrm{MIN}\}$, namely,

$$\hat{Z}_k[\Gamma_k]=Z[\mathcal{A}]+\sum_{s=0}^{L_g}\bigg(\int d\theta_{s-1}d\theta_s\mathcal{C}^{*}_{\mathcal{A}_{s-1}}(\theta_{s-1})\hat{\mathcal{Z}}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\theta_{s-1};\theta_s)\mathcal{C}^{\mathcal{A}_s}(\theta_s)(-1)^{\varepsilon_{\mathcal{A}_{s-1}}+s}$$
$$+O(\mathcal{C}^{\mathcal{A}_s})\bigg)=\int d\theta S_{\mathrm{L};k}\left((\Gamma_k,\partial_\theta\Gamma_k)(\theta),\theta\right). \tag{46}$$

Given the superfields $\mathcal{C}^{\mathcal{A}_s}$ introduced as simple ghosts $\mathcal{C}^{\alpha_s}$, alhough used for a description of a general gauge algebra, the representation of a solution to the generating equation $\left\{\hat{Z}_k,\hat{Z}_k\right\}=0$ as an expansion in powers of $\mathcal{C}^{\mathcal{A}_s}$ can be controlled by an additional *generalized ghost number*, gh_g, $\mathrm{gh}_g(\hat{Z}_k)=0$, coinciding with the standard ghost number only in the sector of $(\Phi^{A_{\rm MIN}},\Phi^{*}_{A_{\rm MIN}})(0)$, for $\mathrm{gh}(\mathcal{A}^I,\mathcal{C}^{\mathcal{A}_s})=(0,1+s)$, and having the spectrum

$$\mathrm{gh}_g(\mathcal{A}^I,\mathcal{C}^{\mathcal{A}_s})=(0,1+s),\;\mathrm{gh}_g(\Phi^{*}_{A_{\rm MIN}})=-1-\mathrm{gh}_g(\Phi^{A_{\rm MIN}}),\;\mathrm{gh}_g(\theta,\partial_\theta)=(0,0).$$

Conditions (27), applied to $S_{\mathrm{L};k}(\theta)$ for $(\varepsilon_P)_{\mathcal{A}_s}=(\varepsilon_P)_I=0$, $s=0,...,L_g$, extract from $\hat{Z}_k$ the functional Z_k in (41), so that the $(\varepsilon_P,\varepsilon)$-even θ-density $S_{\mathrm{L};k}(\theta)$ lifts the function $S_{\mathrm{H};k}(\theta)\in C^\infty(\Pi T^{*}\mathcal{M}_{\min})$ to the superalgebra $C^\infty(\Pi T(\Pi T^{*}\mathcal{M}_{\rm MIN})\times\theta)$. In general, $S_{\mathrm{L};k}(\theta)$ does not obey the generalized master equation (34) with antibracket (38) acting on $C^\infty(\Pi T(\Pi T^{*}\mathcal{M}_{\rm MIN})\times\theta)$,

$$(S_{\mathrm{L};k}(\theta),S_{\mathrm{L};k}(\theta))^{(\Gamma_k,\partial_\theta\Gamma_k)}_\theta=\tilde{f}\left((\Gamma_k,\partial_\theta\Gamma_k)(\theta),\theta\right),\;\tilde{f}(\theta)\in\ker\{\partial_\theta\},\;k=\mathrm{MIN}. \tag{47}$$

4.3 Local Quantization

In this subsection, we shall define, in terms of the above actions, a generating functional of Green's functions, $\mathsf{Z}(\theta)$, and an effective action, $\mathsf{\Gamma}(\theta)$, using an invariant description of super(anti)fields on a general antisymplectic manifold. An essential feature in introducing $\mathsf{Z}(\theta)$ and $\mathsf{\Gamma}(\theta)$ is the choice of Darboux coordinates $(\varphi,\varphi^{*})(\theta)$ compatible with the properties of the quantum action.

Leaving aside the realization of a reducible LSM on $\Pi T^{*}\mathcal{M}_{\rm ext}$, we now suppose that the model is described by a quantum action, $W(\theta,\hbar)=W(\theta)$, defined on an arbitrary antisymplectic manifold $\mathcal{N}$

without connection, $\dim\mathcal{N} = \dim\Pi T^*\mathcal{M}_{\rm ext} = (\overline{n} + (n_-, n_+) + \sum_{r=0}^{L}(2r+3)(\overline{m}_r + (m_{r-}, m_{r+}))$, with local coordinates $\Gamma^p(\theta)$ and a density function $\rho(\Gamma(\theta))$. A local antibracket, an invariant volume element, $d\mu(\Gamma(\theta))$, and a nilpotent second-order operator, $\Delta^{\mathcal{N}}(\theta)$, are defined in terms of an $(\varepsilon_P, \varepsilon)$-odd Poisson bivector, $\omega^{pq}(\Gamma(\theta)) = (\Gamma^p(\theta),\ \Gamma^q(\theta))^{\mathcal{N}}_\theta$, namely,

$$d\mu(\Gamma(\theta)) = \rho(\Gamma(\theta))d\Gamma(\theta),\ \Delta^{\mathcal{N}}(\theta) = \frac{1}{2}(-1)^{\varepsilon(\Gamma^q)}\rho^{-1}\omega_{qp}(\theta)\left(\Gamma^p(\theta), \rho\left(\Gamma^q(\theta),\ \ \right)^{\mathcal{N}}_\theta\right)^{\mathcal{N}}_\theta. \tag{48}$$

The definition of a generating functional of Green's functions $\mathsf{Z}\left((\partial_\theta\varphi^*, \varphi^*, \partial_\theta\varphi, \mathcal{I})(\theta)\right) \equiv \mathsf{Z}(\theta)$ as a path integral, for a fixed θ, is possible, within perturbation theory, by introducing on $\mathcal{N}$ the Darboux coordinates, $\Gamma^p(\theta) = (\varphi^a, \varphi^*_a)(\theta)$, in a vicinity of solutions of the equations $\partial W(\theta)/\partial\Gamma^p(\theta) = 0$, so that $\rho = 1$ and $\omega^{pq}(\theta) = \operatorname{antidiag}(-\delta^a_b, \delta^a_b)$. The function

$$\begin{aligned}\mathsf{Z}(\theta) = \int d\mu\left(\tilde{\Gamma}(\theta)\right) d\Lambda(\theta) \exp\Big\{(i/\hbar)\Big[W\left(\tilde{\Gamma}(\theta), \hbar\right) + X\left((\tilde{\varphi}, \tilde{\varphi}^* - \varphi^*, \Lambda, \Lambda^*)(\theta), \hbar\right)|_{\Lambda^*=0} \\ -\ ((\partial_\theta\varphi^*_a)\tilde{\varphi}^a + \tilde{\varphi}^*_a\partial^r_\theta\varphi^a - \mathcal{I}_a\Lambda^a)(\theta)\Big]\Big\}\end{aligned} \tag{49}$$

depends on an extended set of sources,

$$\begin{aligned}&(\partial_\theta\varphi^*_a, \partial^r_\theta\varphi^a, \mathcal{I}_a)(\theta) = (-J_a, \lambda^a, I_{0a} + I_{1a}\theta),\\ &(\vec{\varepsilon}, \mathrm{gh})\partial_\theta\varphi^*_a = (\vec{\varepsilon}, \mathrm{gh})\mathcal{I}_a + ((1,0,1),1) = (\vec{\varepsilon}, -\mathrm{gh})\varphi^a,\end{aligned}$$

to the superfields $(\varphi^a, \varphi^*_a, \Lambda^a)(\theta)$, where $\Lambda^a(\theta) = (\lambda^a_0 + \lambda^a_1\theta)$ are Lagrangian multipliers to independent non-Abelian hypergauges, see [11],

$$\begin{aligned}&G_a(\Gamma(\theta)), a = 1, ..., \ k = n + \sum_{r=0}^{L}(2r+3)m_r,\ k = k_+ + k_-,\\ &\operatorname{rank}\|\partial G_a(\theta)/\partial\Gamma^p(\theta)\|_{\partial W/\partial\Gamma = G = 0} = \overline{l},\ l = l_+ + l_- = k.\end{aligned}$$

The functions $G_a(\Gamma(\theta))$, $(\vec{\varepsilon}, \mathrm{gh})G_a = (\vec{\varepsilon}, \mathrm{gh})\mathcal{I}_a$, determine a boundary condition for the gauge-fixing action, $X(\theta) = X\left((\Gamma, \Lambda, \Lambda^*)(\theta), \hbar\right)$,

$$\partial_r X(\theta)/\partial\Lambda^a(\theta)|_{\Lambda^*=\hbar=0} = G_a(\theta),$$

defined on the direct sum $\mathcal{N}_{\rm tot} = \mathcal{N}\oplus\Pi T^*\mathcal{K}$ of the manifolds $\mathcal{N}$ and $\Pi T^*\mathcal{K} = \{(\Lambda^a, \Lambda^*_a)(\theta)\}$. Hypergauges in involution, $(G_a(\theta), G_b(\theta))^{\mathcal{N}}_\theta = G_c(\theta)U^c_{ab}(\Gamma(\theta))$, obey different types of unimodularity relations [11], depending on a set of equations for which $X(\theta)$ may be a solution, independently from $W(\theta)$, in terms of the antibracket $(\,\cdot\,,\,\cdot\,)_\theta = (\,\cdot\,,\,\cdot\,)^{\mathcal{N}}_\theta + (\,\cdot\,,\,\cdot\,)^{\mathcal{K}}_\theta$ and the operator $\Delta(\theta) = (\Delta^{\mathcal{N}} + \Delta^{\mathcal{K}})(\theta)$, trivially lifted from $\mathcal{N}$ to $\mathcal{N}_{\rm tot}$,

$$1)\ (E(\theta), E(\theta))_\theta = 0,\ \Delta(\theta)E(\theta) = 0;\ 2)\ \Delta(\theta)\exp\left[\frac{i}{\hbar}E(\theta)\right] = 0,\ E \in \{W, X\}. \tag{50}$$

The functions $G_a(\theta)$, assumed to be solvable with respect to $\varphi^*_a(\theta)$, determine a Lagrangian surface, $\Lambda_g = \{(\varphi^*, \Lambda)(\theta)\} \subset \mathcal{N}_{\rm tot}$, on which the restriction $X(\theta)|_{\Lambda_g}$ is non-degenerate. Given this, integration over $(\tilde{\varphi}^*, \Lambda)(\theta)$ in eq. (49) determines a function, for $\partial_\theta\varphi^a = \mathcal{I}_a = 0$, whose restriction to the Lagrangian surface $\Lambda = \{\varphi(\theta)\} \subset \mathcal{N}$ is also non-degenerate.

In [6, 7], a peculiarity of the generating functional of Green's functions $Z[\Phi^*]$ and of the vacuum functional Z is the dependence of the gauge fermion $\Psi[\Phi]$ and quantum action $S[\Phi, \Phi^*]$ on the components λ^A of superfields $\Phi^A(\theta)$ in the multiplet $(\Phi^A, \Phi^*_A)(\theta) = (\phi^A + \lambda^A\theta, \phi^*_A - \theta J_A)$, where the variables $(\phi^A, \phi^*_A, \lambda^A, J_A)$ constitute the complete set of variables of the BV method [1]. Another feature of [6, 7] is that the structure of superantifields $\Phi^*_A(\theta)$ and the explicit form of $Z[\Phi^*]$ allow one to introduce in a non-contradictory manner, although violating the superfield content of the variables,[7] an effective action

[7] By *violation* of the superfield content, we understand the fact that the derivative of $Z[\Phi^*]$, which defines the effective action in a Legendre transformation, is calculated with respect to only one superfield component, namely, the θ-component of $\Phi^*_A(\theta)$, so that the resulting effective action depends only on ϕ^A and ϕ^*_A, which can be formally expressed as $P_0(\theta)(\Phi^A, \Phi^*_A)(\theta) = (\phi^A, \phi^*_A)$.

Γ, by using a Legendre transformation of $\ln Z[\Phi^*]$ with respect to $P_1(\theta)\Phi^*_A(\theta)$,[8]

$$\Gamma[P_0(\Phi,\Phi^*)] = \frac{\hbar}{i}\ln Z[\Phi^*] + \partial_\theta\left\{[P_1(\theta)\Phi^*_A(\theta)]\,\Phi^A(\theta)\right\},\ \Phi^A(\theta) = -\frac{\hbar}{i}\frac{\delta \ln Z[\Phi^*]}{\delta\left(P_1(\theta)\Phi^*_A(\theta)\right)}, \tag{51}$$

with the standard Ward identity $(\Gamma,\Gamma)=0$ in terms of a superantibracket[6].

In view of the properties of $(W,X)(\theta)$, one can introduce an effective action $\mathsf{\Gamma}(\theta)\equiv\mathsf{\Gamma}(\varphi,\varphi^*,\partial^r_\theta\varphi,\mathcal{I})(\theta)$ defined, in the usual manner, by means of a Legendre transformation of $\ln\mathsf{Z}(\theta)$ with respect to $\partial_\theta\varphi^*_a(\theta)$,

$$\mathsf{\Gamma}(\theta) = \frac{\hbar}{i}\ln\mathsf{Z}(\theta) + \left((\partial_\theta\varphi^*_a)\varphi^a\right)(\theta),\ \varphi^a(\theta) = -\frac{\hbar}{i}\frac{\partial_l \ln\mathsf{Z}(\theta)}{\partial(\partial_\theta\varphi^*_a(\theta))}. \tag{52}$$

The analysis of the properties of $(\mathsf{Z},\mathsf{\Gamma})(\theta)$ is based on the following θ-nonintegrable Hamiltonian-like system, which contains an arbitrary $(\varepsilon_P,\varepsilon)$-even $C^\infty(\mathcal{N}_{\rm tot})$-function, $R(\theta)=R\left((\tilde{\Gamma},\Lambda,\Lambda^*)(\theta),\hbar\right)$, with a vanishing ghost number:

$$\begin{cases} \partial^r_\theta\tilde{\Gamma}^p(\theta) = -i\hbar\, T^{-1}(\theta)\left(\tilde{\Gamma}^p(\theta), T(\theta)R(\theta)\right)_\theta\Big|_{\Lambda^*=0}, \\ \partial^r_\theta\Lambda^a(\theta) = -2i\hbar\, T^{-1}(\theta)(\Lambda^a(\theta),T(\theta)R(\theta))_\theta|_{\Lambda^*=0}, \\ \partial^r_\theta\left(\varphi^*_a,\Lambda^*_a\right)(\theta) = 0, \end{cases} \tag{53}$$

where the function $T\left((\tilde{\Gamma},\Lambda,\Lambda^*)(\theta),\hbar\right)\equiv T(\theta)$ has the form $T(\theta)=\exp\left[(i/\hbar)\,(W-X)(\theta)\right]$. Let us enumerate the properties of $(\mathsf{Z},\mathsf{\Gamma})(\theta)$.

1. The integrand in (49) is invariant, for $\partial_\theta\varphi^*=\partial_\theta\varphi=\mathcal{I}=0$, with respect to the *superfield BRST transformations*

$$\tilde{\Gamma}_{\rm tot}(\theta) = (\tilde{\Gamma},\Lambda,\Lambda^*)(\theta) \to \left(\tilde{\Gamma}_{\rm tot}+\delta_\mu\tilde{\Gamma}_{\rm tot}\right)(\theta),\ \delta_\mu\tilde{\Gamma}_{\rm tot}(\theta) = \left(\partial^r_\theta\tilde{\Gamma}_{\rm tot}\right)\Big|_{\check{\Gamma}_{\rm tot}}\mu, \tag{54}$$

having the form of a θ-shift by a constant parameter μ along an arbitrary solution $\check{\Gamma}_{\rm tot}(\theta)$ of the system (53), or, equivalently, along a vector field determined by the r.h.s. of (53), for $R(\theta)=1$. Here, the arguments of $(W,X)(\theta)$ are the same as in definition (49). The above statement can be verified with the help of the identities

$$\partial_r X(\theta)/\partial F(\theta)|_{\Lambda^*=0} = \partial_r(X(\theta)|_{\Lambda^*=0})/\partial F(\theta),\ F\in\{\Gamma^p,\Lambda^a\}.$$

Notice that the system (53), for $R(\theta)=$ const, admits the integral $(W+X)(\theta)$ in case W and X obey the first system in (50).

2. The vacuum function $\mathsf{Z}_X(\theta)\equiv\mathsf{Z}(0,\varphi^*,0,0)(\theta)$ is gauge-independent, namely, it does not change when $X(\theta)$ is replaced by an $(X+\Delta X)(\theta)$ which obeys the same system in (50) that holds for $X(\theta)$ and conforms to nondegeneracy on the surface Λ_g. Indeed, this hypothesis implies that the variation $\Delta X(\theta)$ obeys a system of linearized equations with a nilpotent operator $Q_j(X)$, $j=1,2$,

$$Q_j(X)\Delta X(\theta)=0,\ \delta_{j1}\Delta(\theta)\Delta X(\theta)=0;\ Q_j(X)=\mathrm{ad}\,X(\theta)-\delta_{j2}(i\hbar\Delta(\theta)), \tag{55}$$

where j is identical to the number that labels that system in eqs. (50) for which $X(\theta)$ is a solution. Using the fact that solutions $X(\theta)$ of every system in (50) are proper, one can prove, by analogy with the theorems of Ref. [22], that the cohomologies of the operator $Q_j(X)$ on the functions $f(\Gamma_{\rm tot}(\theta))\in C^\infty(\mathcal{N}_{\rm tot})$ vanishing for $\Gamma_{\rm tot}(\theta)=0$ are trivial. Hence, the general solution of eq. (55) has the form

$$\Delta X(\theta)=Q_j(X)\Delta Y(\theta),\ \left(\vec{\varepsilon},\mathrm{gh},\frac{\partial}{\partial\theta}\right)\Delta Y(\theta)=((1,0,1),-1,0),\ \Delta Y(\theta)|_{\Gamma_{\rm tot}=0}=0, \tag{56}$$

with a certain $\Delta Y(\theta)$. Now, making in $\mathsf{Z}_{X+\Delta X}(\theta)$ a change of variables induced by a θ-shift by a constant μ, corresponding to the system (53), and choosing

$$2R(\theta)\mu = \Delta Y(\theta),$$

[8]Here, $P_1(\theta)$ and the operator $\delta/\delta\left(P_1(\theta)\Phi^*_A(\theta)\right)$ in (51) are, respectively, the projector from the system $\{P_a(\theta)=\delta_{a0}(1-\theta\partial_\theta)+\delta_{a1}\theta\partial_\theta, a=0,1\}$ on the supermanifold with coordinates $(\Phi^A,\Phi^*_A)(\theta)$ and the superfield variational derivative with respect to $P_1(\theta)\Phi^*_A(\theta)$.

we find that $\mathsf{Z}_{X+\Delta X}(\theta) = \mathsf{Z}_X(\theta)$ and conclude that the S-matrix is gauge-independent in view of the equivalence theorem [23][9].

The above proof shows that the system (53) encodes, due to (54), the BRST transformations for $R(\theta) = \mathrm{const}$, and, at the same time, the continuous anticanonical transformations in an infinitesimal form, with the scalar fermionic generating function $R(\theta)\mu$, where $R(\theta)$ is arbitrary and μ is constant.

Equivalently, following the ideas of Subsection 4.2, the above characteristics of the generating functional of Green's functions can be derived from a Hamiltonian-like system presented in terms of a superfield even Poisson bracket in general coordinates (see footnote 9),

$$\left\{ \begin{array}{l} \partial^r_\theta \tilde{\Gamma}^p(\theta) = -\left\{\tilde{\Gamma}^p(\theta), Z^W[\tilde{\Gamma}] - (Z^X + i\hbar Z^R)[\tilde{\Gamma}_{\mathrm{tot}}]\right\}\Big|_{\Lambda^*=0}, \\ \partial^r_\theta \Lambda^a(\theta) = -2\left\{\Lambda^a(\theta), Z^W[\tilde{\Gamma}] - (Z^X + i\hbar Z^R)[\tilde{\Gamma}_{\mathrm{tot}}]\right\}\Big|_{\Lambda^*=0}, \\ \partial^r_\theta\left(\varphi^*_a, \Lambda^*_a\right)(\theta) = 0 \end{array} \right. \tag{57}$$

with a linear combination of fermionic functionals, corresponding to the above actions, and a bosonic function by the rule

$$Z^E[\Gamma_{\mathrm{tot}}] = -\partial_\theta E(\Gamma_{\mathrm{tot}}(\theta), \hbar),\ E \in \{W, X, R\}. \tag{58}$$

If the actions $(W, X)(\theta)$ obey the first system in (50), then the functionals Z^W, Z^X, formally playing the role of the usual and *gauge-fixing* BRST charges, are nilpotent with respect to the even Poisson bracket $\{\cdot\,,\cdot\} = \{\cdot\,,\cdot\}^{\Pi T\mathcal{N}} + \{\cdot\,,\cdot\}^{\Pi T\mathcal{K}}$. Here, for instance, the first bracket in the sum is defined on arbitrary functionals over $\Pi T\mathcal{N} \times \{\theta\}$, via a θ-local extension of the odd bracket $(\cdot\,,\cdot)^{\Pi T\mathcal{N}}_\theta$ in (38), as follows:

$$\begin{array}{l} \{F_1, F_2\}^{\Pi T\mathcal{N}} \equiv \displaystyle\int d\theta \frac{\delta_r F_1}{\delta\Gamma^p(\theta)}\omega^{pq}(\Gamma(\theta))\frac{\delta_l F_2}{\delta\Gamma^q(\theta)} = \partial_\theta(\mathcal{F}_1(\theta), \mathcal{F}_2(\theta))^{\Pi T\mathcal{N}}_\theta, \\ (\mathcal{F}_1(\theta), \mathcal{F}_2(\theta))^{\Pi T\mathcal{N}}_\theta \equiv ((\mathcal{L}^r_p\mathcal{F}_1)\omega^{pq}(\Gamma(\theta))\mathcal{L}^l_q\mathcal{F}_2)(\theta),\ F_t[\Gamma] = \partial_\theta\mathcal{F}_t((\Gamma, \partial_\theta\Gamma)(\theta), \theta), \end{array} \tag{59}$$

where $\mathcal{L}^l_q(\theta)$ is the left-hand Euler–Lagrange superfield derivative with respect to $\Gamma^q(\theta)$.[10]

Therefore, as in the case of the HS in (44), we arrive at the interpretation of BRST transformations, for a gauge theory with non-Abelian hypergauges in Lagrangian formalism, in terms of the formal "BRST charges" Z^W, Z^X, as well as in terms of the functional Z^R and the even Poisson bracket[11]. The system (57) encodes the BRST transformations, for $Z^R = 0$, and, at the same time, the BRST and continuous canonical transformations with the bosonic generating functional $Z^R\mu$, for an arbitrary Z^R and a constant μ.

3. The functions $(\mathsf{Z}, \mathsf{\Gamma})(\theta)$ obey the Ward identities

$$\begin{array}{l} \displaystyle\left[\left\{\partial_\theta\varphi^*_a(\theta) - \left(\frac{\partial W}{\partial\tilde{\varphi}^a(\theta)}\right)\left(i\hbar\frac{\partial_l}{\partial(\partial_\theta\varphi^*)}, i\hbar\frac{\partial_r}{\partial(\partial^r_\theta\varphi)}\right)\right\}\frac{\partial_l}{\partial\varphi^*_a(\theta)}\right. \\ \displaystyle\left.+\frac{i}{\hbar}\mathcal{I}_a(\theta)\frac{\partial_l}{\partial\Lambda^*_a(\theta)}\,X\left(i\hbar\frac{\partial_l}{\partial(\partial_\theta\varphi^*)}, i\hbar\frac{\partial_r}{\partial(\partial^r_\theta\varphi)} - \varphi^*, \frac{\hbar}{i}\frac{\partial_l}{\partial\mathcal{I}}, \Lambda^*\right)\Big|_{\Lambda^*_a=0}\right]\mathsf{Z}(\theta) = 0, \end{array} \tag{60}$$

$$\begin{array}{l} \displaystyle\mathcal{I}_a(\theta)\frac{\partial_l}{\partial\Lambda^*_a(\theta)}\,X\left(\varphi^b + i\hbar(\mathsf{\Gamma}''^{-1})^{bc}\frac{\partial_l}{\partial\varphi^c}, i\hbar\frac{\partial_r}{\partial(\partial^r_\theta\varphi)} - \frac{\partial_r\mathsf{\Gamma}}{\partial(\partial^r_\theta\varphi)} - \varphi^*, \frac{\partial_l\mathsf{\Gamma}}{\partial\mathcal{I}} + \frac{\hbar}{i}\frac{\partial_l}{\partial\mathcal{I}}, \Lambda^*\right)\Big|_{\Lambda^*_a=0} \\ \displaystyle-\left[\left(\frac{\partial W}{\partial\tilde{\varphi}^a(\theta)}\right)\left(\varphi^b + i\hbar(\mathsf{\Gamma}''^{-1})^{bc}\frac{\partial_l}{\partial\varphi^c}, i\hbar\frac{\partial_r}{\partial(\partial^r_\theta\varphi)} - \frac{\partial_r\mathsf{\Gamma}}{\partial(\partial^r_\theta\varphi)}\right)\right]\frac{\partial_l\mathsf{\Gamma}(\theta)}{\partial\varphi^*_a(\theta)} + \frac{1}{2}\left(\mathsf{\Gamma}(\theta), \mathsf{\Gamma}(\theta)\right)^{(\mathsf{\Gamma})}_\theta = 0, \end{array} \tag{61}$$

with the notation $\mathsf{\Gamma}''_{ab}(\theta) \equiv \frac{\partial_l}{\partial\varphi^a(\theta)}\frac{\partial_r}{\partial\varphi^b(\theta)}\mathsf{\Gamma}(\theta)$, $\mathsf{\Gamma}''_{ac}(\theta)(\mathsf{\Gamma}''^{-1})^{cb}(\theta) = \delta_a{}^b$. Namely, in the symmetric form of the above identities we have extended the standard set of sources $\partial_\theta\varphi^*_a(\theta)$ used in the definition of the generating functional of Green's functions in Abelian hypergauges.

The technique used in deriving the above identities is analogous to the corresponding procedures of Refs. [24, 25], applied, in the BV [1] and Batalin–Lavrov–Tyutin [22] methods, respectively, to the problem of gauge dependence in theories with composite fields. Thus, identities (60) and (61) follow from

[9]Properties 1, 2 of $\mathsf{Z}_X(\theta)|_{\varphi^*=0}$ are valid for arbitrary $\rho(\theta)$, $\Gamma^p(\theta)$ on the manifold $\mathcal{N}$.

[10]The antibracket $(\cdot\,,\cdot)^{\Pi T\mathcal{N}}_\theta$ coinciding, for $\mathcal{N} = \Pi T\mathcal{M}_k$, with $(\cdot\,,\cdot)^{(\Gamma_k, \partial_\theta\Gamma_k)}_\theta$, $k = \mathrm{ext}$, in (38) lifts the operator $\Delta^{\mathcal{N}}$ in (48) to the nilpotent operator $\Delta^{\Pi T\mathcal{N}}$ acting on $C^\infty(\Pi T\mathcal{N} \times \{\theta\})$, defined exactly as $\Delta^{\mathcal{N}}(\theta)$, although in terms of the antibracket (59).

[11]The construction of the latter bracket is different from that of [5], where an odd superfield Poisson bracket was derived from a (t, θ)-local even bracket; however, it is similar to the construction of Ref. [15]: see eqs. (27).

the corresponding system in (50) for $(W, X)(\theta)$. For instance, making the functional averaging of the second system in (50), for $X(\theta)$,

$$\int d\Lambda(\theta) d\mu\left(\check{\Gamma}(\theta)\right) \exp\left[\frac{i}{\hbar}\left(W - (\partial_\theta \varphi^*_a)\tilde{\varphi}^a - \tilde{\varphi}^*_a \partial^r_\theta \varphi^a + \mathcal{I}_a \Lambda^a\right)(\theta)\right]$$
$$\times \left\{\Delta(\theta) \exp\left[\frac{i}{\hbar} X\left((\tilde{\varphi}, \tilde{\varphi}^* - \varphi^*, \Lambda, \Lambda^*)(\theta), \hbar\right)\right]\right\}_{\Lambda^*=0} = 0, \tag{62}$$

and integrating by parts in (62), with allowance for $(\partial/\partial\tilde{\varphi}^* + \partial/\partial\varphi^*)X(\theta) = 0$, we obtain identity (60). Identities (60) and (61) take the standard form with $\partial_\theta \varphi^a = \mathcal{I}_a(\theta) = \theta = 0$, which becomes more involved by the quantities $(\partial_\theta W(\theta)/\partial \tilde{\varphi}^a(\theta))$, although in the case of non-Abelian hypergauges.

In the special case of Abelian hypergauges, $G_A\left((\Phi, \Phi^*)(\theta)\right) = \Phi^*_A(\theta) - \partial\Psi(\Phi(\theta))/\partial\Phi^A(\theta) = 0$, related to the change of variables (35), for $(\varphi, \varphi^*, W) = (\Phi, \Phi^*, S_{\mathrm{H;ext}})$, $\partial_\theta \Phi^A = \mathcal{I}_A = 0$ (locally, $\mathcal{N} = \Pi T^* \mathcal{M}_{\mathrm{ext}}$), the object $\mathsf{Z}(\partial_\theta \Phi^*, \Phi^*)(\theta)$ takes the form

$$\mathsf{Z}\left(\partial_\theta \Phi^*, \Phi^*\right)(\theta) = \int d\Phi(\theta) \exp\left\{\frac{i}{\hbar}\left[S^\Psi_{\mathrm{H}}\left(\Gamma(\theta), \hbar\right) - \left((\partial_\theta \Phi^*_A)\Phi^A\right)(\theta)\right]\right\}. \tag{63}$$

A θ-local BRST transformation for $\mathsf{Z}\left(\partial_\theta \Phi^*, \Phi^*\right)(\theta)$ is given for an HS defined on $\Pi T^* \mathcal{M}_{\mathrm{ext}}$, with the Hamiltonian $S^\Psi_{\mathrm{H}}(\theta, \hbar)$ and a solution $\check{\Gamma}(\theta)$, by the change of variables

$$\Gamma^p(\theta) \to \Gamma^{(1)p}(\theta) = \exp\left[\mu s^{l(\Psi)}(\theta)\right] \Gamma^p(\theta),\ s^{l(\Psi)}(\theta) \equiv \frac{\partial}{\partial\theta} - \mathrm{ad} S^\Psi_{\mathrm{H}}(\theta, \hbar). \tag{64}$$

Transformation (64) with a constant μ is anticanonical, with $\mathrm{Ber}\|\frac{\partial\Gamma^{(1)}(\theta)}{\partial\Gamma(\theta)}\| = \mathrm{Ber}\|\frac{\partial\Phi^{(1)}(\theta)}{\partial\Phi(\theta)}\| = 1$, if $S^\Psi_{\mathrm{H}}(\theta, \hbar)$ is subject to the first system in (50).

The obvious permutation rule of the functional integral, $\varepsilon(d\Phi(\theta)) = 0$,

$$\partial_\theta \int d\Phi(\theta) \mathcal{F}\left((\Phi, \Phi^*)(\theta), \theta\right) = \int d\Phi(\theta) \left[\frac{\partial}{\partial\theta} + (\partial_\theta V)(\theta)\right] \mathcal{F}(\theta),\ \partial_\theta V(\theta) = \partial_\theta \Phi^*_A(\theta) \frac{\partial}{\partial \Phi^*_A(\theta)},$$

yields, for $i\hbar \partial^r_\theta \ln \mathsf{Z}(\theta) = (\partial_\theta \Phi^*_A \partial^r_\theta \Phi^A)(\theta) - \partial^r_\theta \mathsf{\Gamma}(\theta)$, the following relations:

$$\partial_\theta \mathsf{Z}(\theta)|_{\check{\Gamma}(\theta)} = (\partial_\theta V)(\theta) \mathsf{Z}(\theta) = 0,\ \partial^r_\theta \mathsf{\Gamma}(\theta)|_{\check{\Gamma}(\theta)} = \left(\mathsf{\Gamma}(\Gamma(\theta)), \mathsf{\Gamma}(\Gamma(\theta))\right)_\theta = 0. \tag{65}$$

When deriving eqs. (65), we have taken into account the fact that the functional averaging of the HS with respect to $\mathsf{Z}(\theta)$ and $\mathsf{\Gamma}(\theta)$ has the form

$$\langle \partial^r_\theta \Gamma^p \rangle|_{\mathsf{Z}} = \left(\frac{\hbar}{i} \mathsf{Z}^{-1} \frac{\partial \mathsf{Z}(\theta)}{\partial \Phi^*_A(\theta)}, -\partial_\theta \Phi^*_A(\theta)\right),\ \langle \partial^r_\theta \Gamma^p \rangle = \left(\langle \Gamma^p(\theta) \rangle, \mathsf{\Gamma}(\langle \Gamma(\theta) \rangle)\right)_\theta = \partial^r_\theta \langle \Gamma^p \rangle, \tag{66}$$

without the sign of average in (65) for $\check{\Gamma}(\theta)$ and $\Gamma^p(\theta)$. Expressions (65) relate the explicit form of the Ward identities in a theory with Abelian hypergauges to the invariance of the generating functional of Green's functions with respect to the superfield BRST transformations.

5 Connection between Lagrangian Quantizations

5.1 Component Formulation and its Relation to Batalin–Vilkovisky, and Batalin–Tyutin Methods

In this section, on the basis of a component form of the local superfield quantization, we shall establish its connection with the BV method and the first-level formalism [11].

The relation of the objects and quantities of θ-local quantization in the Lagrangian and Hamiltonian formulations of an LSM with the conventional description of a gauge field theory is established through a component representation of the variables $\Gamma^{P_{\mathrm{MIN}}}_{\mathrm{MIN}}$, $\Gamma^{p_k}_k$, Λ^a, $\mathcal{I}_a$, $\Gamma^{p_k}_k(\theta) = \Gamma^{p_k}_{0k} + \Gamma^{p_k}_{1k}\theta$, $k = \mathrm{tot}$, under the restriction $\theta = 0$, for instance, $(\mathcal{M}, \mathcal{N}_k, \Lambda^a, \mathcal{I}_a) \to (\widetilde{\mathcal{M}}, \mathcal{N}_k|_{\theta=0} = \{\Gamma^{p_k}_{0k}\}, \lambda^a_0, I_{0a})$. The extraction of a standard field model from a classical formulation of a general gauge theory is realized, in addition to $\theta = 0$, by various kinds of eliminating the functions $\partial_\theta \mathcal{A}^I(\theta)$, $\mathcal{A}^*_I(\theta)$ and those of the superfields

$\mathcal{A}^I(\theta)$ which contain functions with an incorrect spin-statistics relation, $\varepsilon_P(\mathcal{A}^I) \neq 0$. One possibile way of such elimination is provided by the conditions $\mathrm{gh}(\mathcal{A}^I) = -1 - \mathrm{gh}(\mathcal{A}^*_I) = 0$, $(\varepsilon_P)_I = 0$, and $(\mathrm{gh}, \partial/\partial\theta)\, S_{\mathrm{L(H)}}(\theta) = (0,0)$, mentioned in Subsection 4.1.

Another possibility is related to the superfield BRST transformations for theories of Yang–Mills type [10, 26, 27], for which a Lagrangian classical action $S_{\mathrm{LYM}}(\theta) = S_{\mathrm{L}}\left(\mathcal{A}, \mathcal{D}_\theta\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{D}_\theta\tilde{\mathcal{A}}\right)(\theta)$ is defined in terms of generalized Yang–Mills superfields, $\mathcal{A}^{Bu}(z)$, $\mathcal{A}^{Bu} = (\mathcal{A}^{\mu u}, \mathcal{C}^u)$, $u = 1, ..., r$, and matter superfields, $\tilde{\mathcal{A}}(z) = (\Psi^\delta, \overline{\Psi}^\sigma, \varphi^f, \varphi^{+g})(z)$, with spinor Ψ^δ, $\overline{\Psi}^\sigma$, δ, $\epsilon = 1, ..., k_1$, and spinless φ^f, φ^{+g}, f, $g = 1, ..., k_2$. The superfields $\mathcal{A}^{Bu}(z)$ and $\tilde{\mathcal{A}}(z)$ are defined on the superspace $\mathcal{M} = \mathbb{R}^{1,3} \times \tilde{P} = \{z^B = (x^\mu, \theta)\}$ and take values, respectively, in the adjoint and vector representation spaces of an r-parametric Lie group. The action $S_{\mathrm{LYM}}(\theta)$ can be written as

$$S_{\mathrm{LYM}}(\theta) = \int d^4x \left[\frac{1}{4}\mathcal{G}_{BC}{}^u\mathcal{G}^{CBu}(-1)^{\varepsilon_B} - i\overline{\Psi}^\sigma\gamma^B\nabla_{B}{}^\sigma_\delta\Psi^\delta - \overline{\nabla}_B{}^h_g\varphi^{+g}\nabla^{Bh}{}_f\varphi^f + M(\tilde{\mathcal{A}})\right](z), \tag{67}$$

with an $\tilde{\mathcal{A}}(z)$-local gauge-invariant polynomial $M(\tilde{\mathcal{A}})$, containing no derivatives over z^B. In expression (67), we have introduced the superfield strength $\mathcal{G}_{BC}{}^u = i[\mathcal{D}_B, \mathcal{D}_C]^u = \partial_B\mathcal{A}^u_C - (-1)^{\varepsilon_B\varepsilon_C}\partial_C\mathcal{A}^u_B + f^{uvw}\mathcal{A}^v_B\mathcal{A}^w_C$, $\partial_B = (\partial_\mu, \partial_\theta)$ and the following covariant derivatives, expressed through the matrix elements of the Hermitian generators $\Gamma^u = \mathrm{diag}\left(T^u, \overline{T}^u, \tau^u, \overline{\tau}^u\right)$ of the corresponding Lie algebra:

$$\left(\mathcal{D}^{uv}_B, \nabla_B{}^\sigma_\delta, \nabla_B{}^e_f, \overline{\nabla}_B{}^g_h\right) = \partial_B\left(\delta^{uv}, \delta^\sigma_\delta, \delta^e_f, \delta^g_h\right) + \left(f^{uwv}, -i(T^w)^\sigma_\delta, -i(\tau^w)^e_f, -i(\overline{\tau}^w)^g_h\right)\mathcal{A}^w_B, \tag{68}$$

where the coupling constant is absorbed into the completely antisymmetric structure coefficients f^{uvw}. We have also used a generalization of Dirac's matrices, $\gamma^B = (\gamma^\mu, \gamma^\theta)$, $\gamma^\theta = (\gamma^\theta)^+ = \xi\mathbb{I}_4$, with a Grassmann scalar ξ, $(\vec{\varepsilon}, \mathrm{gh})\xi = ((1,0,1), -1)$. The $\vec{\varepsilon}$-grading and ghost number are nonvanishing for the superfields $(\Psi, \overline{\Psi}, \mathcal{C}^u)$, namely, $\vec{\varepsilon}(\Psi, \overline{\Psi}) = (0,1,1)$, $\vec{\varepsilon}(\mathcal{C}^u) = (1,0,1)$, $\mathrm{gh}(\mathcal{C}^u) = 1$. The functional $Z[\mathcal{A}, \tilde{\mathcal{A}}] = \partial_\theta S_{\mathrm{LYM}}(\theta)$ is invariant under the infinitesimal general gauge transformations

$$\delta_g\mathcal{A}^I(\theta) = \delta_g(\mathcal{A}^{Bu}; \tilde{\mathcal{A}})(z) = -\int d^5z_0\left(\mathcal{D}^{Buv}(z)(-1)^{\varepsilon_B}; i\Gamma^v\tilde{\mathcal{A}}(z)(-1)^{\varepsilon(\tilde{\mathcal{A}})}\right)\delta(z - z_0)\xi^v(z_0), \tag{69}$$

with arbitrary bosonic ($\vec{\varepsilon}_{\mathcal{A}_0} = \vec{0}$) functions $\xi^v(z_0)$ on $\mathcal{M}$, and with functionally-independent generators $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta, \theta_0) \equiv \hat{\mathcal{R}}^I_{\mathcal{A}_0}(\mathcal{A}(\theta), \theta, \theta_0)$. The condensed indices I, $\mathcal{A}_0$ of the theory in question, $(I; \mathcal{A}_0) = ((B, u, \delta, \epsilon, f, h, x); (v, x_0))$, conform to the relations, $\overline{N} > \overline{n}$, $\overline{M} = \overline{m}$, $(\overline{m}, \overline{M}) = (\overline{m}_0, \overline{M}_0)$, provided that

$$\overline{N} = (4r + 2k_2, r + 8k_1),\ \overline{M} = (r, 0),\ \overline{n} = \overline{N} - (0, r),$$

which hold for a reduced theory with the action $S_{\mathrm{YM}}(\theta) = -S_{\mathrm{LYM}}\left(\mathcal{A}, 0, \tilde{\mathcal{A}}, 0\right)(\theta)$ on $\mathcal{M}_{\mathrm{cl}} = \{\mathcal{A}^{\mu u}, \tilde{\mathcal{A}}\}(z)$[12], in view of special *horizontality conditions* for the strength $\mathcal{G}_{BC}{}^u$ and certain subsidiary conditions for the matter superfields $\tilde{\mathcal{A}}(z)$ in [10, 26],

$$\mathcal{G}_{BC}{}^u(z) = \mathcal{G}_{\mu\nu}{}^u(z),\ \left(\nabla_\theta{}^\eta_\delta\Psi^\delta, \overline{\nabla}_\theta{}^\varrho_\sigma\overline{\Psi}^\sigma, \nabla_\theta{}^f_e\varphi^e, \overline{\nabla}_\theta{}^h_g\overline{\varphi}^g\right)(z) = (0,0,0,0). \tag{70}$$

To extract a standard component model defined on $\mathcal{M}_{\mathrm{cl}}|_{\theta=0}$ from a Hamiltonian LSM, it is sufficient to eliminate, for $\theta = 0$, the antifields $\mathcal{A}^*_I(\theta)$ of a Yang–Mills type theory, by analogy with the prescription (70), i.e., by taking into account the relation between $\mathcal{A}^*_I(\theta)$ and $\partial_\theta\mathcal{A}^I(\theta)$, see Section 3.

For the restricted LSM used in the Feynman rules of Section 4, the reduction to the model in the framework of the multilevel formalism of Ref. [11] is provided by the conditions

$$\theta = 0,\ \partial_\theta\varphi^*_a = \partial_\theta\varphi^a = \varphi^*_a = \mathcal{I}_a = 0. \tag{71}$$

In this case, the first-level functional integral $Z^{(1)}$ and its symmetry transformations [11], with the notation λ^a_0 instead of π^a for Lagrangian multipliers in [11],

$$Z^{(1)} = \int d\lambda_0 d\Gamma_0 M(\Gamma_0)\exp\left\{\frac{i}{\hbar}\left(W(\Gamma_0) + G_a(\Gamma_0)\lambda^a_0\right)\right\},$$

$$\begin{cases} \delta\Gamma^p_0 = (\Gamma^p_0, -W + G_a\lambda^a_0)\mu, \\ \delta\lambda^a_0 = \left(-U^a_{cb}\lambda^b_0\lambda^c_0(-1)^{\varepsilon_c} + 2i\hbar V^a_b\lambda^b_0 + 2(i\hbar)^2\tilde{G}^a\right)\mu, \end{cases}$$

[12]For $\theta = 0$, the functional $S_{\mathrm{YM}}(0) = S_{0\mathrm{YM}}$ coincides with the corresponding classical action of [29].

under the identification $(\rho, \omega^{pq})(\Gamma_0) = (M, E^{pq})(\Gamma_0)$, implying the coincidence of $(\cdot\,,\cdot)_\theta|_{\theta=0}$ and $\Delta(0)$ with their counterparts of [11], coincide with $Z_X(0)|_{\varphi_0^*=0}$ and with the BRST transformations $\delta_\mu\Gamma_{0\mathrm{tot}}$ (having the opposite signs) generated by the system (53) for $R(\theta) = 1$. This coincidence is guaranteed by the choice of $X(\theta)$ in the form

$$X(\theta) = \left\{G_a(\Gamma)\Lambda^a - \Lambda_a^*\left[\frac{1}{2}U^a_{cb}(\Gamma)\Lambda^b\Lambda^c(-1)^{\varepsilon_c} - i\hbar V^a_b(\Gamma)\Lambda^b - (i\hbar)^2\tilde{G}^a(\Gamma)\right]\right\}(\theta) + o(\Lambda^*), \qquad (72)$$

where $(V^a_b, \tilde{G}^a)(\theta)$, together with $(U^a_{cb}, G_a)(\theta)$, define the unimodularity relations [11]. The relation of the θ-local quantization to the generating functional of Green's function $Z[J,\phi^*]$ of the BV method [1] is evident after identifying $\mathsf{Z}\left(\partial_\theta\Phi^*, \Phi^*\right)(0) = Z[J,\phi^*]$ in (63), where the action $S^\Psi_{\mathrm{H}}(\Gamma_0,\hbar)$ of (35) obeys eq. (36).

The following aspect of the restriction $\theta = 0$ consists in the representation of an arbitrary function $\mathcal{F}(\theta) = \mathcal{F}\left((\Gamma, \partial_\theta\Gamma)(\theta), \theta\right) \in C^\infty\left(\Pi T\mathcal{N} \times \{\theta\}\right)$ by a functional $F[\Gamma]$ of the superfield methods [6, 7] (in case $\Gamma^p = (\Phi^A, \Phi^*_A)$, see the Introduction)

$$F[\Gamma] = \int d\theta\theta\mathcal{F}(\theta) = \mathcal{F}\left(\Gamma(0), \partial_\theta\Gamma, 0\right) \equiv \mathcal{F}(\Gamma_0, \Gamma_1)\,. \qquad (73)$$

In the first place, formula (73) implies the independence of $F[\Gamma]$ from $\partial^r_\theta\Gamma^p(\theta) = \Gamma^p_1$, in case $F(\theta) = F(\Gamma(\theta), \theta)$. Secondly, formula (73) is fundamental in establishing a relation between the θ-local antibracket $(\cdot\,,\cdot)^{\mathcal{N}}_\theta$ and operator $\Delta^{\mathcal{N}}(\theta)$, acting on $C^\infty\left(N \times \{\theta\}\right)$, with a generalization to arbitrary $(\Gamma, \omega^{pq}, \rho)(\theta)$ of the flat functional operations $(\cdot\,,\cdot)$, Δ of Refs. [6, 7], identical to their counterparts of the BV method in case $\Gamma^p = (\Phi^A, \Phi^*_A)$, $\omega^{pq}(\Gamma(\theta)) = \mathrm{antidiag}\left(-\delta^A_B, \delta^A_B\right)$, $\rho(\theta) = 1$, and in case of a different odd Poisson bivector, $\tilde{\omega}^{pq}(\Gamma(\theta), \theta') = (1 + \theta'\partial_\theta)\omega^{pq}(\theta)$. The correspondence follows from

$$\begin{aligned}
&\left(\mathcal{F}(\theta), \mathcal{G}(\theta)\right)^{\mathcal{N}}_\theta\big|_{\theta=0} = \frac{\delta_r\mathcal{F}(\Gamma_0)}{\delta\Gamma^p_0}\omega^{pq}(\Gamma_0)\frac{\delta_l\mathcal{G}(\Gamma_0)}{\delta\Gamma^q_0} = \left(F[\Gamma], G[\Gamma]\right)^{\mathcal{N}},\\
&\left(F[\Gamma], G[\Gamma]\right)^{\mathcal{N}} = \partial_\theta\left[\frac{\delta_r F[\Gamma]}{\delta\Gamma^p(\theta)}\partial_{\theta'}\left(\tilde{\omega}^{pq}(\Gamma(\theta),\theta')\frac{\delta_l G[\Gamma]}{\delta\Gamma^q(\theta')}\right)\right](-1)^{\varepsilon(\Gamma^p)+1}, \qquad (74)\\
&\Delta^{\mathcal{N}}(\theta)\mathcal{F}(\theta)\big|_{\theta=0} = \Delta^{\mathcal{N}}(0)\mathcal{F}(\Gamma_0) = \Delta^{\mathcal{N}}F[\Gamma],\\
&\Delta^{\mathcal{N}} = \frac{1}{2}(-1)^{\varepsilon(\Gamma^q)}\partial_\theta\partial_{\theta'}\left[\rho^{-1}[\Gamma]\tilde{\omega}_{qp}(\theta',\theta)\left(\Gamma^p(\theta), \rho[\Gamma]\left(\Gamma^q(\theta'), \cdot\right)^{\mathcal{N}}\right)^{\mathcal{N}}\right], \qquad (75)
\end{aligned}$$

where $(\rho[\Gamma], \tilde{\omega}_{pq}(\theta',\theta)) = (\rho(\Gamma_0), \theta'\theta\omega_{pq}(\theta))$ and

$$\int d\theta''\tilde{\omega}^{pd}(\theta',\theta'')\tilde{\omega}_{dq}(\theta'',\theta) = \theta\delta^p{}_q.$$

When establishing the correspondence with the operations $(\cdot\,,\cdot)$, Δ of [6, 7] in (74), (75), we have used a relation between the superfield and component derivatives:

$$\delta_l/\delta\Gamma^p(\theta) = (-1)^{\varepsilon(\Gamma^p)}\left(\theta\delta_l/\delta\Gamma^p_0 - \delta_l/\delta\Gamma^p_1\right),\ \Gamma^p_1 = (\lambda^A, -(-1)^{\varepsilon_A}J_A).$$

In general coordinates, the action of the sum and difference of the operators $\partial_\theta(V \pm U)^{\mathcal{N}}(0)$, $\mathcal{N} = \Pi T^*\mathcal{M}_{\mathrm{ext}}|_{\theta=0}$, reduced to

$$\partial_\theta(V \pm U)(0) = \partial_\theta\Phi^*_A(\theta)\partial/\partial\Phi^*_A(0) \pm \partial_\theta\Phi^A(\theta)\partial_l/\partial\Phi^A(0),$$

is identical to the action of the generalized sum and difference of the functional counterparts V, U in [6]:

$$\begin{aligned}
&\partial_\theta(V - (-1)^{\mathrm{t}}U)^{\mathcal{N}}(\theta)\mathcal{F}(\theta)\big|_{\theta=0} = \left(\mathcal{S}^{\mathrm{t}}(\theta), \mathcal{F}(\theta)\right)^{\mathcal{N}}_\theta\big|_{\theta=0}\\
&= (V - (-1)^{\mathrm{t}}U)^{\mathcal{N}}F[\Gamma] = \left(S^{\mathrm{t}}[\Gamma], F[\Gamma]\right)^{\mathcal{N}},\ \mathrm{t} = 1, 2,\\
&\mathcal{S}^{\mathrm{t}}(\theta) = (\partial_\theta\Gamma^p)\omega^{\mathrm{t}}_{pq}(\Gamma(\theta))\Gamma^q(\theta),\ S^{\mathrm{t}}[\Gamma] = \partial_\theta\left\{\Gamma^p(\theta)\partial_{\theta'}\partial_\theta\left[\tilde{\omega}^{\mathrm{t}}_{pq}(\theta,\theta')\Gamma^q(\theta')\right]\right\} = \mathcal{S}^{\mathrm{t}}(0), \qquad (76)
\end{aligned}$$

where the functions $\omega^{\mathrm{t}}_{pq}(\theta), \tilde{\omega}^{\mathrm{t}}_{pq}(\theta,\theta')$, coinciding for $\mathrm{t} = 1$ with $\omega_{pq}(\theta)$ and $\tilde{\omega}_{pq}(\theta,\theta')$, are defined by the relations

$$\tilde{\omega}^{\mathrm{t}}_{pq}(\theta,\theta') = \theta\theta'\omega^{\mathrm{t}}_{pq}(\theta') = -(-1)^{\mathrm{t}+\varepsilon(\Gamma^p)\varepsilon(\Gamma^q)}\tilde{\omega}^{\mathrm{t}}_{qp}(\theta',\theta),\ \omega^{\mathrm{t}}_{pq}(\theta) = (-1)^{\varepsilon(\Gamma^p)\varepsilon(\Gamma^q)+\mathrm{t}}\omega^{\mathrm{t}}_{qp}(\theta)\,.$$

The $\tilde{\varepsilon}$-bosonic quantities $\mathcal{S}^{\mathrm{t}}(\theta)$ and $\mathcal{S}^{\mathrm{t}}[\Gamma]$, with a vanishing ghost number, play the role of the symmetric Sp(2)-tensor S_{ab} $(a,b=1,2)$ and anti-Hamiltonian S_0 of Ref. [28], which determine (through extended antibrackets) the first-order operators of the modified triplectic algebra. In this case, the additional functions $\omega^2_{pq}(\theta)$, $\tilde{\omega}^2_{pq}(\theta,\theta')$ may be considered as quantities that define another non-antisymplectic (non-Riemannian) nondegenerate structure on $\mathcal{N}$. The θ-local functional operators $\{\Delta^{\mathcal{N}}, V^{\mathcal{N}}, U^{\mathcal{N}}\}(\theta)$ anticommute for a fixed θ,

$$[E^{\mathcal{N}}_{i}(\theta), E^{\mathcal{N}}_{j}(\theta)]_{+}=0,\ i,j=1,2,3,\ (E_1,E_2,E_3)=(\Delta\,,V,U), \tag{77}$$

provided that $\mathcal{S}^{\mathrm{t}}(\theta)$, or $\mathcal{S}^{\mathrm{t}}[\Gamma]$, is subject to

$$\Delta^{\mathcal{N}}(\theta)\mathcal{S}^{\mathrm{t}}(\theta)=0,\ (\mathcal{S}^{\mathrm{u}}(\theta),\mathcal{S}^{\mathrm{v}}(\theta))^{\mathcal{N}}_{\theta}=0,\ \mathrm{t},\mathrm{u},\mathrm{v}=1,2. \tag{78}$$

Relations (78), which hold, due to eqs. (74)–(77), also for functional objects (those without θ-dependence), follow from the well-known properties of the antibracket (*bilinearity, graded antisymmetry, Leibniz rule, Jacobi identity*), and from the rule of antibracket generation by the operator $\Delta^{\mathcal{N}}(\theta)$. The system (78) determines the geometry of $\mathcal{N}$ by restricting the choice of both quantities $\omega^{\mathrm{t}}_{pq}(\theta)$, $\tilde{\omega}^{\mathrm{t}}_{pq}(\theta,\theta')$. Notice that a solution of eqs. (78) always exists, for instance, $\omega^{\mathrm{t}}_{pq}(\theta)=\mathrm{antidiag}\left(\delta^A_B,(-1)^{\mathrm{t}}\delta^A_B\right)$.

6 Conclusion

Let us summarize the main results of the present work:

1. We have proposed a θ-local description of an arbitrary reducible superfield theory as a natural extension of a standard gauge theory, defined on a configuration space $\mathcal{M}_{\mathrm{cl}}|_{\theta=0}$ of classical fields A^i, to a superfield model defined on extended cotangent, $\Pi T^{*}\mathcal{M}_{\mathrm{CL}}\times\{\theta\}$, and tangent, $\Pi T\mathcal{M}_{\mathrm{CL}}\times\{\theta\}$, odd bundles, in the respective Hamiltonian and Lagrangian formulations. It is shown that the conservation, under the θ-evolution defined by the Hamiltonian or Lagrangian system providing a superfield extension of the usual extremals, of the Hamiltonian action $S_H\left((\mathcal{A},\mathcal{A}^{*})(\theta),\theta\right)$, or, equivalently, of an odd analogue of the energy, $S_E\left((\mathcal{A},\partial_\theta\mathcal{A})(\theta),\theta\right)$, is equivalent, due to Noether's first theorem, to the validity of a Hamiltonian or Lagrangian master equation, respectively.

2. Using non-Abelian hypergauges, we have constructed a θ-local superfield formulation of Lagrangian quantization of a reducible gauge model, selected from a general superfield model by conditions of the explicit θ-independence of the classical action and the vanishing of ghost number and auxiliary Grassmann parity (related to θ) for the action and $\mathcal{A}^i(\theta)$. In particular, we have proposed a new superfield algorithm for constructing a first approximation to the quantum action in powers of ghosts of the minimal sector, on the basis of interpreting the reducibility relations as special gauge transformations of ghosts, transformed in an HS with the Hamiltonian chosen as the quantum action. To investigate the properties of BRST invariance and gauge-independence in a superfield form for the introduced generating functionals of Green's functions (including the effective action), we have used *two equivalent* Hamiltonian-like systems. The first system is defined by a θ-local antibracket, in terms of a quantum action, a gauge-fixing action, and an arbitrary θ-local boson function, while the second (dual) system is defined by an even Poisson bracket, in terms of fermion functionals corresponding to the above functions. The two systems permit one to describe the BRST transformations and the continuous (anti)canonical transformations in a manner analogous to the relation between these transformations in the superfield Hamiltonian formalism [5]. We emphasize that, as a basis for the local quantization, we have intensely used the first-level formalism of [11], whose central ingredient is the vacuum functional (however, without recourse to the gauge-fixing action in an explicit form).

3. We have considered the problem of a *dual description* of an L-stage-reducible gauge theory, in terms of a BRST charge for a formal dynamical system with first-class constraints of $(L+1)$-stage-reducibility. It is shown that this problem is a particular case of describing an embedding of a reducible special gauge theory into a general gauge theory of the same stage of reducibility.

4. We have established the coincidence of the first-level functional integral $Z^{(1)}$ in [11] with the local vacuum function of the proposed quantization scheme, in case $\theta=0$ and $\varphi^{*}(\theta)=0$, $\mathrm{Z}_X(0)|_{\varphi^{*}_0=0}$.

5. From the obtained results there follow the generating functional of Green's functions and the effective action of the first-level formalism [11].

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