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Pseudoclassical description of scalar particle in non-Abelian background and path-integral representations

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Abstract

Path-integral representations for a scalar particle propagator in non-Abelian external backgrounds are derived. To this aim, we generalize the procedure proposed by Gitman and Schvartsman 1993 of path-integral construction to any representation of $SU(N)$ given in terms of antisymmetric generators. And for arbitrary representations of $SU(N)$, we present an alternative construction by means of fermionic coherent states. From the path-integral representations we derive pseudoclassical actions for a scalar particle placed in non-Abelian backgrounds. These actions are classically analyzed and then quantized to prove their consistency.

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1 Introduction

QFT with external backgrounds is a good approach for describing many physical situations and effects. If the external background is strong enough it has to be taken into account non-perturbatively. The corresponding methods for QED are well developed and were fruitfully applied for a number of calculations, see e.g. [1] and citations therein. The external background concept in non-Abelian QFT is less developed and meets some difficulties (there is no gauge invariant way of introducing a non-Abelian external field). However, the undeniable existence of physical situations where there is a sufficiently strong quantized non-Abelian field often serves as a physical justification for treating this field as an external classical field, in spite of the above mentioned problem. Interesting physically meaningful results obtained in this conceptual framework serve as an additional justification for it. We can point out calculations of one-loop effective actions in constant non-Abelian external fields [2, 3, 4] that were used for constructing the true QCD vacuum, see [4, 5, 6, 7]. One also ought to mention the description of phase-transitions in cosmological QCD [8], non-pertubative parton production from vacuum by a classical $SU(3)$ [9] and $SU(2)$ [10] chromoelectric field, boundary conditions and topological effects of the vacuum in the presence of a non-homogenous external magnetic field in the form of a flux tube [11, 12], and so on.

The key objects in nonperturbative (with respect to the background) QFT with a non-Abelian background are scalar and spinning particle propagators in the corresponding non-Abelian external

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field. Exact solutions for such objects allow one to obtain by an integration one-loop results for various physical quantities. Moreover, path-integral representations for the propagators may be useful in obtaining exact solutions, which could then be used in calculations. Manifold path-integral representations for scalar and spinning particle propagators were constructed and calculated for various Abelian backgrounds in [13, 14, 15, 16, 17, 18, 19]. It turned out that such representations are also useful for deriving the so-called pseudoclassical actions for spinning particles, see [16, 17, 20]. Some path-integral representations for propagators in non-Abelian backgrounds and problems related to the pseudoclassical description of isospin were studied in [21, 22, 14]. We recall that a classical theory for a Yang-Mills particle was first constructed from the classical limit of the Yang-Mills field equations by Wong [23]. Afterwards, Chen and Dresden [24] showed that the Yang-Mills field equations imply the equations of motion for a test particle with isotopic spin in a way similar as the Einstein equations imply the equations of a massive test particle. Casalbuoni et.al. [22] obtained a gauge-invariant Lagrangian description of scalar and spinning particles with isotopic spin, where Grassmann variables describe the internal degrees of freedom at the classical level, so that quantization gives finite-dimensional representations of the gauge group. Balachandran et.al. [21] applied Dirac quantization to a pseudoclassical Lagrangian formulation of scalar and spinning particles interacting with a non-Abelian gauge field, and additionally developed the method we use here to obtain the irreducible representations of isospin. In [18], the isospinor structure of the propagator of a scalar relativistic particle in the fundamental representation of $SU(2)$ is derived from a path-integral representation using methods developed for the case of the spinning particle.

In the present article we return once again to these problems for the case of a scalar particle with isospin placed in various non-Abelian external backgrounds. We point out that a quantized scalar field in a non-Abelian background has been put forward as a tentative explanation of QCD confinement by means of a massive scalar particle (dilaton) [25], and also appears in the form of fundamental scalars coupled to gauge curvature terms in string theory [26].

We construct path-integral representations for the scalar particle propagator from two approaches: one is a generalization of the procedure proposed in [18] to any representation of $SU(N)$ given in terms of antisymmetric generators, while the other is a constructed using fermionic coherent states valid for arbitrary representations of $SU(N)$. The latter approach is a modification of the path-integral representation of the Dirac propagator by means of fermionic coherent states presented in [14]. In both cases we derive the pseudoclassical actions for a scalar particle in non-Abelian backgrounds, and quantize them to prove their consistency. In the Appendix, we put some technical details and proofs. The developed techniques can be easily generalized to the case of a spinning particle in non-Abelian and gravitational backgrounds. Such a generalization is the subject of our next publication.

2 Propagator representations

The causal propagator for a relativistic scalar particle interacting with a $su(N)$ valued external field $\mathcal{A}_\mu$ in Minkowski spacetime (in natural units $\hbar = c = 1$) is described by the equation

$$\left(\mathcal{P}^2 - m^2\right)^\alpha_{\ \beta} D^\beta_{\ \gamma}(x,y) = -\delta^\alpha_\gamma \delta^4(x-y) \, , \; \mathcal{P}_\mu = i\partial_\mu - q\mathcal{A}_\mu \, , \tag{1}$$

where $\mathcal{A}_\mu = A^a_\mu t^\alpha_{a\beta}$ is a linear combination of the traceless hermitian matrices $t^\alpha_{a\beta}$, $a = 1,..,N^2-1$ which are the generators of the Lie algebra $su(N)$ in an $n \times n$ irreducible matrix representation whose indices are labeled by greek letters from the beginning of the alphabet, α,β,γ,etc., $\alpha = 1, ..., N$. Since

$SU(N)$ is a compact group, there is a basis where its structure constants are totally antisymmetric and purely imaginary,

$$[t_a, t_b] = f_{ab}^c t_c,\; f_{ab}^c \equiv f_{[abc]}\,, \tag{2}$$

and the generators can be normalized as $\mathrm{tr}\,(t_a t_b) = 1/2\delta_{ab}$.

In the following we will consider two different realizations of the Lie algebra (2) of $su(N)$. The first realization will be in terms of creation and annihilation operators defined on a suitable Fock space, and the second realization will be in terms of the generators of a suitable Clifford algebra.

I. Let us consider the first realization. Consider an abstract Hilbert space $\mathcal{H}$ which is the direct product of the usual representation space for the Heisenberg algebra, whose basis vectors are denoted as $|x\rangle$,

$$\hat{x}^\mu |x\rangle = x^\mu |x\rangle\;,\;\langle x|y\rangle = \delta^4(x-y)\;,\;\int d^4x\,|x\rangle\langle x| = I\,,$$
$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu\;,\;\langle x|\,\hat{p}_\mu\,|y\rangle = -i\partial_\mu \delta^4(x-y)\;, \tag{3}$$

and an abstract Hilbert space V which we do not specify for the time being, but whose orthonormal basis vectors are $|\alpha\rangle$, $\alpha = 1, ..., n$,

$$\langle\alpha|\beta\rangle = \delta_{\alpha\beta}\,,\;\sum_{\alpha=1}^{n}|\alpha\rangle\langle\alpha| = I\,. \tag{4}$$

Thus, the abstract Hilbert space $\mathcal{H} = H \otimes V$ has the orthonormal basis $|x,\alpha\rangle = |x\rangle \otimes |\alpha\rangle$, $\langle x,\alpha|y,\beta\rangle = \delta^4(x-y)\,\delta_{\alpha\beta}$.

Next, we interpret the matrix operators appearing in (1), as matrix elements of operators in $\mathcal{H}$. With this in mind, the propagator $D^\alpha_\beta(x,y)$ is the matrix element of an abstract operator $\hat{D}$,

$$D(x,y)^\alpha_\beta = \langle x,\alpha|\,\hat{D}\,|y,\beta\rangle\;, \tag{5}$$

and the generators $t^\alpha_{a\beta}$ are matrix elements of the operators $\hat{t}_a$,

$$\langle\alpha|\,\hat{t}_a\,|\beta\rangle = t^\alpha_{a\beta}\,.$$

We note that if the matrix elements of the operators $\hat{t}_a$ are generators of a representation the algebra $su(N)$, then so are the operators themselves:

$$[t_a, t_b]^\alpha_\beta = f^c_{ab} t^\alpha_{c\beta} \Leftrightarrow \left[\hat{t}_a, \hat{t}_b\right] = f^c_{ab}\hat{t}_c\,. \tag{6}$$

Using the operators just defined, one can write (1) in operator form,

$$\left(\hat{P}^2 - m^2\right)\hat{D} = -I\,,$$

where

$$\hat{P}_\mu = -\hat{p}_\mu - q\hat{A}_\mu\,,\;\hat{A}_\mu = A^a_\mu(\hat{x})\,\hat{t}_a\,,\;\langle x,\alpha|\,\hat{P}_\mu\,|y,\beta\rangle = \left(i\partial_\mu\delta_{\alpha\beta} - qA_\mu(x)\,t^\alpha_{a\beta}\right)\delta^4(x-y)\,.$$

Thus, one can formally write the inverse of the operator $\hat{D}$,

$$\hat{D} = -\left(\hat{P}^2 - m^2 + i\varepsilon\right)^{-1}\,,$$

by means of the proper time representation

$$\hat{D} = i\int_0^\infty d\lambda e^{-i\hat{H}(\lambda)}\,,\ \hat{H} = -\lambda\left(\hat{P}^2 - m^2 + i\varepsilon\right)\,. \tag{7}$$

Let us now further specify $\mathcal{H}$ by defining V as the one-particle sector of the Fock space for the fermionic creation and annihilation operators $a^\dagger$ and a,

$${}_{\alpha}|0\rangle = 0\,,\ {}^{\dagger}_{\alpha}|0\rangle = |\alpha\rangle\,,$$

which satisfy the algebra

$$\left[{}^{\dagger}_{\alpha},{}_{\beta}\right]_+ = \delta_{\alpha\beta}\,,\ \left[{}^{\dagger}_{\alpha},{}^{\dagger}_{\beta}\right]_+ = \left[{}_{\alpha},{}_{\beta}\right]_+ = 0\,. \tag{8}$$

Then it is possible to represent the operators $\hat{t}_a$ as

$$\hat{t}_a = {}^{\dagger}_{\alpha}t^{\alpha}_{a\beta}{}_{\beta}\,,\ t^{\alpha}_{a\beta} = \langle\alpha|\,\hat{t}_a\,|\beta\rangle\,. \tag{9}$$

Here it is important to observe that $\hat{t}_a$ are generators of a representation of $su\,(N)$,

$$\left[\hat{t}_a,\hat{t}_b\right] = f^{c}_{ab}\hat{t}_c\,,$$

since their matrix elements $t^{\alpha}_{a\beta}$ satisfy the $su\,(N)$ commutation relations (6). In addition, tracelessness and hermiticity of $t^{\alpha}_{a\beta}$ imply the same for the operators $\hat{t}_a$,

$$\mathrm{tr}\hat{t}_a \equiv \sum_{\alpha=1}^{M}\langle\alpha|\,\hat{t}_a\,|\alpha\rangle = t^{\alpha}_{a\alpha} = 0$$

$$\hat{t}^{\dagger}_a = \left({}^{\dagger}_{\alpha}t^{\alpha}_{a\beta}{}_{\beta}\right)^{\dagger} = {}^{\dagger}_{\beta}\bar{t}^{\alpha}_{a\beta}{}_{\alpha} = \hat{t}_a\,,$$

where the $\dagger$-involution of the abstract operator algebra complex-conjugates the matrix entries of $t^{\alpha}_{a\beta}$ in the above. Finally, we note that $\hat{t}_a$ conserves the number of particles. Using the representation (9) for the generators t_a and following [27, 14], we now introduce coherent states $|\chi\rangle$ and $\langle\bar{\chi}|$ defined by the exponential of the fermion operators and † acting on the vacuum:

$$|\chi\rangle = D\,(\chi)\,|0\rangle\,,\ \langle\bar{\chi}| = |\chi\rangle^{\dagger}\,,\ D\,(\chi) = e^{{}^{\dagger}\chi - \bar{\chi}}\,,\ \left[{}_{\alpha},D\,(\chi)\right]_- = \chi_\alpha D\,(\chi)\,,$$

where χ_α and $\bar{\chi}_\alpha = \chi^{\dagger}_{\alpha}$ are Grassmann numbers that commute with the vacuum state. Consequently, these states satisfy

$${}^{\alpha}|\chi\rangle = \chi^{\alpha}|\chi\rangle\,,\ \langle\bar{\chi}|\,{}^{\dagger}_{\alpha} = \langle\bar{\chi}|\,\bar{\chi}_{\alpha}\,,$$

$$\langle\bar{\chi}|\,\xi\rangle = e^{\frac{1}{2}\left(\chi\bar{\chi}+\xi\bar{\xi}-2\xi\bar{\chi}\right)}\,,\ \int\prod_{\alpha=1}^{N}d\bar{\chi}_{\alpha}d\chi^{\alpha}\,|\chi\rangle\langle\bar{\chi}| = \hat{1}_V\,,\ \int d\chi\chi = \int d\bar{\chi}\bar{\chi} = 1\,.$$

Using the above identity resolutions, it is possible to relate matrix elements from the one-particle sector fock-space basis $|\alpha\rangle$ to the coherent basis $|\chi\rangle$,

$$\langle\alpha|\cdot|\beta\rangle = \int\prod_{\sigma,\kappa=1}^{N}d\bar{\chi}'_{\sigma}d\chi'^{\sigma}d\bar{\chi}_{\kappa}d\chi^{\kappa}e^{\frac{1}{2}\left(\chi'\bar{\chi}'+\chi\bar{\chi}\right)}\chi'^{\alpha}\,\langle\bar{\chi}'|\cdot|\chi\rangle\,\bar{\chi}_{\beta}\,, \tag{10}$$

where we have used $\langle \bar{\chi} | \alpha \rangle = \bar{\chi}_\alpha \exp \frac{1}{2}\chi\bar{\chi}$. As a consequence, we are able to recast the original form of the propagator (5), as matrix elements of one-particle Fock states, in terms of matrix elements of the coherent states,

$$D(x,y)^\alpha_\beta = \int \prod_{\sigma,\kappa=1}^{N} d\bar{\chi}'_\sigma d\chi'^\sigma d\bar{\chi}_\kappa d\chi^\kappa e^{\frac{1}{2}(\chi'\bar{\chi}'+\chi\bar{\chi})} \chi'^\alpha \langle x, \bar{\chi}' | \hat{D} | y, \chi \rangle \bar{\chi}_\beta \,. \tag{11}$$

In the next section, the matrix elements $\langle x, \bar{\chi}' | \hat{D} | y, \chi \rangle$ will be used to to obtain a path-integral representation for the propagator.

II. Another possible interpretation of the propagator $D(x,y)$ appearing in (1) can be simply as the matrix elements

$$D(x,y)^i{}_j = \langle x | \hat{D}^i{}_j | y \rangle$$

of the basis elements $|x\rangle$ of the abstract Hilbert space H. The abstract operator $\hat{D}$ acquires indices directly from the matrices of the generators of $su(N)$. Notice we have relabeled the indices of the matrix representation of $su(N)$. The new indices i and j denote the matrix entries of a new set of generators T_a,

$$T_a = \frac{1}{4}\Gamma_\alpha t^\alpha_{a\beta}\Gamma_\beta \,, \; [\Gamma_\alpha, \Gamma_\beta] = 2\delta_{\alpha\beta} \,. \tag{12}$$

These generators are very convenient for obtaining path-integral representations of the propagator using techniques adapted from the spinning particle case. However, for T_a satisfying (2), this is a representation only if the matrices t_a are antisymmetric, $t_a^T = -t_a$. This drawback can be circumvented if we take the t_a matrices in the adjoint representation $t^c_{ab} = f^c_{ab}$. Besides, there are be situations where it is possible to choose antisymmetric t_a for different irreducible representations. For instance, in the case of $SU(2)$, it is always possible to choose antisymmetric t_a for the integer spin s representations. In this case, $\alpha, \beta = 1, ..., 2s+1$and $i, j = 1, ..., 2^s$. In the general case, in the adjoint representation, $\alpha, \beta = 1, ..., N^2-1$ and thus $i, j = 1, ..., 2^{[(N^2-1)/2]}$. In the familiar case of the adjoint representation of $su(2)$, one has

$$T_i = \frac{i}{4}\varepsilon_{ijk}\Gamma_k\Gamma_j = -\frac{i}{4}\varepsilon_{ijk}\Gamma_j\Gamma_k \,, \; i,j,k = 1,2,3 \,,$$

where the Γ's satisfy $[\Gamma_i, \Gamma_j]_+ = 2\delta_{ij}$ and are order 2 matrices, so they can be chosen to be the Pauli matrices, $\Gamma_i = \sigma_i$ [1],

$$T_i = -\frac{i}{4}\varepsilon_{ijk}\sigma_j\sigma_k = \frac{1}{2}\sigma_i \,. \tag{13}$$

The generators T_i are hermitian and traceless, and they satisfy the $su(2)$ algebra

$$[T_i, T_j] = i\varepsilon_{ijk}T_k \,.$$

This case is special, because the choice of the adjoint representation for t_a gives T_a in the fundamental representation. Another special situation occurs with $SU(4)$, where we can choose t_a to be antisymmetric matrices of order 6, since $su(4) \simeq so(6)$. Thus one has even or odd spinors of $so(6)$ with 4 components, giving by means of a method described in sections 3.1 and 4.1, the fundamental representation of $SU(4)$.

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[1] $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

3 Path integral in coherent states representation

3.1 Path integral

Our goal in this section is to write a path-integral representation for

$$D_{\chi}\left(x,\bar{\chi}';y,\chi\right)\equiv\langle x,\bar{\chi}'|\,\hat{D}\,y,\chi\rangle=i\int_{0}^{\infty}d\lambda\,\langle x,\bar{\chi}'|\,e^{-i\hat{H}(\lambda)}\,|y,\chi\rangle \tag{14}$$

We insert $N-1$ identity resolutions $I=\int dxd\bar{\chi}d\chi\,|x,\chi\rangle\,\langle x,\bar{\chi}|$ and N integration over λ:

$$\begin{aligned}D_{\chi}\left(x,\bar{\chi}';y,\chi\right)=\lim_{N\to\infty}i\int_{0}^{\infty}d\lambda_0\int\left(\prod_{k=1}^{N-1}dx_k d\bar{\chi}_k d\chi_k\right)d\lambda_1\cdots d\lambda_N\\ \prod_{k=1}^{N}\langle x_k,\bar{\chi}_k|\,e^{-i\hat{H}(\lambda_k)/N}\,|x_{k-1},\chi_{k-1}\rangle\,\delta\left(\lambda_k-\lambda_{k-1}\right),\end{aligned} \tag{15}$$

where $x_N=x$, $\bar{\chi}_N=\bar{\chi}'$, $x_0=y$ and $\chi_0=\chi$. In order to evaluate the general matrix element appearing in (15), one must choose a definite ordering prescription for the operators in $\hat{H}$. In particular, one must solve the ordering ambiguity of the four-fermion term in $\hat{P}^2$. In [14], an additional identity resolution is inserted between the $\hat{P}$ operators as a solution to the ordering problem. We do not know to which ordering prescription this corresponds, and conventional ordering prescriptions such as Weyl ordering and normal ordering are not gauge-invariant. In the sequel we show that Weyl ordering is not gauge-invariant, and compute the resulting effective action. As shown in the Appendix (39), the Hamiltonian operator differs from the Weyl-ordered[2] expression by the term $\lambda\frac{q^2}{4}\mathrm{tr}\left(t_a t_b\right){}^{a}_{\mu}{}^{\mu b}$. This action, apart from the gauge-breaking term, is identical to the one that would be obtained by doubling the time partition.

Applying the midpoint rule (41) for the general matrix elements gives

$$\begin{aligned}\langle x_k,\bar{\chi}_k|\,\hat{H}\left(\lambda_k\right)|x_{k-1},\chi_{k-1}\rangle&=\int\frac{dp_k}{(2\pi)^4}d\bar{\eta}_k d\eta_k\,\langle x_k,\bar{\chi}_k|\,p_k,\eta_k\rangle\left(H_W\left(\lambda_k\right)+Q\left(\lambda_k\right)\right)\langle p_k,\bar{\eta}_k|\,x_{k-1},\chi_{k-1}\rangle\,,\\ H_W\left(\lambda_k\right)&\equiv H_W\left(\lambda_k,\frac{x_k+x_{k-1}}{2},p_k,\bar{\eta}_k,\frac{\eta_k+\chi_{k-1}}{2}\right),\\ Q\left(\lambda_k\right)&\equiv\lambda_k\frac{q^2}{4}\mathrm{tr}\left(t_a t_b\right)A^a_\mu\left(\frac{x_k+x_{k-1}}{2}\right)A^{b\mu}\left(\frac{x_k+x_{k-1}}{2}\right),\end{aligned}$$

where H_W is the Weyl-symbol of $\hat{H}_W$. Substituting the delta functions $\delta\left(\lambda_k-\lambda_{k-1}\right)$ by their integral representations and using the integral representations of the fermionic delta (43) for the χ and $\bar{\chi}$integrations, we have

$$\begin{aligned}D_{\chi}\left(x,\bar{\chi}';y,\chi\right)=\lim_{N\to\infty}i\int_{0}^{\infty}d\lambda_0\int\left(\prod_{k=1}^{N-1}dx_k\right)\left(\prod_{k=1}^{N}\frac{dp_k}{(2\pi)^4}d\lambda_k\frac{d\pi_k}{(2\pi)}d\bar{\eta}_k d\eta_k\right)\exp\frac{1}{2}\left(\chi'\bar{\chi}'-\eta_N\bar{\eta}_N+2\bar{\chi}'\eta_N\right)\\ \exp i\sum_{k=1}^{N}\left\{p_k\frac{(x_k-x_{k-1})}{\Delta t}+\pi_k\frac{(\lambda_k-\lambda_{k-1})}{\Delta t}-\frac{i}{2}\frac{(\eta_k-\eta_{k-1})}{\Delta t}\bar{\eta}_k-\frac{i}{2}\frac{(\bar{\eta}_k-\bar{\eta}_{k-1})}{\Delta t}\eta_{k-1}-H_W\left(\lambda_k\right)-Q\left(\lambda_k\right)\right\}\Delta t,\end{aligned}$$

[2]Weyl ordering here means total symmetrization in bosonic degrees of freedom, and total antisymmetrization in fermionic degrees of freedom.

where $H_W(\lambda_k) = H_W\left(\lambda_k, \frac{x_k+x_{k-1}}{2}, p_k, \bar{\eta}_k, \frac{\eta_k+\eta_{k-1}}{2}\right)$, $\eta_0 = \chi$. The term $\chi'\bar{\chi}' - \eta_N\bar{\eta}_N + 2\bar{\chi}'\eta_N$ comes from $\langle\bar{\chi}'|\,\eta_N\rangle$, and in the limit $N \to \infty$ will reduce to $2\bar{\chi}'\eta(1)$. Taking the limit $N \to \infty$ ($\Delta t \to 0$) and renaming $\eta \to \chi$ and $\bar{\eta} \to \bar{\chi}$, one has

$$D_\chi(x, \bar{\chi}'; y, \chi) = i\int_0^\infty d\lambda_0 \int DxDpD\lambda D\pi D\bar{\chi}D\chi \exp iS_{eff} \exp \bar{\chi}(1)\chi(1)\,,$$
$$S_{eff} = \int_0^1 dt\left(p\dot{x} + \pi\dot{\lambda} + \frac{i}{2}(\bar{\chi}\dot{\chi} - \dot{\bar{\chi}}\chi) + \lambda\left(\left(p_\mu + qA_\mu^a I_a\right)^2 - m^2\right) - \frac{q^2}{4}\lambda\mathrm{tr}\left(t_a t_b\right) A_\mu^a A^{b\mu}\right)\,, \quad (16)$$

where $I_a = \bar{\chi}t_a\chi$, and the functional integration is performed over the paths $x^\mu(t)$, $p_\mu(t)$, $\lambda(t)$, $\pi(t)$, $\bar{\chi}(t)$ and $\chi(t)$, with boundary values $x^\mu(0) = y^\mu$, $x^\mu(1) = x^\mu$, $\lambda(0) = \lambda_0$, $\bar{\chi}(1) = \bar{\chi}'$ and $\chi(0) = \chi$.

Since the path integral is translation-invariant, one can integrate over the momenta p_μ by shifting $p \mapsto p + \tilde{p}$, where $\tilde{p} = -\dot{x}/2\lambda - qA^a I_a$ is the solution to the classical equation $\dot{x} = \partial H_{eff}/\partial p$. One finds after making the substitution $2\lambda = e$, the Lagrangian form of the path integral:

$$D_\chi(x, \bar{\chi}'; y, \chi) = i\int_0^\infty d\lambda_0 \int DxDeD\pi D\bar{\chi}D\chi M[e, x] \exp i\left(S_{eff} + S_G\right) \exp\left(\bar{\chi}(1)\chi(1)\right)\,,$$
$$S_{eff} = \int_0^1 dt\left(-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - q\dot{x}^\mu A_\mu^a I_a + \frac{i}{2}(\bar{\chi}\dot{\chi} - \dot{\bar{\chi}}\chi)\right)\,, \quad (17)$$

with the Lagrangian measure and reparametrization gauge-fixing term S_G

$$M[e, x] = \int Dp \exp\frac{i}{2}\int_0^1 e\left(p^2 - \frac{q^2}{4}\mathrm{tr}t_a t_b A_\mu^a A^{\mu b}\right) dt \quad (18)$$
$$S_G = \int_0^1 \pi\dot{e}d\tau \quad (19)$$

Thus, the path-integral representation for the propagator can be derived with an unambiguous ordering prescription (Weyl-ordering) at the cost of defining a gauge non-invariant measure.

3.2 Pseudoclassical action

The action functional S_{eff} in (17),

$$S_{eff} = \int_0^1 dt\left(-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - q\dot{x}^\mu A_\mu^a I_a + \frac{i}{2}(\bar{\chi}\dot{\chi} - \dot{\bar{\chi}}\chi)\right)\,,\; I_a = \bar{\chi}t_a\chi\,, \quad (20)$$

is reparametrization invariant,

$$\delta_\epsilon S_{eff} = 0\,,\; \delta_\epsilon x = \epsilon\dot{x}\,,\; \delta_\epsilon e = \frac{d}{dt}(\epsilon e)\,,\; \delta_\epsilon\chi = \epsilon\dot{\chi}\,,\; \delta_\epsilon\bar{\chi} = \epsilon\dot{\bar{\chi}}\,. \quad (21)$$

In the gauge $e = \sqrt{\dot{x}^2}/m$ it coincides with the action given in [22, 21] describing a scalar relativistic particle with anticommuting coordinates in a representation of a symmetry group G, whose equations of motion are

$$m\frac{d}{dt}\frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} = q\dot{x}^\nu F_{\mu\nu}^a I_a\,,\; D_t\chi^\alpha \equiv \frac{d}{dt}\chi^\alpha + iq\dot{x}^\mu A_\mu^a t_{\alpha\beta}^\alpha\chi^\beta = 0\,, \quad (22)$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + iqf^a_{bc}A^b_\mu A^c_\nu$ is the field-strength and D_t is the covariant derivative.

For canonical analysis purposes[3], however, it is better to start from the reparametrization invariant action (20). Since this action does not contain derivatives of the einbein, it is best to consider it as a velocity (see [29]), and not introduce its conjugate momentum. One thus arrives at the following Hamiltonian,

$$H = -\frac{e}{2}T - \dot{\chi}_\alpha\phi_\alpha - \dot{\bar{\chi}}_\alpha\bar{\phi}_\alpha\,,$$

where the set of constraints $\Phi = \{T, \phi, \bar{\phi}\}$,

$$T = \left(p_\mu + qA^a_\mu I_a\right)^2 - m^2\,,\ \phi_\alpha = \pi_\alpha - \frac{i}{2}\bar{\chi}_\alpha\,,\ \bar{\phi}_\alpha = \bar{\pi}_\alpha - \frac{i}{2}\chi_\alpha\,,$$

defines a degenerate supermatrix $\{\Phi, \Phi\}$. The constraint algebra is simplified if we consider an equivalent set of constraints $\left\{\tilde{T}, \phi, \bar{\phi}\right\}$, where $\tilde{T}$ is obtained from T through the shifts $\chi \to \chi - i\bar{\phi}$ and $\bar{\chi} \to \bar{\chi} - i\phi$,

$$\left\{\tilde{T}, \phi_\alpha\right\} = \left\{\tilde{T}, \bar{\phi}_\alpha\right\} = 0\,,\ \{\phi_\alpha, \bar{\phi}_\beta\} = -i\delta_{\alpha\beta}\,.$$

The new Hamiltonian with redefined Lagrange multipliers is

$$\tilde{H} = \Lambda\tilde{T} + \Lambda_\alpha\phi_\alpha + \bar{\Lambda}_\alpha\bar{\phi}\,,$$

giving the following time-evolution for the constraints,

$$\frac{d}{dt}\tilde{T} = 0\,,\ \frac{d}{dt}\phi_\alpha = i\bar{\Lambda}_\alpha\,,\ \frac{d}{dt}\bar{\phi}_\alpha = i\Lambda_\alpha\,,$$

so the condition of conservations of the constraints in time simply determines Λ and $\bar{\Lambda}$. The equations of motion for the independent variables $\eta = (x^\mu, p_\mu, \chi_\alpha, \bar{\chi}_\alpha)$ are given by

$$\dot{\eta} = \left\{\eta, \Lambda\tilde{T}\right\}_{D(\phi)}\,,\ \phi_\alpha = \bar{\phi}_\alpha = \tilde{T} = 0\,,$$

where the Dirac brackets have been constructed with regard to the second-class constraint set $\{\phi; \bar{\phi}\}$.Using well known properties of the Dirac brackets, the equations of motion become

$$\dot{\eta} = \{\eta, \Lambda T\}_{D(\phi)}\,,\ \phi_\alpha = \bar{\phi}_\alpha = T = 0\,,$$

And the nonzero brackets between independent variables are

$$\{x^\mu, p_\nu\}_{D(\phi)} = \delta^\mu_\nu\,,\ \{\chi_\alpha, \bar{\chi}_\beta\}_{D(\phi)} = -i\delta_{\alpha\beta}\,. \tag{23}$$

Moreover, the I_a are covariantly constant generators of $SU(N)$,

$$\{I_a, I_b\}_{D(\phi)} = -if^c_{ab}I_c\,,\ D_tI_a \equiv \frac{d}{dt}I_a + iq\dot{x}^\mu A^b_\mu f^c_{ab}I_c = 0\,, \tag{24}$$

hence are called isospin.

[3]Definitions and conventions are those used in [28].

From (23), we see that the Grassmann operators will generate a creation-annihilation operator algebra,

$$\chi_\alpha \to a_\alpha \,, \; \bar{\chi}_\alpha \to a_\alpha^\dagger \,, \; \left[a_\alpha, a_\beta^\dagger\right]_+ = \delta_{\alpha\beta} \,. \tag{25}$$

The Hilbert space $\mathcal{H}$ can be realized as the direct product of a representation space for the Heisenberg algebra and the 2^n-dimensional Fock space of the creation and annihilation operators,

$$|x;\alpha_1 \cdots \alpha_p\rangle = a_{\alpha_1}^\dagger \cdots a_{\alpha_p}^\dagger \,|x;0\rangle \in \mathcal{H} \,, \; p = 0,..,n \,. \tag{26}$$

As is well-known, the group $SO(2n)$ preserves the commutation relations (25), and the $so(2n)$ generators in the above representation are given by $c_{\alpha\beta} = \left[a_\alpha, a_\beta^\dagger\right]/2$, $a_\alpha a_\beta$ and $a_\alpha^\dagger a_\beta^\dagger$. The $c_{\alpha\beta}$ belong to the $u(n)$ subalgebra of $so(2n)$. The n operators $c_{\alpha\beta}$ for $\alpha = \beta$ form the Cartan subalgebra of $so(2n)$.

The representation (26) is a 2^n-dimensional spinor representation of $so(2n)$, and its irreducible representations are given by states with an even or odd number of creation operators, corresponding to the 2^{n-1}-dimensional Weyl (semi-spinor) representations of $so(2n)$. These states can be further decomposed in irreducible representations of $su(N)$, since the isospin generators $\hat{t}_a$ are a linear combination of the $so(2n)$ generators,

$$\hat{t}_a = t_{a\alpha\beta} a_\alpha^\dagger a_\beta = t_{a\alpha\beta}\left(-2c_{\beta\alpha} + \delta_{\alpha\beta}\right) = -2t_{a\alpha\beta} c_{\beta\alpha} \,.$$

Therefore, we see that the $\hat{t}_a$ generate a $su(N)$ subalgebra of $so(2n)$.

In general, in order to determine the $SU(N)$ content of the wave function, one proceeds as in [21]: given t_a an irreducible representation of $su(N)$ in terms of $n \times n$ matrices, the wave function belongs to a (Weyl) semi-spinor representation of $so(2n)$. Then, one decomposes the set of Cartan generators of $su(N)$ (a maximal set of commuting generators) in terms of the n Cartan generators of $so(2n)$. In the special case of the representation (26), one can choose the operators $c_\alpha = \left[a_\alpha, a_\alpha^\dagger\right]/2$, $\alpha = 1,...,n$ as the maximum set of commuting generators of $so(2n)$. For instance, in the case of $SU(2)$ one can take the isospin projection I_1 and for $SU(3)$ one can take the isospin projection I_1 and the hypercharge Y to characterize irreducible representations. One then decomposes isospin generators in terms of the c_α to obtain their eigenvalues for the spinor representation of $so(2n)$to which the wave function belongs. The range of these eigenvalues gives the irreducible representations of $SU(N)$. Therefore, to each given n-dimensional irreducible representation t_a of $SU(N)$, the wave function will belong to a 2^{n-1}-dimensional representation of $SO(2n)$ (a semi-spinor representation), which decomposes into irreducible representations of $SU(N)$ as determined by the isospin generators $\hat{t}_a$.

In the special case of $SU(2)$, since it is of rank 1, the Cartan subalgebra is generated by a single element, say t_3, whose matrix representation in a basis of isospin s eigenstates is of the form $t_3 = \text{diag}(s, s-1, ..., -s+1, -s)$. The decomposition in Cartan generators of $so(4s+2)$ of the isospin operator $\hat{t}_3$ is as follows,

$$\hat{t}_3 = sc_1 + (s-1)\,c_2 + \cdots + (-s)\,c_{2s+1} \,.$$

Each c_α can take either of the values plus or minus 1/2. However, the wave function is in a state of either an even number of plus $+1/2$ (even Weyl spinor) or an odd number of $+1/2$ (odd Weyl spinor). For instance, for $s = 1/2$, the possible eigenvalues of $t_3$for the even representation is twice 0, giving two scalar representations; and for the odd representation is $\pm 1/2$, giving the isospin 1/2 representation. For integer spin, even and odd representations decompose in the same way, and the largest representation is of spin $(s+1)\,s/2$. For example, $s=1$ gives the eigenvalues $1, 0, -1$ and again 0, giving the isospin 1 representation plus a scalar. We summarize the results for some values of isospin in the table below,

isospin	symmetry group	representation dimension	decomposition(even;odd)
0	$SO(2)$	1	$\underline{0}$
1/2	$SO(4)$	2	$2\times\underline{0}\,;\,\underline{\tfrac{1}{2}}$
1	$SO(6)$	4	$\underline{0}+\underline{1}$
3/2	$SO(8)$	8	$3\times\underline{0}+\underline{2}\,;\,2\times\underline{\tfrac{3}{2}}$
2	$SO(10)$	16	$\underline{0}+\underline{1}+\underline{2}+\underline{3}$

Thus, in order to obtain the fundamental representation of $SU(2)$ upon quantization, one must choose the Hilbert space to be the odd Weyl spinor representation of $SO(4)$ of two-component spinors. In this case, one gets from the constraint T the Dirac quantization condition

$$\hat{T}\phi = \left[\left(\hat{p}_\mu + qA^a_\mu t_a\right)^2 - m^2\right]\phi(x) = 0\,, \tag{27}$$

which is precisely the wave equation in (1) for $t_a = \frac{1}{2}\sigma_a$.

It also possible to arrive at these results starting from the classical action (20). In the following, it will be convenient to express the Grassmann variables χ in terms of their real and imaginary parts,

$$\chi_\alpha = \frac{1}{\sqrt{2}}\left(\chi_{1\alpha} + i\chi_{2\alpha}\right)\,,$$

so that we are left with the real variables

$$\chi_{1\alpha} = \frac{1}{\sqrt{2}}\left(\chi_\alpha + \bar{\chi}_\alpha\right)\,,\ \chi_{2\alpha} = \frac{1}{i\sqrt{2}}\left(\chi_\alpha - \bar{\chi}_\alpha\right)\,. \tag{28}$$

In this way, the Grassmanian kinetic term becomes

$$L_{kin} = \frac{i}{4}\left(\chi_1\dot{\chi}_1 - \dot{\chi}_1\chi_1 + \chi_2\dot{\chi}_2 - \dot{\chi}_2\chi_2\right)\,.$$

L_{kin} is invariant under transformations induced by $R_{\alpha\beta} = -i\left(\chi_{1\alpha}\chi_{1\beta} + \chi_{2\alpha}\chi_{2\beta}\right)$ and $S_{\alpha\beta} = -i\left(\chi_{1\alpha}\chi_{2\beta} + \chi_{1\beta}\chi_{2\alpha}\right)$

$$\delta_\omega\chi_{i\alpha} \equiv \left\{\frac{1}{2}\omega_{\beta\gamma}R_{\beta\gamma}, \chi_{i\alpha}\right\}_{D(\phi)} = \omega_{\alpha\beta}\chi_{i\beta}\,,$$

$$\delta_\lambda\chi_{i\alpha} \equiv \left\{\frac{1}{2}\lambda_{\beta\gamma}S_{\beta\gamma}, \chi_{i\alpha}\right\}_{D(\phi)} = (-1)^{i+1}\lambda_{\alpha\beta}\chi_{i\beta}$$

where the Dirac brackets for the real variables follows from the old variables' brackets (23) and their expression in terms of the real variables (28),

$$\{\chi_{1\alpha}, \chi_{1\beta}\}_{D(\phi)} = \{\chi_{2\alpha}, \chi_{2\beta}\}_{D(\phi)} = -i\delta_{\alpha\beta}\,,\ \{\chi_{1\alpha}, \chi_{2\beta}\}_{D(\phi)} = 0\,.$$

The symmetry generators $R_{\alpha\beta}$ and $S_{\alpha\beta}$ satisfy the Lie algebra

$$\{R_{\alpha\beta}, R_{\gamma\delta}\}_{D(\phi)} = \delta_{\alpha\gamma}R_{\beta\delta} + \delta_{\beta\delta}R_{\alpha\gamma} - \delta_{\alpha\delta}R_{\beta\gamma} - \delta_{\beta\gamma}R_{\alpha\delta}\,,$$

$$\{S_{\alpha\beta}, S_{\gamma\delta}\}_{D(\phi)} = \delta_{\alpha\gamma}R_{\beta\delta} + \delta_{\beta\delta}R_{\alpha\gamma} + \delta_{\alpha\delta}R_{\beta\gamma} + \delta_{\beta\gamma}R_{\alpha\delta}\,,$$

$$\{R_{\alpha\beta}, S_{\gamma\delta}\}_{D(\phi)} = \delta_{\alpha\gamma}S_{\beta\delta} - \delta_{\beta\delta}S_{\alpha\gamma} + \delta_{\alpha\delta}S_{\beta\gamma} - \delta_{\beta\gamma}S_{\alpha\delta}\,.$$

Above we recognize the commutation relations of the combination of the $o(2n)$ generators L_{ij}, $i,j = 1,...,2n$,

$$R_{\alpha\beta} = L_{2\alpha-1,2\beta-1} + L_{2\alpha,2\beta}\,, \quad S_{\alpha\beta} = L_{2\alpha,2\beta-1} - L_{2\alpha-1,2\beta} - \delta_{\alpha\beta}\,.$$

Moreover, from the following decomposition of the generators I_a in terms of the symmetric and antisymmetric part of t_a,

$$\begin{aligned} I_a &= t_{a(\alpha\beta)}\left(\bar{\chi}_\alpha\chi_\beta + \bar{\chi}_\beta\chi_\alpha\right) + t_{a[\alpha\beta]}\left(\bar{\chi}_\alpha\chi_\beta - \bar{\chi}_\beta\chi_\alpha\right) \\ &= \frac{i}{2}t_{a(\alpha\beta)}\left(\chi_{1\alpha}\chi_{2\beta} + \chi_{1\beta}\chi_{2\alpha}\right) + \frac{1}{2}t_{a[\alpha\beta]}\left(\chi_{1\alpha}\chi_{1\beta} + \chi_{2\alpha}\chi_{2\beta}\right)\,, \\ &= -\frac{1}{2}t_{a(\alpha\beta)}S_{\alpha\beta} + \frac{i}{2}t_{a[\alpha\beta]}R_{\alpha\beta} \end{aligned} \tag{29}$$

we again find the I_a are a linear combination of $R_{\alpha\beta}$ and $S_{\alpha\beta}$, which is to say that the I_a are the generators of the subalgebra $su(N)$ of $so(2n)$. It is $so(2n)$ and not $o(2n)$, because the trace part of $S_{\alpha\beta}$ in the expansion of I_a gives no contribution, since the t_a are traceless.

4 Path integral in Clifford algebra representation

4.1 Path integral

We use the representation (12) for the generators T_a and standard techniques [16, 30] from the spinning particle case, adapted to our present problem, to represent the causal propagator. In this case, the indices α, β and γ label the matrix entries of the Γ-matrices, that is, they label the representation space for the Clifford algebra. The propertime representation for the operator $\hat{D}$ (7) in the position representation is

$$D\left(x_{out}, x_{in}\right) = i\int_0^\infty \langle x_{out}|\, e^{-i\hat{H}(\lambda)}\, |x_{in}\rangle\, d\lambda\,. \tag{30}$$

Next a discretization is made inserting $N-1$ identity resolutions $I = \int dx\,|x\rangle\langle x|$ in the above expression,

$$\begin{aligned} D\left(x_{out}, x_{in}\right) &= \lim_{N\to\infty} i\int_0^\infty d\lambda_0 \int_{-\infty}^{\infty}\left(\prod_{i=1}^{N-1} dx_i\right) d\lambda_1\cdots d\lambda_N \\ &\prod_{i=1}^{N}\langle x_i|\, e^{-i\hat{H}(\lambda_i)/N}\,|x_{i-1}\rangle\,\delta\left(\lambda_i - \lambda_{i-1}\right) \end{aligned} \tag{31}$$

where $x_N = x_{out}$ and $x_0 = x_{in}$. Applying the symmetric or Weyl correspondence to the general matrix element, one has

$$\langle x_i|\, e^{-i\hat{H}(\lambda_i)/N}\,|x_{i-1}\rangle = \int \frac{dp_i}{(2\pi)^4}\exp\left(-\frac{i}{N}H\left(\lambda_i, \frac{x_1+x_2}{2}, p_i\right)\right) e^{i(x_i - x_{i-1})p_i}\,, \tag{32}$$

where H is the Weyl symbol of $\hat{H}$,

$$H\left(\lambda, x, p\right) = \lambda\left[m^2 - \left(p_i^2 + qp_i^\mu A_\mu^a\left(x\right)T_a\right)^2\right]\,.$$

As in the spinning particle case [16], one assigns to each matrix T_a its 'time' $\tau_j = j\Delta\tau$, so that the time-ordered (31) becomes, for $1/N \equiv \Delta\tau$,

$$D(x_{out}, x_{in}) = \lim_{\Delta\tau\to 0} iT \int_0^\infty d\lambda \int_{-\infty}^{\infty} \left(\prod_{i=1}^{N-1} dx_i\right) \left(\prod_{i=1}^{N} \frac{dp_i}{(2\pi)^4} d\lambda_i \frac{d\pi_i}{2\pi}\right)$$
$$\times \exp i \sum_{i=1}^{N} S_i (x_i, x_{i-1}, p_i, \lambda_i, \pi_i) \,, \tag{33}$$

where

$$S_i = \left(\frac{x_i - x_{i-1}}{\Delta\tau} \cdot p_i - H\left(\lambda_i, \frac{x_i + x_{i-1}}{2}, p_i\right) + \pi_i \frac{\lambda_i - \lambda_{i-1}}{\Delta\tau}\right) \Delta\tau \,. \tag{34}$$

In the limit $\Delta\tau \to 0$, $S_i \to S_H[x, p; \tau_{in}, \tau_{out}]$ is the Hamiltonian action, a functional of the trajectory $(x(t), p(t))$ in phase space, defined in the proper-time interval $[\tau_{in}, \tau_{out}]$, and (33) is the discrete version of the following path integral in the Hamiltonian form:

$$D(x_{out}, x_{in}) = iT \int_0^\infty d\lambda_0 \int_{x_{in}}^{x_{out}} Dx \int Dp \int_{\lambda_0} D\lambda D\pi \exp i \int_{\tau_{in}}^{\tau_{out}} \left(\dot{x} \cdot p - H(\lambda, x, p) + \pi\dot{\lambda}\right) d\tau \,. \tag{35}$$

Following [16], we introduce odd sources $\rho_a(\tau)$, anticommuting with the Γ-matrices, and rewrite 35 as

$$D(x_{out}, x_{in}) = i \int_0^\infty d\lambda_0 \int_{x_{in}}^{x_{out}} Dx \int Dp \int_{\lambda_0} D\lambda D\pi \exp i \int_0^1 \left[\lambda\left(\left(p_\mu + \frac{q}{4} t^\alpha_{a\beta} A^a_\mu \frac{\delta_l}{\delta\rho_\alpha}\frac{\delta_l}{\delta\rho_\beta}\right)^2 - m^2\right)\right.$$
$$\left. p \cdot \dot{x} + \pi\dot{\lambda} d\tau\right] \times T \int_0^1 \rho_\alpha(\tau) \Gamma^\alpha d\tau \bigg|_{\rho=0} \,,$$

where for simplicity we have made $\tau_{in} = 0$ and $\tau_{out} = 1$. It is possible to present the last term on the right-hand side of the above equation as a path integral [16, 30],

$$T \int_0^1 \rho_\alpha(\tau) \Gamma^\alpha d\tau = \exp\left(i\Gamma^\alpha \frac{\partial_l}{\partial\theta^\alpha}\right)$$
$$\times \int_{\psi(0)+\psi(1)=\theta} \exp\left[\int_0^1 \left(\psi^\alpha(\tau) \dot{\psi}_\alpha(\tau) - i2\rho_\alpha(\tau) \psi^\alpha(\tau)\right) d\tau + \psi^\alpha(1) \psi_\alpha(0)\right] \mathcal{D}\psi \bigg|_{\theta=0}$$
$$\mathcal{D}\psi = D\psi \left[\int_{\psi(0)+\psi(1)=0} \exp \int_0^1 \psi^\alpha(\tau) \dot{\psi}_\alpha(\tau) d\tau\right]^{-1} \,,$$

where θ are odd constants, anticommuting with the Γ-matrices. Then, we arrive at the Hamiltonian path-integral representation for the propagator:

$$
\begin{aligned}
D\left(x_{out}, x_{in}\right) = i \exp\left(i\Gamma^{\alpha}\frac{\partial_l}{\partial\theta^{\alpha}}\right)\int_0^{\infty} d\lambda_0 \int_{x_{in}}^{x_{out}} Dx \int Dp \int_{\lambda_0} D\lambda D\pi \\
\int \exp\left\{i\int_0^1 \left[\lambda\left(\left(p_\mu - qt^{\alpha}_{\alpha\beta}A^a_\mu\psi_\alpha\psi_\beta\right)^2 - m^2\right)\right.\right. \\
\left.\left. -i\psi^{\alpha}\dot{\psi}_\alpha + p\cdot\dot{x} + \pi\dot{\lambda}\right] d\tau + \psi^{\alpha}(1)\,\psi_\alpha(0)\right\} \mathcal{D}\psi\Big|_{\theta=0}\,, \\
x(0) = x_{in}\,,\; x(1) = x_{out}\,,\; \lambda(0) = \lambda_0\,,\; \psi(0) + \psi(1) = \theta\,.
\end{aligned}
$$

Integrating over the momenta, one finds the Lagrangian path-integral representation:

$$
\begin{aligned}
D\left(x_{out}, x_{in}\right) = i\exp\left(i\Gamma^{\alpha}\frac{\partial_l}{\partial\theta^{\alpha}}\right)\int_0^{\infty} de_0 \int \exp\left\{i\left(S_{eff} + S_G\right) + \psi^{\alpha}(1)\,\psi_\alpha(0)\right\} M\left[e,x\right] DxDeD\pi\mathcal{D}\psi|_{\theta=0} \\
S_{eff} = i\int_0^1 \left(-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 + qt^{\alpha}_{\alpha\beta}\dot{x}^{\mu}A^a_\mu\psi_\alpha\psi_\beta - i\psi^{\alpha}\dot{\psi}_\alpha\right) \\
x(0) = x_{in}\,,\; x(1) = x_{out}\,,\; e(0) = e_0\,,\; \psi(0) + \psi(1) = \theta\,,
\end{aligned}
\tag{36}
$$

where the measure $M\left[e,x\right]$ and S_G are

$$
M\left[e,x\right] = \int Dp \exp\frac{i}{2}\int_0^1 ep^2 d\tau\,,\; S_G = \int_0^1 \pi\dot{e}\,.
$$

4.2 Pseudoclassical action

Let us consider the reparametrization invariant action from (36) with the rescaling $\psi \to i/\sqrt{2}\psi$,

$$
S_{eff} = \int dx^4\left(-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - q\dot{x}^{\mu}A^a_\mu I_a + \frac{i}{2}\psi^{\alpha}\dot{\psi}_\alpha\right)\,,\; I_a = \frac{1}{2}t^{\alpha}_{\alpha\beta}\psi_\alpha\psi_\beta\,. \tag{37}
$$

The above action is essentially the one written in [22] in the case the set of Grassmann variables ψ belong to the adjoint representation of a compact simple group G ($t^{\alpha}_{\alpha\beta} = f^b_{ac}$), and in [21] for ψ in a representation with antisymmetric generators t_a. The equations of motion in the gauge $e = \sqrt{\dot{x}^2}/m$ are

$$
\begin{aligned}
\frac{d}{dt}\left(m\frac{\dot{x}^{\mu}}{\sqrt{\dot{x}^2}}\right) = qF^a_{\mu\nu}\dot{x}^{\nu}I_a\,,\; D_t\psi^{\alpha} \equiv \frac{d}{dt}\psi^{\alpha} + iq\dot{x}^{\mu}A^a_\mu t^{\alpha}_{\alpha\beta}\psi^{\beta} = 0\,, \\
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + iqf^a_{bc}A^b_\mu A^c_\nu\,,
\end{aligned}
$$

where $F^a_{\mu\nu}$ is the non-Abelian field strength.

Next, we follow a similar canonical analysis path than the one taken in the case of the coherent representation, this time with t_a denoting $n \times n$ antisymmetric matrices. As expected, the Hamiltonian is proportional to constraints,

$$
H = -\frac{e}{2}T - \dot{\psi}^{\alpha}\phi_\alpha
$$

where

$$T = \left(p_\mu + qA_\mu^a I_a\right)^2 - m^2\,,\ \phi_\alpha = \pi_\alpha - \frac{i}{2}\psi_\alpha$$

After redefining T through the shift $\psi \to \psi - i\phi$, $T \to \tilde{T}$, the constraint algebra becomes

$$\left\{\tilde{T}, \phi_\alpha\right\} = 0\,,\ \{\phi_\alpha, \phi_\beta\} = -i\delta_{\alpha\beta}\,.$$

The set $\Phi = \left\{\tilde{T}, \phi\right\}$ is first-class, and the evolution of the independent variables $\eta = (x, p, \psi)$ is

$$\dot{\eta} = \{\eta, \Lambda T\}_{D(\phi)} = 0\,,\ T = \phi_\alpha = 0\,,$$

where the Dirac brackets are defined with respect to the second-class constraint set $\{\phi\}$. The Dirac commutator of the independent variables is

$$\{x^\mu, p_\nu\}_{D(\phi)} = \delta^\mu_\nu\,,\ \{\psi_\alpha, \psi_\beta\}_{D(\phi)} = -i\delta_{\alpha\beta}\,,$$

The isospin quantities I_a satisfy the Lie algebra of $SU(N)$ after quantization and are covariantly constant:

$$\{I_a, I_b\}_{D(\phi)} = -if^c_{ab}I_c\,,\ D_\tau I^c = \frac{d}{d\tau}I^c + iq\dot{x}^\mu A^a_\mu f^c_{ab} I^b = 0\,.$$

It is clear that upon quantization the Grassmann variables ψ_α generate Clifford algebra with n generators and positive-definite inner product. And thus the physical states ϕ are $2^{[n/2]}$-component vectors satisfying

$$\left[\left(\hat{p}_\mu + qA^a_{\mu a}\right)^2 - m^2\right]\phi(x) = 0\,, \tag{38}$$

where the quantum-mechanical isospin operators $_a = \frac{1}{4}t^\alpha_{a\beta}\Gamma_\alpha\Gamma_\beta$ are precisely those introduced in (12) and they satisfy the $su(N)$ algebra (2)

$$[_a, _b] = f^c_{abc}\,.$$

Let us draw a similar analysis of the isospin content for the classical theory as the one given in section 3.1. Here, the Grassmannian kinetic terms in the action are invariant under the transformations generated by $R_{\alpha\beta} = -i\psi_\alpha\psi_\beta$,

$$\delta_\omega\psi_\alpha = \left\{\frac{1}{2}\omega_{\beta\gamma}R_{\beta\gamma}, \psi_\alpha\right\}_{D(\phi)} = \omega_{\alpha\beta}\psi_\beta\,,\ \omega_{\alpha\beta} = -\omega_{\beta\alpha}\,,$$

which give a representation for the Lie algebra $so(n)$:

$$\{R_{\alpha\beta}, R_{\gamma\delta}\}_{D(\phi)} = \delta_{\alpha\gamma}R_{\beta\delta} - \delta_{\beta\gamma}R_{\alpha\delta} - \delta_{\alpha\delta}R_{\beta\gamma} + \delta_{\beta\delta}R_{\alpha\gamma}\,.$$

So the the generators I_a are a linear combination of the generators $L_{\alpha\beta}$ of $so(n)$, and therefore they generate an $su(N)$ subalgebra of $so(n)$. In order to determine the $SU(N)$ content of the wave function, one proceeds as in [21]: given t_a an irreducible representation of $su(N)$ in terms of $n \times n$ **antisymmetric** matrices, it is clear ψ_α gives rise to a Clifford algebra in the quantum theory, so the wave function can be taken to belong to a spinor representation of $so(n)$. One then calculates the eigenvalues of a maximal set of commuting generators I_a for this representation, and thus determines how it decomposes in irreducible representations of $SU(N)$. Therefore, to each given n-dimensional irreducible representation t_a of $SU(N)$, the wave function will belong to a $2^{[n/2]}$-dimensional representation of $SO(n)$, which decomposes into irreducible representations of $SU(N)$ as determined by the isospin generators I_a.

For example, in the case of $SU(2)$, one can find a basis for which t_a are antisymmetric, and in which I_1 decomposes as

$$I_1 = L_{23} + 2L_{45} + \cdots + sL_{2s,2s+1} .$$

The wave function is a 2^s-component spinor of $SO(2s+1)$. Below we give the $SU(2)$ decomposition of the spinor $SO(2s+1)$ representation for some values of isospin:

isospin	symmetry group	representation dimension	decomposition
1	$SO(3)$	2	$\underline{\frac{1}{2}}$
2	$SO(5)$	4	$\underline{\frac{3}{2}}$
3	$SO(7)$	8	$\underline{0} + \underline{3}$

5 Summary

We have described two methods of generating classical actions for a scalar particle with isospin via path-integral representations of the causal propagator. Dirac quantization of these actions produce the corresponding wave equations for various possible representations of $SU(N)$. By means of a judicious choice of the pseudo-classical action and the representation of the $su(N)$ algebra in the action, it is possible to obtain the wave action for any desired isospin.

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A Weyl ordering of operators and functions in the Berezin algebra

Let us write the Hamiltonian operator (7) explicitly:

$$\hat{H} = -\lambda \left(\hat{p}^2 + qt_{a\alpha\beta}\left(\hat{p}^{\mu}{}^{a}_{\mu} + {}^{a}_{\mu}\hat{p}^{\mu}\right){}^{\dagger}_{\alpha\beta} + q^2 t_{a\alpha\beta} t_{b\gamma\delta}{}^{a\,\mu b\dagger}_{\mu}{}_{\alpha\beta}{}^{\dagger}_{\gamma\delta} - m^2\right) .$$

Total symmetrization in $\hat{x}$and $\hat{p}$, and total antisymmetrization in $a^{\dagger}$ and a gives the Weyl-ordered Hamiltonian operator $\hat{H}_W$:

$$\hat{H}_W = -\lambda \left(\hat{p}^2 + \frac{q}{2} t_{a\alpha\beta}\left(\hat{p}^{\mu}{}^{a}_{\mu} + {}^{a}_{\mu}\hat{p}^{\mu}\right)\left[{}^{\dagger}_{\alpha}, {}_{\beta}\right] + q^2 t_{a\alpha\beta} t_{b\gamma\delta}{}^{a\,\mu b}_{\mu}\left({}^{\dagger}_{\alpha\beta}{}^{\dagger}_{\gamma\delta}\right)_W - m^2\right) , \quad (39)$$

where the four-fermion term is given by

$${}^{\dagger}_{\alpha\beta}{}^{\dagger}_{\gamma\delta} = \left({}^{\dagger}_{\alpha\beta}{}^{\dagger}_{\gamma\delta}\right)_W + \frac{1}{2}\delta_{\gamma\delta}\left({}^{\dagger}_{\alpha\beta}\right)_W - \frac{1}{2}\delta_{\delta\alpha}\left({}^{\dagger}_{\gamma\beta}\right)_W + \delta_{\alpha\beta}\left({}^{\dagger}_{\gamma\delta}\right)_W + \frac{1}{2}\delta_{\gamma\beta}\left({}^{\dagger}_{\alpha\delta}\right)_W - \frac{1}{4}\delta_{\delta\alpha}\delta_{\gamma\beta} - \delta_{\alpha\beta}\delta_{\gamma\delta}$$

Using the tracelessness of the matrices t_a and antisymmetry of the structure constants f_{abc}, we have

$$\hat{H} = \hat{H}_W + \lambda\frac{q^2}{4}\mathrm{tr}\left(t_a t_b\right){}^{a\,\mu b}_{\mu}$$

Thus, the Hamiltonian is the sum of a Weyl-ordered expression plus a gauge non-invariant contribution. The Weyl-symbol corresponding to $\hat{H}_W$ is

$$H_W = -\lambda \left(p^2 + 2qt_{a\alpha\beta}\left(p^\mu A^a_\mu\right)\bar{\chi}_\alpha\chi_\beta + q^2 t_{a\alpha\beta}t_{b\gamma\delta}A^a_\mu A^{\mu b}\bar{\chi}_\alpha\chi_\beta\bar{\chi}_\gamma\chi_\delta - m^2\right) \tag{40}$$

A.1 Proof[4] of fermionic midpoint rule.

If $F\left(,^\dagger\right)$ is any Weyl-ordered polynomial in and †, then

$$\langle\bar{\chi}|\,F\left(,^\dagger\right)|\chi\rangle = \int d\bar{\eta}d\eta\,\langle\bar{\chi}|\,\eta\rangle\,F\left(\frac{\chi+\eta}{2},\bar{\eta}\right)\langle\bar{\eta}\,|\chi\rangle\,, \tag{41}$$

$$= \int d\bar{\eta}d\eta\,\langle\bar{\chi}|\,\eta\rangle\,F\left(\chi,\frac{\bar{\chi}+\bar{\eta}}{2}\right)\langle\bar{\eta}\,|\chi\rangle\,, \tag{42}$$

Let us prove the identity (41). The proof of the second identity is analogous. First, consider $F\left(^\dagger\right)$ a polynomial in creation operators. Clearly, F is Weyl-ordered, and (41) is trivially satisfied,

$$\langle\bar{\chi}|\,F\left(^\dagger\right)|\chi\rangle = \int d\bar{\eta}d\eta\,\langle\bar{\chi}|\,\eta\rangle\,F\left(\bar{\chi}\right)\langle\bar{\eta}\,|\chi\rangle\,.$$

Now, for $F\left(,^\dagger\right) = \frac{1}{2}\left({}_\alpha f\left(^\dagger\right) + (-1)^{\varepsilon(f)} f\left(^\dagger\right){}_\alpha\right)$, where the plus or minus sign depends on the parity of $f\left(^\dagger\right)$, (41) is easily seen to hold. Any Weyl-ordered polynomial can be obtained by repeated antisymmetrizations of the form $F = \frac{1}{2}\left({}_\alpha f \pm f_\alpha\right)$ where $f\left(,^\dagger\right)$ is Weyl-ordered. Therefore, let us prove (41) inductively, by assuming it holds for $f\left(,^\dagger\right)$ and proving it is also true for $F = \frac{1}{2}\left({}_\alpha f \pm f_\alpha\right)$,

$$\begin{aligned}
\langle\bar{\chi}|\,\frac{1}{2}\left({}_\alpha f \pm f_\alpha\right)|\chi\rangle &= \int d\bar{\eta}d\eta\frac{1}{2}\left(\left(\langle\bar{\chi}|\,{}_\alpha\,|\eta\rangle\,\langle\bar{\eta}|\,f\,|\chi\rangle \pm \langle\bar{\chi}|\,\eta\rangle\,\langle\bar{\eta}|\,f_\alpha\,|\chi\rangle\right)\right.\\
&= \int d\bar{\eta}d\eta\,\langle\bar{\chi}|\,\eta\rangle\,\frac{\eta_\alpha+\chi_\alpha}{2}\,\langle\bar{\eta}|\,f\,|\chi\rangle\\
&= \int d\bar{\eta}d\eta d\bar{\xi}d\xi\,\langle\bar{\chi}|\,\eta\rangle\,\langle\bar{\eta}|\,\xi\rangle\,\frac{\chi_\alpha+\eta_\alpha}{2}f\left(\frac{\chi+\xi}{2},\bar{\xi}\right)\langle\bar{\xi}\,|\chi\rangle\\
&= d\bar{\xi}d\xi\,\langle\bar{\chi}|\,\xi\rangle\,\frac{\chi_\alpha+\xi_\alpha}{2}f\left(\frac{\chi+\xi}{2},\bar{\xi}\right)\langle\bar{\xi}\,|\chi\rangle
\end{aligned}$$

where in the last equality we used the identity

$$\int d\bar{\eta}d\eta\,\langle\bar{\alpha}|\,\eta\rangle\,\langle\bar{\eta}|\,\beta\rangle\,f\left(\eta\right) = \langle\bar{\alpha}|\,\beta\rangle\,f\left(\beta\right)\,. \tag{43}$$

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