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Space-time noncommutativity with a bifermionic parameter

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Abstract

We consider a Moyal plane and propose to make the noncommutativity parameter $\Theta^{\mu\nu}$ bifermionic, i.e., composed of two fermionic (Grassmann odd) parameters. The Moyal product then contains a finite number of derivatives, which allows to avoid many difficulties of the standard approach. As an example of realization of such an idea, we construct a two-dimensional field theory model based on the Moyal product with a bifermionic parameter and show that it has a locally conserved energy-momentum tensor. The model has no problems with the canonical quantization and appears to be renormalizable.

Noncommutativity enters field theory from many sides [1], including string theory and Quantum Hall Effect. In [2] it was shown in particular that a combination of quantum mechanics and classical gravity leads to uncertainty relations between space-time coordinates, so that these coordinates become noncommutative (NC). It is rather possible that on a more fundamental level of a fully quantized field theory the NC parameter is *Grassmann even and bifermionic*, i.e., is a composite of two fermionic (Grassmann odd) parameters. This possibility will be studied in this paper with a simple example of two dimensional quantum field theory. We will show that having a bifermionic NC parameter helps to overcome, or at least to by-pass, many difficulties of the standard approach to NC field theories.

Previously, a bifermionic NC parameter appeared in the context of a construction of Lorentz covariant supersymmetric brackets [3]. However, consequences of such NC parameter for quantum field theory were not discussed.

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The standard Moyal product is non-local. As a consequence, field theories based on this product are also non-local. Therefore, many essential properties of local theories disappear in their NC counterparts. For example, it is hard to define a locally conserved energy-momentum tensor [4]. Even more problems appear in space-time NC theories. The presence on an infinite number of time derivatives makes the standard canonical Hamiltonian approach hardly possible. One can instead develop an Ostrogradski-type formalism [5] (see also [6]) at the expense of introducing a new evolution parameter in addition to the usual time coordinate. Unfortunately, the canonical field operators then depend on this additional parameter thus preventing us from using the full power of canonical methods. Note, that one can define a Poisson structure which is suitable for analyzing gauge symmetries in space-time NC theories by looking at their constraint algebra [7]. Although this method is rather useful in working with particular models [8], it is hard to say whether it can be applied to quantization. As a consequence of these difficulties with constructing a canonical Hamiltonian formalism, the unitarity in space-time noncommutative theories is not guaranteed, see [10]. There is a rather radical approach to NC theories, based on twisted quantization rules, which avoids the problems mentioned above, but also removes practically all effects of the noncommutativity (see [11] and references therein).

We suggest the following geometry. Consider a real plane \mathbb{R}^2 with the coordinates being Grassmann even elements of Berezin algebra (see [9]) (bosonic variables of pseudoclassical mechanics) and a star product such that $x^\mu \star x^\nu - x^\nu \star x^\mu = i\Theta^{\mu\nu}$, and $\Theta^{\mu\nu}$ is a real Grassmann even constant. Note, that in two dimensions this inevitably means that space and time do not commute. We take the simplest possible bifermionic representation for $\Theta^{\mu\nu}$, namely

$$\Theta^{\mu\nu} = i\theta^\mu\theta^\nu, \quad (1)$$

where θ^μ are constant real fermions (Grassmann odd constants). Bifermionic constants appear naturally in pseudoclassical models of relativistic particles with spin, see e.g. [12, 13]. In such models these constants become numbers upon quantization¹

The usual Moyal product with the NC parameter given by (1) terminates at the second term

$$\begin{aligned} f_1 \star f_2 &= \exp\left(\frac{i}{2}\Theta^{\mu\nu}\partial_\mu^x\partial_\nu^y\right) f_1(x)f_2(y)|_{y=x} \\ &= f_1 \cdot f_2 - \frac{1}{2}\theta^0\theta^1(\partial_0 f_1\partial_1 f_2 - \partial_1 f_1\partial_0 f_2). \end{aligned} \quad (2)$$

Symmetrized star-product therefore equals to the ordinary one. In particular, $f \star f = f \cdot f$.

¹For completeness we note that there is an alternative approach [14] based on introduction of additional fields, which leads, however, to equivalent results, see [15] for a discussion.

Let us construct a field theory model on the NC manifold we have just defined. We take the action in the form

$$S = \int d^2x \left(\frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 + \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m_1^2 \varphi_1^2 - \frac{1}{2}m_2^2 \varphi_2^2 - \frac{1}{2}m^2 \varphi^2 - ei[\varphi_1, \varphi_2]_\star \star \varphi \star \varphi - \lambda \varphi_\star^n \right). \quad (3)$$

The signature of space-time is $(+-)$. φ , φ_1 and φ_2 are real scalar fields. Let us explain briefly where different terms come from. We need at least two scalar fields, φ_1 and φ_2 , to be able to construct a non-trivial star product $[\varphi_1, \varphi_2]_\star = \varphi_1 \star \varphi_2 - \varphi_2 \star \varphi_1$. The space-time integral of a star-commutator vanishes, therefore to obtain a nonzero term in the action we have to multiply it by another field, which we choose as $\varphi \star \varphi$. Star-commutator of two real fields is imaginary, and we multiply our interaction term by an imaginary coupling constant ie . We add standard kinetic and mass terms for all fields and a self interaction term $\lambda \varphi_\star^n$ with n -th star-power of φ . The last term is needed to make the quantum theory based on (3) non-trivial, see below.

By using (2) one can rewrite the interaction terms in (3) as

$$S_{\text{int}} = \int d^2x (ei\theta^0 \theta^1 (\partial_0 \varphi_1 \partial_1 \varphi_2 - \partial_1 \varphi_1 \partial_0 \varphi_2) \varphi^2 - \lambda \varphi^n). \quad (4)$$

We can also rewrite this term in a manifestly Lorentz-invariant form:

$$S_{\text{int}} = \int d^2x \left(-\frac{1}{2} ei \epsilon_{\rho\sigma} \theta^\rho \theta^\sigma \epsilon^{\mu\nu} \partial_\mu \varphi_1 \partial_\nu \varphi_2 - \lambda \varphi^n \right). \quad (5)$$

Note, that since $\epsilon_{\rho\sigma}$ is Lorentz invariant, one must not transform the external parameters θ^ρ in order to achieve invariance of the action. This is a peculiar feature of two-dimensional theories. In higher dimensions one should probably use the twisted Poincare invariance like in NC theories with generic Θ [16].

The equations of motion for this model read:

$$\begin{aligned} -\square \varphi - m^2 \varphi - \lambda n \varphi^{n-1} + 2ie\theta^0 \theta^1 \varphi (\partial_0 \varphi_1 \partial_1 \varphi_2 - \partial_1 \varphi_1 \partial_0 \varphi_2) &= 0, \\ -\square \varphi_1 - m_1^2 \varphi_1 - 2ie\theta^0 \theta^1 \varphi (\partial_0 \varphi \partial_1 \varphi_2 - \partial_0 \varphi_2 \partial_1 \varphi) &= 0, \\ -\square \varphi_2 - m_2^2 \varphi_2 - 2ie\theta^0 \theta^1 \varphi (\partial_0 \varphi_1 \partial_1 \varphi - \partial_1 \varphi_1 \partial_0 \varphi) &= 0. \end{aligned} \quad (6)$$

In NC theories it is problematic to define a locally conserved energy-momentum tensor [4]. In our model no such problems arise. Since the action (3) contains first derivatives at most, the standard energy-momentum tensor

$$T^\nu_\mu = \varphi_{;\mu} \frac{\delta \mathcal{L}}{\delta \varphi_{;\nu}} - \delta^\nu_\mu \mathcal{L} \quad (7)$$

(where \mathcal{L} is the Lagrangian density and $\varphi_{;\mu} \equiv \partial_\mu \varphi$) is locally conserved. Indeed, by taking the components

$$\begin{aligned}
T_0^0 &= \frac{1}{2}((\partial_0 \varphi)^2 + (\partial_1 \varphi)^2 + m^2 \varphi^2 + (\partial_0 \varphi_1)^2 + (\partial_1 \varphi_1)^2 + m_1^2 \varphi_1^2 \\
&\quad + (\partial_0 \varphi_2)^2 + (\partial_1 \varphi_2)^2 + m_2^2 \varphi_2^2) + \lambda \varphi^n, \\
T_1^1 &= \frac{1}{2}(-(\partial_0 \varphi)^2 - (\partial_1 \varphi)^2 + m^2 \varphi^2 - (\partial_0 \varphi_1)^2 - (\partial_1 \varphi_1)^2 + m_1^2 \varphi_1^2 \\
&\quad - (\partial_0 \varphi_2)^2 - (\partial_1 \varphi_2)^2 + m_2^2 \varphi_2^2) + \lambda \varphi^n, \\
T_1^0 &= -T_0^1 = \partial_0 \varphi \partial_1 \varphi + \partial_0 \varphi_1 \partial_1 \varphi_1 + \partial_0 \varphi_2 \partial_1 \varphi_2
\end{aligned} \tag{8}$$

and using the equations of motion (6) one can easily check the conservation equation

$$\partial_\mu T^\mu_\nu = 0. \tag{9}$$

It is interesting to note that the energy-momentum tensor (8) does not contain the NC coupling $e\theta^0\theta^1$. This can be understood looking at (5). In curved-space generalizations the indices μ, ν of the partial derivatives should be regarded as curved (world) ones, so that $e^{\mu\nu}$ becomes a tensor with the components $\pm g^{-1/2}$ (with g being determinant of the metric). If the indices of $\theta^{0,1}$ are regarded as flat (tangential), then $\epsilon_{\rho\sigma} = \pm 1$, the whole first term in (5) does not depend on the metric and, therefore, does not contribute to the metric stress-energy tensor.

Since the model contains first time derivatives at most, the Hamiltonian analysis proceeds without any complications. The canonical momenta are defined as variational derivatives of the action w.r.t. temporal derivatives of corresponding fields

$$\begin{aligned}
\pi &= \partial_0 \varphi, \\
\pi_1 &= \partial_0 \varphi_1 + ei\theta^0\theta^1 \partial_1 \varphi_2 \varphi^2, \\
\pi_2 &= \partial_0 \varphi_2 - ei\theta^0\theta^1 \partial_1 \varphi_1 \varphi^2.
\end{aligned} \tag{10}$$

No constraints appear in this model.

The canonical Hamiltonian reads

$$\begin{aligned}
H &= \int dx (\pi \partial_0 \varphi + \pi_1 \partial_0 \varphi_1 + \pi_2 \partial_0 \varphi_2 - \mathcal{L}) \\
&= \int dx \left(\frac{1}{2}(\pi^2 + (\partial_1 \varphi)^2 + m^2 \varphi^2 + \pi_1^2 + (\partial_1 \varphi_1)^2 + m_1^2 \varphi_1^2 \right. \\
&\quad \left. + \pi_2^2 + (\partial_1 \varphi_2)^2 + m_2^2 \varphi_2^2) + \lambda \varphi^n + ei\theta^0\theta^1 (-\pi_1 \partial_1 \varphi_2 + \pi_2 \partial_1 \varphi_1) \varphi^2 \right).
\end{aligned} \tag{11}$$

Note, that after substituting in (11) the canonical momenta in terms of temporal derivatives of the fields by means of (10) one obtains the space integral of T_0^0 .

The canonical Hamiltonian is conserved by the construction, which can also be checked by direct calculations.

Let us now proceed with the quantization. Since there are no constraints, we simply replace the fields and their momenta by operators (which will be denoted by the same letters with hats) that obey the canonical commutation relations:

$$[\hat{\varphi}(t, x), \hat{\pi}(t, x')] = [\hat{\varphi}_1(t, x), \hat{\pi}_1(t, x')] = [\hat{\varphi}_2(t, x), \hat{\pi}_2(t, x')] = i\hbar\delta(x - x'). \quad (12)$$

The Hamiltonian (11) can be transformed into an operator without any ordering ambiguities. The Hamiltonian operator \hat{H} obtained in this way is not positive definite. However, the evolution operator constructed from \hat{H} can be treated perturbatively w.r.t. the charge e , and any perturbation series terminates at the first term. One can then pass to the path integral formulation and define Feynman rules in a perfectly standard way. One can think of θ^μ as of constant fermion fields. Note, that in the contrast to [12, 15] we have no constraints here which fix the value of the bifermionic parameter to a number. Upon quantization in the effective action our fermions θ^μ are replaced by their averages (precisely as usual fermionic fields). However, a trace of their Grassmann nature remains: the perturbative series terminate. Therefore, in deriving the Feynman diagrams from the path integral one can still consider θ^μ as a fermion (again, precisely as in the case of fields) and set it to a number at the end of the calculation of quantum amplitudes.

Let us study renormalization of our model. Take $n = 4$ in (3). If we neglect for a while the interaction described by the charge e we obtain a model consisting of two free massive scalars (φ_1 and φ_2) and another massive scalar with a φ^4 selfinteraction. This model is known to be multiplicatively renormalizable. Now, let us add the missing term. Due to the fermionic nature of θ^0 and θ^1 the vertex generated by this interaction can enter a Feynman diagram at most once. It is easy to see that the only possible one-particle irreducible power-counting divergent diagram with the charge e is the one of Fig. 1. However, because of the structure $(\partial_0\varphi_1\partial_1\varphi_2 - \partial_1\varphi_1\partial_0\varphi_2)$ in the interaction term, the diagram is proportional to $k_0k_1 - k_1k_0 = 0$, i.e. this diagram vanishes identically. We conclude, that noncommutativity does not bring any new divergences in the model, so that it remains renormalizable. Note, that there are non-trivial finite diagrams involving the noncommutative interaction. For example, by using the φ^4 vertex one can add an arbitrary number of pairs of external φ legs to the diagram of Fig. 1.

To summarize, we have suggested a space-time NC model with a bifermionic NC parameter. In this model the Moyal product terminates at the second term, and we have no problems caused by the nonlocality. In particular, this model has a locally conserved energy-momentum tensor and a well defined Hamiltonian. Also, we do not see any problem with quantization by the canonical methods and by the path integral². Moreover, this model appears to be renormalizable.

What we have presented here is just a general idea which has to be investigated further to put it on firmer grounds. For example, the unitarity has to be checked

²Another example of a space-time NC theory having no problems with the path integral to all order of perturbation theory is an integrable 2D gravity [17].

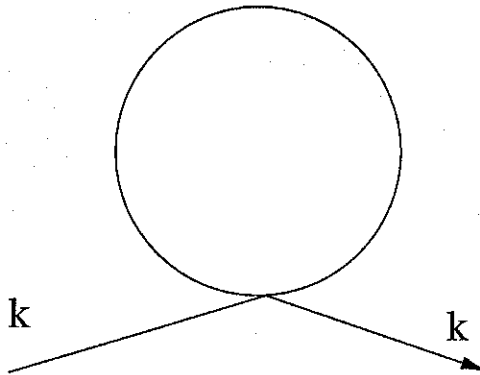


Figure 1: The only power-counting divergent diagram with the new interaction. Left and right legs correspond to φ_1 and φ_2 respectively. The closed loop corresponds to φ .

explicitly. Although, due to the locality, there should be no UV/IR mixing [18], the IR behavior of our model has to be explored. We used the simplest possible representation for the bifermionic constant. Perhaps physically more preferable would be to represent it through constant spinors (instead of constant vector fermions) as $\Theta^{\mu\nu} \propto \bar{\psi}[\gamma^\mu, \gamma^\nu]\psi$. In two dimension and for Majorana fermions this representation is essentially equivalent to (1), but in higher dimensions it opens many new possibilities, in particular, for constructing supersymmetric models.

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