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Stationary solutions for the linear sigma model and another kind of spontaneous symmetry breaking resulting in two coupling constants

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Abstract

The linear sigma model is re-investigated in the framework of the Gaussian (variational) approximation. Classes of analytical solutions are found for the time-dependent case. Several particular situations are discussed which can be relevant for different systems such as (relativistic) heavy ions collisions. The renormalization of the ultraviolet divergences can lead to a different kind of spontaneous symmetry breaking in which two different renormalized coupling constants emerge. Two different and independent possible reasons for this are investigated respectively in static and time dependent calculations. While in the static calculation two renormalization scale parameters are introduced in the time dependent picture only one is considered in a particular prescription for the temporal evolution.

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1 Introduction

In the Wigner-Weyl realization of a symmetry in a field theory there are multiplet(s) of degenerated states - such as particles with equal masses and quantum numbers. However it can happen that the ground state is not invariant under the whole transformation group although the Lagrangian is. This occurs in the Nambu-Goldstone realization. Non zero expected value of one or more fields (which will be called condensate(s)) appear in the ground state for this (spontaneous) symmetry breaking. Non degenerate states emerge, eg. particles with different masses. In this case the symmetry is said to be spontaneously (or dynamically) broken (SSB) [1]. Zero energy excitations appear for SSB of global symmetries by the Goldstone theorem [2]. The lightest hadrons are known to respect approximately chiral symmetry: $SU_L(2) \times SU_R(2)$ which is believed to be spontaneously broken down to $SU(2)$ from phenomenology and theoretical reasons. This is expected to occur with the formation of a scalar isoscalar $< \bar{q}_R q_L + \bar{q}_L q_R >$ condensate in the QCD vacuum [1, 4, 5, 6] not only in the light quark sector but in all the $SU(N_f)$ flavor and chiral group [7, 8, 9, 10, 11], besides the other condensates expected to be in the QCD vacuum [12, 13]. Hadronic models for strong interacting systems are expected and required to exhibit the relevant features of low energy QCD, respecting the main symmetries (and properties) of the fundamental theory with its (observed or not) degrees of freedom. However, tree-level hadronic models can embody higher order effects of Quantum Chromodynamics - perturbative or not. Besides that calculations on discretized space-time provide a powerful way to the investigation of links between the two levels of strong interactions. One of the relevant issues for theses links is the understanding of the behavior of hadronic properties (masses and coupling constants) in different dynamical situations which are expected to be directly related to the symmetry and fundamental properties. In particular much attention has been given to dynamical effects in experimental conditions in which strong interacting processes are investigated. These issues are far more important in experiments where global many body observables are investigated such as in relativistic heavy ion collisions (rhic) and high energy heavy ions collisions (hehic). Hydrodynamical descriptions of many particle systems can be expected to be directly related to the field theoretical approach. For this, the time and spatial evolution of these systems must be addressed in a variety of approaches. Different aspects of the time evolution of systems undergoing spontaneous symmetry breaking have been addressed in many works [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61]. In this sense, this work tries to provides elements which can be helpful to describe effects in experimental situations at the hadronic level, when quarks and gluons are confined.

The linear sigma model [14, 15] has been extensively investigated and in many respects provides excelent description of experimental observations. Although there are scalar mesons which have been observed in different processes [16, 17, 18] there are no real evidence whether their structure is consistent with the quark-antiquark expected to be present in a chiral partners of the pseudo-scalar mesons (being that their properties are studied nowadays in the vacuum and in the nuclear medium) [9, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. Furthermore connections between the linear sigma model and low-energy QCD can be expected to occur and they have been discussed along last 30 years [36, 37, 38, 39, 40]. The non linear realization of chiral symmetry [1, 41, 42, 43] will not be worked out here. The couplings to, for example, vector mesons are determined by the realization of chiral symmetry [39, 44, 45] and although these subjects are not the main aim of this work some aspects will be shortly addressed. Hadron masses, interactions and the manifestation of the symmetry in the vacuum (and in the medium) also impose severe constraints for these formulations. Non perturbative effects have an extremely important role in such bound states and their interactions although they are not exactly calculable in the strong coupling regime of QCD so far (unless, in principle, in lattice calculation). Once again, hopefully, the investigation of the hadronic models, in the (low and high energies) confined regime may very useful to shed light on effects observed effects and corresponding structure of the QCD.

Nevertheless it is reasonable to ask questions such as: can the linear sigma model be an appropriate framework to consider the dynamics and properties of the scalars even if they may not be described by quark-anti-quark states as it is found in some investigations? It will be assumed along this work the scalars mesons are coupled to the pseudoscalar ones within the linear realization of chiral symmetry. Although in this picture the lightest scalar field is expected to develop a non zero ("classical") expected value in the vacuum, the scalar condensate $< \sigma >_{vac} > \equiv \bar{\sigma}_0 \propto < q\bar{q} >$, it may happen that the expected value of the scalar meson field may have other components in its complete quark-gluon structure. The Gell-Mann–Oakes-Renner relation accounts for this very plausible link between the quark and observed low energy degrees of freedom:

$$m_q < \bar{q}q > = -f_\pi^2 m_\pi^2. \tag{1}$$

This condensate is therefore to be identified to the pion decay constant f_π and to the chiral radius in the usual Nambu (Goldstone) picture. These three quantities reduce to the same value in the usual analysis of the linear sigma model (LSM) at the tree level. The pion masses, as well as the quark masses, are obtained by breaking explicitely the invariance of the Langrangian with additional term(s) as, for instance, $\mathcal{L}_{sb} = c\sigma$, where $c \propto \mu_\pi^2$ although other forms can be also suitable like proposed in the present work and elsewhere [72].

In this paper the Linear Sigma Model (LSM) with pions and sigma is re-investigated within the variational approximation with trial Gaussian wavefunctionals [62, 63, 64, 65]. The time dependent variational principle will also be considered as worked out in [66] to calculate the equations of motion. This will allow for the investigation of time dependent and static effects related to the spontaneous symmetry breaking. The equations of motion reduce to the GAP equations in the static limit which determine the physical masses and ground state of the system. These GAP equations are found from the minimization of an averaged energy density with respect to the trial variational parameters: the classical fields $\bar{\sigma} =< \sigma >$, $\bar{\pi} =< \pi >= 0$ and the corresponding two point functions $G_S =< \sigma^2 >$, $G_P =< \pi^2 >$. The Golsdstone theorem is satisfied. Particular solutions for the time-dependent calculation are found which do not introduce further ultraviolet divergences in the ground state (local) static limit. They give rise to modifications in the model parameters such as masses and couplings possibly corresponding to situations present in experimental investigations of high energy and relativistic heavy-ion collisions. The elimination of the ultraviolet (UV) divergences can be done with the inclusion of one or two renormalization scale parameters. These different procedures are considered respectively in the time-dependent and static calculations. With this, there may occur another kind of dynamical symmetry breaking in which more than one renormalized coupling constant appear. Without the Lagrangian term which break chiral symmetry explicitely the pion mass is set to zero with the renormalization mass scale(s) (or eventually the renormalized parameters) or written as directly proportional to $\lambda < 0|\pi|0 >^2 \to 0$. The divergence of the axial current is calculated resulting a sort of 'averaged PCAC' (in terms of an averaged value of the pion field $\bar{\pi}$) which is associated to the method which deals with averages. The paper is organized as follows. In the next section the linear sigma model is presented, its ground state is described at the tree level and the usual Gaussian variational approximation is presented as usually done. In section 3 particular stationary-like analytical solutions for the time dependent calculation are found and partially compared to the results of the time independent calculation. The renormalization is performed in section 4, in two ways: in the ususal way or with the introduction of two renormalization scale parameters. In section 5 the divergence of the (vector and axial) currents are calculated. In the final part there is a summary.

2 Linear Sigma Model

The Lagrangian density of the Linear Sigma Model (LSM), with a sigma and pions (σ, π), chirally coupled to vector and axial fields can be given by [39]:

$$\mathcal{L} = \frac{1}{2}\left(\mathcal{D}_\mu\sigma.\mathcal{D}^\mu\sigma + \mathcal{D}_\mu\pi.\mathcal{D}^\mu\pi\right) - \frac{\lambda}{4}\left((\sigma)^2 + (\pi)^2 - v^2\right)^2 + \frac{a}{4}\left(\mathcal{F}^i_{\mu\nu L}.\mathcal{F}_i^{\nu\mu}{}_L + \mathcal{F}^i_{\mu\nu R}.\mathcal{F}_i^{\nu\mu}{}_R\right), \tag{2}$$

where v is a constant associated to the bare masses and coupling (λ) - the chiral radius. The choice of how the chiral vector and axial fields are coupled is not the main subject of this work. Other couplings in the linear sigma model Lagrangian can be considered [39, 1, 44] without modifying the conclusions of this work. Choices for the covariant derivatives and the "left-right components" kinetic tensor (with a constant coefficient a) will be given respectively by:

$$\begin{aligned}
\mathcal{D}_\mu\sigma &= \partial_\mu\sigma + \gamma(A_\mu{}^i_R - A_\mu{}^i_L).\pi^i, \\
\mathcal{D}_\mu\pi^i &= \partial_\mu\pi^i + \gamma\left((\vec{A}^R_\mu + \vec{A}^L_\mu)\times\pi\right)^i - \gamma\sigma\left((A^R_\mu)^i - (A^L_\mu)^i\right), \\
\mathcal{F}^L &= \partial_\mu A^L_\nu - \partial_\nu A^L_\mu + \gamma A^L_\mu \times A^L_\nu,
\end{aligned} \tag{3}$$

where the two components, right and left, are the chiral combinations of the isovector and iso-axial vector fields expected to correspond to the mesons ρ and A_1 (with $\vec{A}^{R(L)}_\mu \propto (\vec{V}_\mu + (-)\vec{A}_\mu)/2)$.

For the parameter v set to zero the model becomes scale invariant at tree level and the sigma mass also goes to zero. The scale invariance is also broken with the inclusion of quantum loop corrections [67, 12]. This will be briefly discussed latter. However to obtain the chiral symmetry breaking, with the chiral scalar condensate $\bar{\sigma}$, this term shows to be necessary. The resulting coupling constants of the model after the shift due to the scalar condensate are given by:

$$g_{\sigma\sigma\sigma\sigma} = g_{\pi\pi\pi\pi} = \frac{\lambda}{4} \qquad g_{\sigma\pi\pi} = 2\lambda\bar{\sigma} \qquad g_{\pi\pi\sigma\sigma} = \frac{\lambda}{2}, \tag{4}$$

They are related to each other due to the chiral symmetry and its spontaneous breakdown. From the usual linear coupling of the scalar field to baryons the nucleonic mass, or part of it, can be generated without breaking explicitely chiral symmetry. For instance: $\Delta m_N \simeq \pm g_S\bar{\sigma}$ *For the isovector and axial fields however the covariant derivatives given above (3) only generate a mass term for the axial field. This is to be associated to the mass diference of the rho and* A_1 *mesons. This mass term from the above covariant derivative (CD), obtained with the shift of the scalar field, is given by:* $m^2_{CD}A^i_\mu A^\mu_i = \gamma^2\bar{\sigma}^2 A^i_\mu A^\mu_i$. *Given that* $\bar{\sigma} = f_\pi \simeq 93\ MeV$ *and* $(m_{A_1} - m_\rho) \sim 450\ MeV$, *the coupling can be fixed,* $\gamma \sim 5$. *The mass of the rho is not obtained from the above Lagrangian density as proportional to* $\bar{\sigma}$. *This variation of the rho mass is expected to occur being possibly in agreement with recent results for the restoration of chiral SSB* *at high energy densities. The rho mass is then expected to decrease with increasing energy densities until the restoration of chiral symmetry is reached [68]. This can be reproduced, for example, with different covariant derivatives and different mechanisms of mass generation may be at work [1, 69, 68, 70, 71]. ????????* These chiral vector fields will not be really discussed along this work. In fact their couplings in chiral models depend on the realization of chiral symmetry.

2.1 Tree level and pion mass

The sigma field is therefore expected to acquire non zero expected value in the vacuum, the scalar condensate which does not endow the pion with mass (like a Goldstone boson). *The minimization of the potential of σ, π with relation to σ and π_a lead to the values of the fields in the ground state. It yields respectively:*

$$\lambda\left(-v^2 + (\sigma^2 + \pi^2)\right)\sigma = 0, \quad \lambda\left(-v^2 + (\sigma^2 + \pi^2)\right)\pi_a = 0. \tag{5}$$

The first equation has three solutions: $\sigma = \bar{\sigma} = 0$, $\sigma = \bar{\sigma} = \pm v$ whereas π_a is set to zero in the ground state. For a certain range of values for v and λ the sigma field develops a non zero ground state classical value $\bar{\sigma} =< \sigma > \to \pm v$ [1]. This parameter constrains the dynamics of the fields becoming the chiral radius. The emergence of this scalar condensate breaks spontaneously the global symmetry and orthogonal fields, the pions, should have zero mass [2] - since the pion mass is in fact non zero (but small) they seem to correspond to quasi-Goldstone bosons. The scalar field is shifted to account for this appearence of the condensate when computing the properties and observables of the corresponding processes. This is usually written as:

$$\sigma \to \bar{\sigma} + s. \tag{6}$$

The following expressions for the pion and sigma masses arise respectively:

$$m_\pi^2 = \lambda\left(\bar{\sigma}^2 - v^2\right) \to 0, \qquad m_\sigma^2 = \lambda\left(3\bar{\sigma}^2 - v^2\right), \tag{7}$$

for which $\bar{\sigma} = \pm v$. ???

Assume that, for some other external reason or coupling, $\bar{\sigma} \neq v$ in the vacuum of the model. The parameter v still is the bare pion decay constant (chiral limit): $v = F_\pi = 88MeV$ [42] while $\bar{\sigma}$ can be the total pion decay constant in the tree level $\bar{\sigma} = f_\pi \simeq 93$MeV. It is found that $\lambda \simeq 43.3$ and $m_\sigma^2 \simeq (628MeV)^2$, for $m_\pi^2 = (140MeV)^2$. This value of the scalar meson mass is close to values found in the PDG tables and in theoretical estimates quoted above. In this picture a pion mass can be obtained by considering the interaction of the pion with the external degrees of freedom (like an effective mass)???????

A chiral symmetry breaking Lagrangian term is usually introduced such that a realistic pion mass is obtained and such that low energy theorems of current algebra are respected [1]. Two different symmetry breaking terms (not completely equivalent) will be preliminary considered:

$$\mathcal{L}_{sb} = c\sigma + \frac{d}{2}\sigma^2, \tag{8}$$

where c, d are constants to be determined from PCAC. For this, it is assumed in a first analysis that each of these terms is responsible for the total pion mass independently. The term proportional to σ^2 can be responsible for modifying the sigma mass and/or the pion mass. To calculate the PCAC, each of the terms are considered separatedly, i.e. $c \neq 0, d = 0$ or $c = 0, d \neq 0$. They are obtained respectively as: $c = -f_\pi m_\pi^2$ and $d = -m_\pi^2$. However the parameter d can also modify the sigma mass yielding $d \propto \Delta m_\sigma^2$ as seen below. The minimization of the sigma-pion potential with both terms simultaneously leads to the following expressions:

$$\begin{aligned} &\lambda\left(-v^2 + (\sigma^2 + \pi^2) + \frac{d}{\lambda}\right)\sigma + c = 0, \\ &\lambda\left(-v^2 + (\sigma^2 + \pi^2)\right)\pi_a = 0. \end{aligned} \tag{9}$$

Whereas c introduces a more complicated mathematical expression for the solution of the condensate $\bar{\sigma}$ (deforming the chiral circle), the parameter d can be regarded as a shift the chiral radius: $v^2 \to v^2 - \frac{d}{\lambda}$, which endows the pion with a mass besides modifying the sigma mass. Consequently the chiral radius is written as:

$$\sigma^2 + \pi^2 = \frac{1}{2}\left(2v^2 - \frac{c}{\lambda} - \frac{d}{\lambda}\sigma\right). \tag{10}$$

The parameter is therefore $d = -m_\pi^2$, as discussed above, corresponding to a deformation of the chiral circle. It also yields a modification of the sigma mass $\Delta m_\sigma^2 = d$. These symmetry breaking terms, either of c or d [72], will not be considered explicitely in the remaining part of this work.

2.2 Gaussian approximation

Quantum fluctuations for the sigma and pion fields will be computed in the frame of the variational approach in the Schroedinger picture using the Gaussian prescription [66, 62, 73, 74, 75, 46, 48, 47] . In this approach the state of the system is described by normalized wave-functional(s) which satisfies the functional Schrödinger equation. Field operators and their respective canonical conjugated momenta ($\hat{\phi}_i = \sigma, \pi$; $\hat{\xi}_i = \hat{\Pi}_\sigma, \hat{\Pi}_\pi$) are applied to the wavefunctional respectively as: $\hat{\phi}_i|\Psi[\phi_i] >= \phi_i|\Psi[\phi_i] >$, $\hat{\xi}_i|\Psi[\phi_i] >= -i\frac{\delta}{\delta\phi_i}|\Psi[\phi_i] >$. In the following, the time-dependent version of the variational principle is described and the static limit considered latter. Given a Gaussian trial wavefunctional, $|\Psi >= |\Psi[\sigma, \pi] >$

containg the variational parameters, the averaged action which yield the time-dependent Schrödinger equation can be calculated. This action is given by:

$$\mathcal{I} = \int dt \ < \Psi[\phi_i] | \left(i\partial_t - H[\phi_i, \Pi_i] \right) | \Psi[\phi_i] > . \tag{11}$$

A trial Gaussian wave-functional is decomposed into two Gaussians, one for each of the the fields (scalar and pseudo-scalar fields). For example, for the sigma field part it can be written:

$$\Psi\left[\sigma(\mathbf{x})\right] = N exp\left\{ -\frac{1}{4} \int d\mathbf{x} d\mathbf{y} (\sigma(\mathbf{x}) - \tilde{\sigma}) \left(G_S^{-1}(\mathbf{x},\mathbf{y}) + i\Sigma_S(\mathbf{x},\mathbf{y}) \right) (\sigma(\mathbf{y}) - \tilde{\sigma}) + i \int d\mathbf{x} \tilde{\xi}_S (\sigma(\mathbf{x}) - \tilde{\sigma}) \right\}, \tag{12}$$

Where N is the normalization, the variational parameters are (i) the vacuum expected value ("condensate", constant) $\tilde{\sigma}(\mathbf{x},t) =< \Psi|\sigma|\Psi >$; (ii) the two point function, representing quantum fluctuations, for the width of the Gaussian: $G_S(\mathbf{x},\mathbf{y};t) =< \Psi|\sigma(\mathbf{x})\sigma(\mathbf{y})|\Psi >$; (iii) their respective conjugated variables for the time dependent formalism are respectively $\tilde{\xi}_S(\mathbf{x},t)$ and $\Sigma_S(\mathbf{x},\mathbf{y};t)$. An analogous expression for the pion wavefunctional is considered with the variational parameters: $G_P^{a,b}(\mathbf{x},\mathbf{y};t) =< \Psi|\pi^a\pi^b|\Psi >$ (where a,b stand for pion index: 1,2,3); $\tilde{\pi}_a =< \Psi|\pi_a|\Psi >= 0$; $\Sigma_P(\mathbf{x},\mathbf{y};t)$ and $\tilde{\xi}_P(\mathbf{x},t)$. The two point function $G_P^{a,b}$ is a matrix in isospin space and it is considered to be diagonal in the particular case worked out here ($G_P^{a,b} = G_P\delta^{a,b}$). This guarantees the explicit chiral and isospin invariances as long as we take $\tilde{\pi} = 0$. However all the calculations are done as if this quantity were non zero, and for drawing conclusions, the limit $\tilde{\pi} \to 0$ is done.

The variation of the averaged action with respect to the Gaussian variational parameters ($\tilde{\sigma}, \tilde{\pi}$, $G_P, G_S, \tilde{\xi}_S, \tilde{\xi}_P$, Σ_S and Σ_P) yield the movement equations. They may be written in the following form:

$$\begin{aligned}
&\frac{\delta\mathcal{I}}{\delta\tilde{\pi}_a} \to \partial_t \tilde{\xi}_a^P(\mathbf{x},t) = 6\lambda \left(-\Delta + \tilde{\pi}^2 + 3G_P + \tilde{\sigma}^2 + G_S - v^2 \right) \delta_{a,b}\, \tilde{\pi}_b(\mathbf{x},t);\\
&\frac{\delta\mathcal{I}}{\delta\tilde{\xi}_P^a} \to \partial_t \tilde{\pi}^a(\mathbf{x},t) = -\tilde{\xi}_P^a(\mathbf{x},t);\\
&\frac{\delta\mathcal{I}}{\delta\tilde{\sigma}} \to \partial_t \tilde{\xi}_S(\mathbf{x},t) = 6\lambda \left(-\Delta + \tilde{\sigma}^2 + 3G_S + \tilde{\pi}^2 + G_P - v^2 \right) \tilde{\sigma}(\mathbf{x},t);\\
&\frac{\delta\mathcal{I}}{\delta\tilde{\xi}_S} \to \partial_t \tilde{\sigma}(\mathbf{x},t) = -\tilde{\xi}_S(\mathbf{x},t);\\
&\frac{\delta\mathcal{I}}{\delta G_S} \to \partial_t \Sigma_S(\mathbf{x},\mathbf{y};t) = 2\Sigma_S^2(\mathbf{x},\mathbf{y};t) - \frac{G_S^{-2}(\mathbf{x},\mathbf{y};t)}{8} + \left(-\frac{\Delta}{2} + \frac{\lambda}{4} \left(6G_S + 6\tilde{\sigma}^2 + 2\tilde{\pi}^2 + 2G_P - 2v^2 \right) \right),\\
&\frac{\delta\mathcal{I}}{\delta G_P^{a,b}} \to \partial_t \Sigma_P^{a,b}(\mathbf{x},\mathbf{y};t) = 2(\Sigma_P^2)^{a,b}(\mathbf{x},\mathbf{y};t) - \frac{G_P^{-2}(\mathbf{x},\mathbf{y},t)}{8} + \left(-\frac{\Delta}{2} + \frac{\lambda}{2} \left(3G_P + \tilde{\sigma}^2 + 3\tilde{\pi}^2 + G_S - v^2 \right) \right) \delta_{a,b};\\
&\frac{\delta\mathcal{I}}{\delta\Sigma_S} \to \partial_t G_S(\mathbf{x},\mathbf{y};t) = 2\left(G_S(\mathbf{x},\mathbf{z};t)\Sigma_S(\mathbf{z},\mathbf{y};t) + \Sigma_S(\mathbf{x},\mathbf{z};t) G_S(\mathbf{z},\mathbf{y};t) \right),\\
&\frac{\delta\mathcal{I}}{\delta\Sigma_P^{a,b}} \to \partial_t G_P^{a,b}(\mathbf{x},\mathbf{y};t) = 2\left(G_P^{a,c}(\mathbf{x},\mathbf{z};t)\Sigma_P^{c,b}(\mathbf{z},\mathbf{y};t) + \Sigma_P^{a,c}(\mathbf{x},\mathbf{z};t) G_P^{c,b}(\mathbf{z},\mathbf{y};t) \right),
\end{aligned} \tag{13}$$

With implicit summations in c and integrations in coordinate. The completely isospin invariant solution corresponds to $\tilde{\pi}_a = 0$. The "static" limit yields the GAP and field equations which define the ground state in the variational approach. In this approach the expected value of the scalar field, to be associated to the chiral scalar condensate, is introduced as a variational parameter.

2.3 Time independent limit: GAP equations

In the time independent limit the time derivatives of the equations of mouvement (13) are set to zero and the remaining equations are those for $\tilde{\sigma}$, $\tilde{\pi}$ and $G_i(\mathbf{x},\mathbf{x})$, assumed to be homogeneous functions. The physical masses μ_i^2 (when $G_{a,b} = G_{a,b}\delta_{a,b}$) can be written as:

$$\begin{aligned} \mu_S^2 &= \lambda\left(3\tilde{\sigma}^2 + \tilde{\pi}^2 + 3G_S + G_P - v^2\right), \\ \mu_P^2 &= \lambda\left(3\tilde{\pi}^2 + \tilde{\sigma}^2 + 3G_P + G_S - v^2\right). \end{aligned} \tag{14}$$

The (diagonal) two point functions G_i can thus be written as:

$$G_i(\mathbf{x},\mathbf{x};\mu_i^2) =< \mathbf{x}|\frac{1}{2\sqrt{\Delta + \mu_i^2}}|\mathbf{x}> . \tag{15}$$

where the above expressions for the masses (14) have been considered. These functions have exactly the form of the perturbative two point Feynman Green's functions with time integrated, the imaginary part with opposite sign and with physical mass. They have ultraviolet (UV) divergences which are to be eliminated to provide reliable results. The time-dependent renormalization is usually believed to be the same of the time-independent: there seems to have no extra ultraviolet divergence due to temporal evolution [75, 46].

The time independent limit of the condensate equations in the case discussed above (13) can be written as:

$$\begin{aligned} \lambda\left(-\Delta + \tilde{\pi}^2 + \tilde{\sigma}^2 + 3G_P + G_S - v^2\right)\tilde{\pi}_a(\mathbf{x}) = 0; \\ \lambda\left(-\Delta + \tilde{\sigma}^2 + \tilde{\pi}^2 + 3G_S + G_P - v^2\right)\tilde{\sigma}(\mathbf{x}) = 0. \end{aligned} \tag{16}$$

With the GAP equations, the sigma and pion masses are found to be directly related to the variational parameters, and they can be re-written in the homogeneous case ($\Delta\tilde{\sigma} = \Delta\tilde{\pi} = 0$) as:

$$\mu_S^2 = 2\lambda\tilde{\sigma}^2, \qquad \mu_P^2 = 2\lambda\tilde{\pi}^2 \to 0. \tag{17}$$

With the inclusion of quantum corrections, the parameter v is not equal to $\tilde{\sigma}$. The zero pion mass is obtained directly in this approach for $\tilde{\pi} = 0$ in agreement with the Goldstone theorem [65, 77, 78]. For the sake of generality some calculations will be done for both $\tilde{\sigma}$ and $\tilde{\pi}$ different from zero.

2.4 Some non-homogeneous solutions

For particular non homogeneous situations, there are interesting solutions for expressions (16). Consider, for instance, that:

$$\Delta\tilde{\sigma} = A_S^2\tilde{\sigma}, \qquad \Delta\tilde{\pi} = A_P^2\tilde{\pi}, \tag{18}$$

where $A_i = A_i(\mathbf{x})$ or constants. The respective GAP equations can be written as:

$$\mu_S^2(\mathbf{x}) \to (\mu_S^2)_{eff}(\mathbf{x}) = 2\tilde{\sigma}^2(\mathbf{x}) + A_S^2, \qquad \mu_P^2(\mathbf{x}) \to (\mu_P^2)_{eff}(\mathbf{x}) = 2\tilde{\pi}^2(\mathbf{x}) + A_P^2, \tag{19}$$

yielding an analogous form of the homogeneous limit (17) with x-dependent effective masses $(\mu_i^2)_{eff}(\mathbf{x})$. Different possibilities arise and the choice of appropriate coordinates (cylindrical, cartesian or spherical) is relevant depending of the corresponding geommetry of the systemb.

For constant imaginary A_i, corresponding to a periodic configuration of $\tilde{\phi}_i$, there results $A_i^2 < 0$, leading to a smaller effective mass. For a continuously varying system which develops inhomogeneities (with real A_i) $\mu_i^{eff} > \mu_i$. For other types of configuration in which A is x-independent at least in a localized region of space, $\mu_s^{eff} \neq \mu_s$. For x-dependent A_i there are a large variety of situations. Some questions arise such as: is this kind of inhomogeneities sizeably present in experimental conditions, for example in the fireballs of rhic and in hehic where the value of the condensate is expected to be modified due to the expected restoration of chiral symmetry? The same question would arise for any other field.

3 Time dependent stationary-like solutions

Before eliminating ultraviolet divergences through renormalization of coupling constant and mass parameters it is interesting to seek a particular classes of solutions of the regularized equations of mouvement (13). In this section stationary-like solutions are searched for the sigma and pion variational parameters (they are denoted by G_i and $\bar{\phi}_i$, where $i = \pi, \sigma$). The following general prescriptions are considered:

$$\dot{G}^i(\mathbf{x},\mathbf{x};t) = a_i(t)G^i(\mathbf{x},\mathbf{x};t), \;\; (i), \qquad \dot{\bar{\phi}}^\alpha(\mathbf{x},t) = \alpha(t)\bar{\phi}^i(t), \;\; (ii), \tag{20}$$

Where a_i, α , $\bar{\phi}^i$ and G^i can be time-dependent or not. For particular solutions it will be suitable to change the notation for: $\dot{G}^i(\mathbf{x},\mathbf{x};t) = a_i(t)G^i(\mathbf{x},\mathbf{x})$, $\quad \dot{\bar{\phi}}^i(\mathbf{x},t) = b_i(t)\bar{\phi}_0^i$. It is kept the possibility of having different "flows" (a_i, b_i) for each of the fields. For real parameters a_i, b_i these prescriptions correspond to amplifications or reductions of the variables with time. The dependences on $\mathbf{x}, t$ will be omitted in most of the expressions from here on.

The prescription (20 (i)) (for $G(t)_i$) is considered for the equations of motion (13). By redefining $\Sigma_i = a_i \Sigma_i^a$ the following equation (for $\dot{G}_i$) is obtained:

$$\left(G_i - \frac{1}{4} G_0^i (\Sigma_a^i)^{-1}\right) \Sigma_i + \Sigma_i \left(G^i - \frac{1}{4} (\Sigma_a^i)^{-1} G_0^i\right) \equiv \Sigma^i(t) G_-^i(t) + G_+^i(t) \Sigma^i(t) = 0. \tag{21}$$

Although this expression is similar to the one obtained within a static (thermal or not) ensemble such as those obtained, for example, in [74] it cannot be casted in that form and indeed corresponds to a time dependent system although "it seems" that $\dot{G}_i = 0$ because the prescriptions for the solutions of expressions (20) allow the rearrangement of the equations. [1]. These (new) variables, (G_+, G_-), can be rewritten in the following form:

$$G^i(t) = \frac{\Sigma_a^i(t)}{2} \left(G_-^i(t) + G_+^i(t)\right), \quad G_0^i(t) = \frac{\Sigma_a^i}{2}(t). \left(G_+^i(t) - G_-^i(t)\right). \tag{22}$$

There is a solution for which G_0^i is a constant and the temporal evolution of G^i is given, according to prescription (20), by $a^i = a^i(t)$. The above expression (22) can also be rewritten as:

$$\tilde{\Sigma}^i(t) G^i(t) + G^i(t) \tilde{\Sigma}^i(t) = 0, \tag{23}$$

where $\tilde{\Sigma}_i(t) = \Sigma_i(t) - a_i/4(t)$ and $\dot{\tilde{\Sigma}}_i(t) = \dot{\Sigma}_i(t) - \dot{a}_i/4(t) \to 0$. Therefore, in spite of $G(\mathbf{x}, \mathbf{y}; t)$ is a function of time, its time dependence, from equations (13), can be given by the form of expression (23) due to a redefinition of the conjugated variable $\tilde{\Sigma}_i$ within the prescription (20). Final solutions for the equations of motion are found substituting these expressions in the equations (13) for $\dot{\Sigma}_i$. Usually this eliminates the variable $\Sigma_i(\mathbf{x}, \mathbf{y}; t)$ although such expression for $\Sigma_i(\mathbf{x}, \mathbf{y}; t)$ cannot always be obtained analitically in the coordinate space. Considering again that $\dot{\tilde{\Sigma}}_i(t) = 0$, it follows that $\tilde{\Sigma}(t) = 0$ and finally: $\Sigma_i(t) = a_i/4(t)$. This prescription yields, after a re-arrangment, the following expression for G_i (where i always stands for σ or π):

$$0 = -\frac{1}{4} G_i^{-2} + \left(\Delta + \mu_i - \frac{a_i^2}{2} + 4\tilde{\Sigma}_i^2\right), \tag{24}$$

where $\tilde{\Sigma}_i = \Sigma_i - a_i/4 \to 0$ in this case.

The prescriptions proposed above can be akin to those a self similar-like system in which the resulting stationary solution keeps a modified effective (and eventually time-dependent) mass $\tilde{\mu}_i(t)$ in the form:

$$\tilde{\mu}_i^2(t) = \mu_i^2(t) - \frac{a_i^2(t)}{2} + 4\tilde{\Sigma_i(t)}^2, \tag{25}$$

[1]However the same kind of prescription can be considered for a thermal (smoothly varying) calculation producing similar expressions. One defines $G_+(\mathbf{x}, by; t)$ and $G_-(\mathbf{x}, \mathbf{y}; t)$ in the same way such that it has nothing to do with thermalized mixed states, although eventually it can be translated into a particular (self-similar) system out of equilibrium.

where $\tilde{\Sigma}_i$ is given, in a more general case, in terms of G by rewriting expressions (21,22). However, from the prescription considered above there is one (trivial) solution $\tilde{\Sigma}_i = 0$ yielding $\Sigma_i(\mathbf{x}, \mathbf{y}; t) = a_i(t)/4\ \delta(\mathbf{x} - \mathbf{y})$.

Rewriting explicitely the above expression (25) in terms of $G_\pm$ and the corresponding mouvement equations for G_i (13) it is obtained that:

$$\begin{aligned} \tilde{\mu}_\sigma^2(t) &= \lambda \left(3\tilde{\sigma}_i^2(t) + 3G_\sigma(t) - \frac{a_S^2}{4}(t) + \tilde{\pi}^2(t) + G_P(t) \right), \\ \tilde{\mu}_\pi^2(t) &= \lambda \left(3\tilde{\pi}_i^2(t) + 3G_\pi(t) - \frac{a_P^2}{4}(t) + \tilde{\sigma}^2(t) + G_S(t) \right), \end{aligned} \tag{26}$$

where G_i stands for the sigma and pion two point functions. Although the condensates $\tilde{\sigma}$ and $\tilde{\pi}$ have not been calculated so far we draw some possible pictures below. Renormalization of ultraviolet divergences in G_i as shown in section 5 does not change the form of the solutions. For real parameters a_i the pion and sigma effective masses are smaller.

In the time dependent solution (or in medium solution) for $\mu_P \to 0$ it follows that $\tilde{\mu}_P \neq 0$, as an effective ("dynamical") pion mass. In this case, the time evolution of the system is basically driven by an effective mass which should be rather time dependent ($a_i(t)$), although solutions for constant a_i may appear. The "effective" pion mass in these situations, where non homogeneities are in the form proposed in the section 3.2 (periodic), can induce a corresponding time evolution of the pion field (and eventually condensate if present) such that its total effective mass acquires particular values, according to expression (17). This can correspond, for example, to a situation in which $(\partial_t^2 - \Delta)\bar{\phi}_i = 0$ and therefore $A_i^2 = a_i^2$. Otherwise the effective pion mass is nonzero even in the so called chiral limit (where $c = d = 0$ in the Lagrangian density given in expression (8)) for such (time dependent) case. This non zero solution for the (effective) pion mass may eventually be part of a dynamical origin from the measured pion mass in the (non trivial) vacuum...

A different kind of solution with a modified total mass in the framework of the variational approach was found in an expanding environment, namely in the $\lambda\phi^4$ model in a de Sitter metric for inflationnary models [58]. Therefore it is reasonable to expect that the Universe expansion may contribute to masses of particles depending of its geommetry.

3.1 Equations of Condensates

So far, only the equations for G_i and Σ_i were investigated. For the equations of motion of the condensates, without writing their eventual x-dependences, the following analogous prescriptions are considered:

$$\dot{\tilde{\pi}}(t) = p(t)\ \tilde{\pi}(\mathbf{x}; t), \qquad \dot{\tilde{\sigma}}(t) = s(t)\ \tilde{\sigma}(\mathbf{x}; t), \tag{27}$$

Where the parameters p, s can be constants or not. The equations of motion (13) can be written in the form:

$$\begin{aligned} \left(-\Delta + \frac{\mu_P^2 - 2\tilde{\pi}^2}{6\lambda} + \frac{\dot{p} + p^2}{6\lambda}\right) \tilde{\pi}_a = 0, \\ \left(-\Delta + \frac{\mu_S^2 - 2\tilde{\sigma}^2}{6\lambda} + \frac{\dot{s} + s^2}{6\lambda}\right) \tilde{\sigma} = 0. \end{aligned} \tag{28}$$

The time derivative of these equations lead to the following equations:

$$\begin{aligned} \left(-\Delta + \frac{\mu_P^2 - 2\tilde{\pi}^2}{6\lambda} + \frac{\dot{p} + p^2}{6\lambda} + \frac{\dot{\mu}_P^2 - 2.(3\tilde{\pi}^2)}{6p\lambda} + \frac{\ddot{p} + 2\dot{p}p}{6p\lambda}\right) \tilde{\pi}_a = 0, \\ \left(-\Delta + \frac{\mu_S^2 - 2\tilde{\sigma}^2}{6\lambda} + \frac{\dot{s} + s^2}{6\lambda} + \frac{\dot{\mu}_S^2}{6s\lambda} + 2\tilde{\sigma}^2 + \frac{\ddot{s} + 2\dot{s}s}{6s\lambda}\right) \tilde{\sigma} = 0. \end{aligned} \tag{29}$$

On the other hand, the time derivative of the GAP expressions (14) yield:

$$\begin{aligned} \dot{\mu}_S^2 = \lambda \left(6s\tilde{\sigma}^2 + 2.(p)\tilde{\pi}^2 + 3a_s G_S + a_p G_P\right), \\ \dot{\mu}_P^2 = \lambda \left(3.(2p)\tilde{\pi}^2 + 2s\tilde{\sigma}^2 + a_s G_S + 3a_p G_P\right). \end{aligned} \tag{30}$$

Some particular solutions are given below. Depending on the prescriptions for a_i, s, p numerical solutions are found. The parameters a_i, s, p can be partially determined or constrained in terms of the parameters of the model (coupling and mass).

3.2 Some solutions

For constant "flow" parameters, $\dot{a}_i = \dot{s} = \dot{p} = 0$, real or complex, (which still yield $G = G(t)$), the equations above together (28,29,30) yield to the following coupled equations:

$$\begin{aligned} \left(-\Delta + \frac{\mu_P^2 - 2\tilde{\pi}^2}{6\lambda} + \frac{p^2}{6\lambda} + \frac{\dot{\mu}_P^2}{6p\lambda} + \right) \tilde{\pi}_a = 0, \\ \left(-\Delta + \frac{\mu_S^2 - 2\lambda\tilde{\sigma}^2}{6\lambda} + \frac{s^2}{6\lambda} + \frac{\dot{\mu}_S^2}{6s\lambda} + \right) \tilde{\sigma} = 0, \end{aligned} \tag{31}$$

where $\dot{\mu}_i^2$ are given by expressions (30). These coupled equations, for the prescriptions proposed above, yield algebraic equations for the parameters instead of differential equations.

Several types of solutions can be searched depending on $\Delta\bar{\phi} \neq 0$ or $\Delta\bar{\phi} = 0$. For example, there is a particular class of solutions for which $-\Delta\bar{\phi}_i = -\ddot{\bar{\phi}}_i$, as noted above ($\bar{\phi}_i = \tilde{\sigma}, \tilde{\pi}$). The resulting expressions have the same form of the static ones for the condensate although the effective masses also would have time and spatial dependences which are required to be compatible with this configuration. Considering only that $\Delta\bar{\phi} = 0$ the solutions will be called homogeneous. In this situation more restrictive condition will be considered in equations (30,31) such that an analytic simplified solution appear:

$$a_s = a_p = 2p = 2s. \tag{32}$$

This conditions mean that: $\partial G_s = 2\partial_t \tilde{\sigma}$ and the same relationship for $\tilde{\pi}(t)$ and $G_P(t)$. This is quite reasonable since from the variational equations: $G_S \propto \tilde{\sigma}^2$ and similarly for the $G_P \propto \tilde{\pi}^2$. It follows that

$$\dot{\mu}_\sigma^2 = a_s(\mu_\sigma^2 + \lambda v^2), \qquad \dot{\mu}_\pi^2 = a_s(\mu_\pi^2 + \lambda v^2), \tag{33}$$

being that the constant v^2 breaks scale invariance already in the Lagrangian density. A solution for these equations is given by:

$$\mu_\pi^2 = -\lambda v^2 + C_p e^{a_s t}, \qquad \mu_\sigma = -\lambda v^2 + C_s e^{a_s t}, \tag{34}$$

where the constant $C_i > 0$ is the only parameter to distinguish the masses at a given time t_i which may be $t_i = 0$. Different values for $s = p = \dot{\tilde{\sigma}}/\tilde{\sigma}$ yield diverse situations as long as the relations among the variables (32) are kept and the physical masses are positive (with $\lambda > 0$). For real parameters, a_S, (which may be for contraction or expansion) these solutions cannot be valid for long times otherwise m_i^2 become negative ($a_i < 0$)or too much large ($a_i > 0$).

The case of a increasingly larger scalar order parameter, $\tilde{\sigma}(t)$ with $a_S > 0$ from an initial time when $\tilde{\sigma} = 0$, the solution (33) can be valid for an interval of time until $\tilde{\sigma}(t) \to v \simeq 93$ MeV, eventually occurring in experimental situations [79, 49]. Considering that sigma and pion masses are zero at $t = 0$, it is reasonable to choose $C_s = \lambda v^2$ and $C_p = \lambda v^2$. It can be written that: $\mu_\sigma^2 = \lambda v^2(-1 + exp(a_S t))$, and a similar expression for the pion mass. This is consistent with a zero mass for the sigma when $\tilde{\sigma} = 0$ and it increases until its value when $\tilde{\sigma} \simeq 93$MeV. In this case the time scale needed for the pion mass reach its real value will be nearly given by:

$$\Delta t \simeq \frac{1}{a_s} Ln \left(1 + \frac{2.27}{\lambda}\right).$$

For a coupling constant $\lambda \simeq 20$ we obtain $\Delta t \simeq 0.11/a_s$fm. Larger the coupling constant λ faster the mass will vary until its value in the (true) vacuum which is an expected behavior. Meanwhile the sigma mass will vary accordingly because the flow was assumed to be the same $a_s = a_p$. The values of the parameters a_i, s, p may be different in a more complete calculation for $\tilde{\pi} = 0$ or not. Other different scenarios can be obtained from the above equations. The equations would also hold for non zero $\tilde{\pi}$ for example in conditions when the disoriented chiral condensates - DCC - could be expected to appear [80]. This kind of modification and time dependence of the masses and couplings can have relevant effects in experimental situations.

4 Elimination of UV Divergences: Renormalization

The Gaussian approximation is self consistent and it corresponds to a sum of the "cactus" Feynman diagrams, Hartree Bogoliubov approximation and to the leading order large N approximation [81, 82]. In the case of zero pion mass an infrared divergence appears in G_P which is not treated.

The regularized expression of the two point functions G_i (where i stands for pion or sigma) with a cutoff Λ in the momentum space, in the limit of large Λ, is given by:

$$G_i = G_i(\mu_i^2) = \frac{1}{8\pi^2}\left\{\Lambda^2 - \frac{\mu_i^2}{2}Ln\left(\frac{4\Lambda^2}{e\mu_i^2}\right)\right\}. \tag{35}$$

The GAP equations with the expression (35), assuming $\tilde{\pi} = 0$ or not, and considering only one cutoff, can be written as:

$$\begin{aligned}
2\mu_S^2 &= -\lambda v^2 + 3\lambda\tilde{\sigma}^2 + \lambda\frac{\Lambda^2}{2\pi^2} - \frac{3\lambda\mu_S^2}{16\pi^2}Ln\left(\frac{4\Lambda^2}{e\mu_S^2}\right) - \frac{\lambda\mu_P^2}{16\pi^2}Ln\left(\frac{4\Lambda^2}{e\mu_P^2}\right) + O(\tilde{\pi}), \\
2\mu_P^2 &= -\lambda v^2 + \lambda\tilde{\sigma}^2 + \lambda\frac{\Lambda^2}{2\pi^2} - \frac{3\lambda\mu_P^2}{16\pi^2}Ln\left(\frac{4\Lambda^2}{e\mu_P^2}\right) - \frac{\lambda\mu_S^2}{16\pi^2}Ln\left(\frac{4\Lambda^2}{e\mu_S^2}\right) + O(\tilde{\pi}) \to 0,
\end{aligned} \tag{36}$$

where the terms containing $\tilde{\pi}^2$ were not written explicitely ($O(\tilde{\pi})$). It is worth to remind the other way of writing μ_P^2, in expression (17) according to which $\mu_P^2 = 0$. There are other ways of eliminating the UV divergences of these equations. We follow the logics of the references quoted above: the GAP equation above with $\mu_s^2(\bar{\sigma} = 0) \equiv M_S^2$ is subtracted from the GAP equation with $\mu_s^2(\bar{\sigma} \neq 0)$. This is also considered for the pion GAP equation with $\mu_P^2(\bar{\pi} \neq 0)$ and $\mu_P^2(\bar{\pi} = 0) \equiv M_P^2$. This makes possible the pion and sigma masses to become equal at some energy scale ($M_S^2 = M_P^2$) or to be always different ($M_S^2 \neq M_P^2$).

To eliminate the UV divergences renormalized parameters are defined in these subtractions of the GAP equations. They can be written in terms of the bare parameters and mass scale parameters (M_S^2, M_P^2). Following this usual reasoning for each of the GAP equations they can be re-written as:

$$\begin{aligned}
\mu_S^2 &= M_S^2 + 3\lambda_{R,S}3\tilde{\sigma}^2 - \frac{3\mu_S^2\lambda_{R,S}}{16\pi^2}Ln\left(\frac{\mu_S^2}{M_S^2}\right) - \frac{\mu_P^2\lambda_{R,S}}{16\pi^2}Ln\left(\frac{\mu_P^2}{M_P^2}\right) + O(\tilde{\pi}), \\
\mu_P^2 &= M_P^2 + \lambda_{R,P}\tilde{\sigma}^2 - \frac{3\mu_P^2\lambda_{R,P}}{16\pi^2}Ln\left(\frac{\mu_P^2}{M_P^2}\right) - \frac{\mu_S^2\lambda_{R,P}}{16\pi^2}Ln\left(\frac{\mu_S^2}{M_S^2}\right) + O(\tilde{\pi}) \to 0.
\end{aligned} \tag{37}$$

In these expressions renormalized parameters determining the form of the renormalized potential were defined by:

$$\begin{aligned}
M_S^2 = \mu_{S,R}^2 &\equiv \lambda_{R,S}\left(-v^2 + \frac{\Lambda^2}{2\pi^2} - \frac{\mu_P^2}{16\pi^2}Ln\left(\frac{4\Lambda^2}{2M_P^2}\right)\right), \\
M_P^2 = \mu_{P,R}^2 &\equiv \lambda_{R,P}\left(-v^2 + \frac{\Lambda^2}{2\pi^2} - \frac{\mu_S^2}{16\pi^2}Ln\left(\frac{4\Lambda^2}{2M_S^2}\right)\right).
\end{aligned} \tag{38}$$

Renormalized coupling constants were also defined, namely:

$$\begin{aligned}\lambda_{R,S} &= \frac{\lambda}{1+\frac{3\lambda}{16\pi^2}Ln\left(\frac{2\Lambda^2}{eM_S^2}\right)},\\ \lambda_{R,P} &= \frac{\lambda}{1+\frac{3\lambda}{16\pi^2}Ln\left(\frac{2\Lambda^2}{eM_P^2}\right)}.\end{aligned} \tag{39}$$

These coupling constants are to be respectively the g_{σ^4} and g_{π^4} couplings. Their combination leads to a $g_{\sigma^2\pi^2}$ coupling constant. From the discussion above in expressions (37) it is seen that the sigma coupling $\lambda_{R,S}$ (for σ^4 vertex-type interaction) can be different from $\lambda_{R,P}$ (for π^4 vertex-type interaction). This means that the coupling $\sigma^2\pi^2$ could have two values depending on the process under consideration: for the sigma or pion self energies μ_i^2, corresponding respectively to the fourth term on the right hand side of each of the two GAP equations (37). In the Gaussian approach there is no coupling like $g_{\sigma\pi^2}$ which is obtained in the tree level of approximation with the usual shift of the scalar field. Each of the above couplings contribute to the renormalized finite theory of the finite energy density.

The resulting relations between the bare coupling constant and the renormalized ones, expressions (39), depend rather on the ratio of the cutoff to the scale parameters Λ^2/M_i^2. Considering that $M_P = M_S$ the couplings satify $\lambda_{R,S} = \lambda_{R,P}$, as it is usually considered. Otherwise for $M_P \neq M_S$, for fixed ratios in the limit of very large cutoff, there are several scenarios. In the limit of $\Lambda \to \infty$ for fixed $\Lambda/M_S < 1$ and $\Lambda/M_P > 1$ (or the inverse situation) it follows that the sign of the renormalized couplings may be even different. These expressions for the renormalized coupling constant seems to exhibit features related to asymptotic freedom as it was shown for the $\lambda\phi^4$ model [83, 73]. This model keeps many similarities to the Linear Sigma model. The appearance of the two couplings (39) may be associated to another kind of spontaneous symmetry breaking in this model.

4.1 Stationary-like situation

Next we show the renormalization of the stationary-like solutions found in the section 4 considering equal mass scale parameters $M = M_S = M_P$. In those cases the equations of motion (13) become GAP-like equations. They can be written as:

$$\begin{aligned}\mu_S^2 &= \lambda\left(3\tilde{\sigma}^2+\tilde{\pi}^2+3G_S(\tilde{\mu}_S^2)+G_P(\tilde{\mu}_P^2)-v^2\right)-\frac{a_S^2}{4},\\ \mu_P^2 &= \lambda\left(3\tilde{\pi}^2+\tilde{\sigma}^2+3G_P(\tilde{\mu}_P^2)+G_S(\tilde{\mu}_S^2)-v^2\right)-\frac{a_P^2}{4},\end{aligned} \tag{40}$$

where $\tilde{\mu}_i^2 = \mu_i^2 - a_i^2/4$. With the expression of the functions G_i in terms of the cutoff Λ and with the extra terms with only one mass scale parameter M the following renormalized expressions is obtained:

$$\begin{array}{l} \mu_S^2 = \mu_{R,S}^2 - \frac{a_s^2}{4} + 3\tilde{\lambda}_{R,S}\tilde{\sigma}^2 - \frac{3\tilde{\mu}_S^2\tilde{\lambda}_{R,S}}{16\pi^2} Ln\left(\frac{\tilde{\mu}_S^2}{M^2}\right) - \frac{\tilde{\mu}_P^2\tilde{\lambda}_{R,S}}{16\pi^2} Ln\left(\frac{\tilde{\mu}_P^2}{M^2}\right) + O(\tilde{\pi}), \\ \mu_P^2 = \mu_{R,P}^2 - \frac{a_P^2}{4} + \tilde{\lambda}_{R,P}\tilde{\sigma}^2 - \frac{3\tilde{\mu}_P^2\tilde{\lambda}_{R,P}}{16\pi^2} Ln\left(\frac{\tilde{\mu}_P^2}{M^2}\right) - \frac{\tilde{\mu}_S^2\tilde{\lambda}_{R,P}}{16\pi^2} Ln\left(\frac{\tilde{\mu}_S^2}{M^2}\right) + O(\tilde{\pi}). \end{array} \quad (41)$$

In these expressions, renormalized masses determining the form of the renormalized potential were defined by:

$$\begin{array}{l} \mu_{S,R}^2 = \tilde{\lambda}_{R,S}\left(-v^2 + \frac{\Lambda^2}{2\pi^2} - \frac{\tilde{\mu}_P^2}{16\pi^2} Ln\left(\frac{4\Lambda^2}{2M^2}\right)\right), \\ \mu_{P,R}^2 = \tilde{\lambda}_{R,P}\left(-v^2 + \frac{\Lambda^2}{2\pi^2} - \frac{\tilde{\mu}_S^2}{16\pi^2} Ln\left(\frac{4\Lambda^2}{2M^2}\right)\right). \end{array} \quad (42)$$

Renormalized coupling constants were also defined, namely:

$$\begin{array}{l} \tilde{\lambda}_{R,S} = \frac{\lambda}{1 + \frac{3\lambda}{16\pi^2}\frac{\tilde{\mu}_s^2}{\mu_s^2 + \frac{a_s^2}{4}} Ln\left(\frac{2\Lambda^2}{eM^2}\right)}, \\ \tilde{\lambda}_{R,P} = \frac{\lambda}{1 + \frac{3\lambda}{16\pi^2}\frac{\tilde{\mu}_p^2}{\mu_p^2 + \frac{a_p^2}{4}} Ln\left(\frac{2\Lambda^2}{eM^2}\right)}. \end{array} \quad (43)$$

In two limits, namely either with $a_i \to 0$ or with $a_P = a_S$ (keeping $\mu_S = \mu_P$), the usual renormalized expressions with only one coupling constant are obtained. This limit of equal coupling constants $\tilde{\lambda}_{R,S} = \tilde{\lambda}_{R,P}$ can also be obtained with two mass scales M_i within the time-dependent picture as shown in the previous section for the static limit. Considering the particular values for the masses given by: $4\tilde{\mu}_i = a_i^2$ the resulting renormalized couplings reduce to $\tilde{\lambda}_{R,i} = \lambda$. In the most general case however there would appear two renormalized coupling constants in this time dependent picture. Therefore two ways of obtaining different renormalized coupling constants were found within the same self-interacting model.

4.2 Other remarks

The Gaussian approximation considers two components for a field: a "classical" one, $\bar{\sigma} =< \sigma >$, whose expected value in the vacuum is non zero like in a condensed state, and a "quantum" one whose expected value in the vacuum is zero and it yields the two point Green's functions $G_i(\mathbf{x},\mathbf{y}) \propto < \phi_i^2 >$. The former appears for a spontaneous symmetry breaking which modifies the ground state that has a privileged direction (or more directions). Usually there are two degenerated points of minimum characteristic of SSB systems [1]. The "quantum part" of the field corresponds to the physical particles through the creation and annihilation operators. These two parts have crossed terms in the energy density [46, 49].

These two components do not have the same "mass" necessarily. They can be calculated respectively with:

$$"m_{\tilde{\sigma}}^2" \text{ from } \left.\frac{d^2\mathcal{H}}{d\tilde{\sigma}^2}\right|_{\tilde{\sigma}=0} \qquad m_\sigma^2 \equiv \mu_S^2 \text{ from } \frac{d\mathcal{H}}{dG_S}, \tag{44}$$

and analogously for the pion. The first of these masses is related to the condition of stability of the effective potential in the broken symmetry direction while from the second one yields the GAP equation, expected to be "physical mass" (14) [73, 46, 48]. In these definitions it was considered a regularized expression for the energy density which are renormalized after the variational procedure as it was shown in the last section. A slightly different approach, which does not change our conclusions, was suggested in [84] for which all the information concerning the phase structure of the model would be given instead by renormalized parameters. However there still is the usual ambiguity of choosing an absolute value for the true vacuum energy density which will not be really addressed in this work [84]. The usual procedure employed in the renormalization scheme is to subtract the expressions of the averaged energy in the symmetric phase of the potential ($\bar{\sigma} = 0$) from $\mathcal{H}$ in the asymmetric phase $\bar{\sigma} \neq 0$ [73, 64]. A mass scale (free) parameter still is to be fixed as it was shown in the last section. The renormalized mass and coupling constant are the same as those obtained from the GAP equations.

4.3 Symmetry Restoration and particle production?

The scalar condensate can be alternatively (or equivalently) introduced by means of a Bogoliubov transformation from the ground state in the symmetric phase [46, 85, 86]. Considering this ground state fulfilled with the condensate, with its quantum/virtual fluctuations, it can be excited producing particles. This can occur in the high energy/relativistic heavy ion collisions in whose conditions the restoration of chiral symmetry is searched. Considering the QCD scalar condensate identified to a measure of the density of quark and antiquarks: $< \bar{q}_R q_L + \bar{q}_L q_R > \simeq (250\text{MeV})^3$ the restoration of chiral symmetry implies a symmetrical production of particles with quark and anti-quark content (either equal number of baryons-anti-baryons and/or mesons with quark-anti-quark structure). This goes along with experimental data in which the baryons/anti-baryons ratio goes to one at the energies in which chiral symmetry is expected to occur [79]. Assume, next, a resulting *fireball* in which quarks deconfine from a rhic with a volume of approximatedly $V \simeq 1500$ fm^3.

From this configuration around 1500 quarks and antiquarks would be excited from the ("evaporated") vacuum eventually favoring the production nearly 750 (light) mesons, lighter than baryons, without computing the eventual formation of equal number of baryons and anti-baryons. This very crude and naive estimate does not include the structure of the colliding nuclei neither does consider that the spatial distribution of the condensate can be modified as the colliding nuclei and particles push the condensate away from the central region of the collision, *squashing* and *smashing* the vacuum [49]. An estimatation of the spatial modifications of the condensate due to the fireball can also be considered.

5 Conserved and partially conserved (averaged) currents

The axial and vector transformation of the fields, scalar σ and pseudoscalar π, with infinitesimal parameters α_a can be written respectively as:

$$\begin{aligned} &\sigma \to \sigma + \alpha_a \pi_a, \qquad \pi_a \to \pi_a - \alpha_a \sigma. \\ &\pi_a \to \pi_a - \beta \pi_a, \qquad \sigma \to \sigma + \beta \sigma. \end{aligned} \tag{45}$$

The divergences of the vector and axial currents are found by performing the corresponding transformations in the Lagrangian density and the associated variations:

$$\partial_\mu j^\mu_{a,axial} = \frac{\partial \mathcal{L}}{\partial \alpha_a}, \qquad \partial_\mu j^\mu_{a,vector} = \frac{\partial \mathcal{L}}{\partial \beta}, \tag{46}$$

Because the Gaussian variational method deals with averaged values, the averaged value of the action was calculated with the trial wave-functionals after the transformations. The divergence of the corresponding current is found with expressions (46), i.e., an averaged value.

The average of the action usually considered for the variational principle can be written as a sum of two terms which in the time independent limit reduces to the average of $-\hat{H}$. The variations with relation to the parameters of the transformations yield a zero divergence of the vector current (conserved current) and the following value for the "averaged divergence of the axial current":

$$< \partial_\mu j^\mu_{a,axial} >= -2\tilde{\sigma}(< \pi^2 > - < \sigma^2 >)\tilde{\pi}_a + \, O(G^{-2}\tilde{\pi}\tilde{\sigma}) + ... \tag{47}$$

Considering equation (14) we can write the above expression as:

$$< \partial_\mu j^\mu_{a,axial} >= -\frac{8}{9} \, \tilde{\sigma} \, \mu_P^2 \, \tilde{\pi}_a \, + \, O(\mu_P^4, \mu_S^4), \tag{48}$$

where $O(\mu^4)$ represents higher order terms in the masses. The pion decay constant, in this level of approximation, can be written as: $f_\pi = -\chi\tilde{\sigma}$, where $\chi = 8/9$ or $\chi = 1$. There is a non zero averaged value of the pion, $\tilde{\pi}_a =< \pi_a >$, as a consequence of the variational method using averages. Without the averages, the divergence of this current would be proportional to the pion field and expression (48) would be, almost exactly, the usual partially conserved axial current (PCAC), except for the numerical factor different from 1. It is interesting to note that the pion mass which is present in expression (48) was not introduced as a symmetry breaking term in the Lagrangian. It is tempting to consider this expression as an "averaged PCAC" with calculable higher order corrections in the pion and sigma masses. For $\tilde{\pi}_a = 0$ (as it occurs in the vacuum) it would result a completely conserved current. This expression seems to keep similarities to an anomaly, similarly as it may occur in other models with the calculation of quantum loops [67, 1].

6 Summary and Comments

Aspects associated to spontaneous symmetry breaking were investigated in the linear sigma model.

Classes of analytical expressions representing solutions of the time dependent linear sigma model were found in which effective masses were defined driving the the time evolution of the system. Two kinds of solutions were explicitely shown: with continuous increase or decrease of the condensate, considering spatial inhomogeinities or not. Pion mass is found to be zero for a chiral invariant Lagrangian in the Gaussian approach in agreement with other works respecting the Goldstone theorem. On the other hand contributions for the masses due to dynamical time dependent reasons were discussed.

The renormalized coupling constant of the linear sigma model can assume different values for the sigma and pion interactions due to two reasons: from the temporal evolution and in the static limit . Two renormalization mass scales were introduced in the static calculation and only one in the time-dependent case for a particular class of solutions. Consequently the ground state (and the temporal dynamics) may not be invariant under the corresponding (chiral) transformations although the Lagrangian is. One may find, however, that due to different breakdowns of the symmetry (the scalar condensate and appearance of different couplings) the masses of the scalar and pseudoscalar particles may become less or more degenerated. Experimental evidences for this effect were not yet presented and can be expected to be present in the light hadronic phenomenology. This picture can be obtained from expressions (41). Similar conclusions were reached for particular (stationary-like) solutions of the time-dependent situation with only one mass scale renormalization parameter.

Some aspects of the ground state fulfilled with the scalar condensate were also briefly analysed. This condensate can be excited in very energetic experimental conditions which squash and smash the vacuum yielding non condensed particles and exotic condensate configurations [49] It was shown that the chiral radius, the parameter which constraint the fields on the functional sphere of the internal chiral space,

v, does not necessarily (always) can be identified to the pion decay constant and to the scalar (sigma) condensate associated to the QCD $< \bar{q}q >$ condensate in agreement with other results [87].

Besides that a sort of averaged partially conserved axial current was deduced as if it were "dynamically generated" for the parameter $\bar{\pi} \neq 0$ resulting in $m_\pi \neq 0$. It is an expression formally similar to the PCAC for really massive pions. However it is rather as a feature of the variational approximation when considering the classical field $\bar{\pi} \neq 0$ during the calculations before setting it to zero. Besides the description of hadronic properties in the vacuum effectively (i.e. not in terms of the quark-gluon degrees of freedom which are confined anyway), this seems to be of interest for other related and intensively subjects studied nowadays such as: the behavior of each of the variables involved in models, like the LSM, at extreme conditions of density, temperature and even isospin *in a hadronic/nuclear medium* to quote few works [35, 88, 89, 70, 90, 91, 86, 65, 84].

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