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Star products made easier

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Abstract

We develop an approach to the deformation quantization on the real plane with an arbitrary Poisson structure which based on Weyl symmetrically ordered operator products. By using a polydifferential representation for deformed coordinates $\hat{x}^j$ we are able to formulate a simple and effective iterative procedure which allowed us to calculate the fourth order star product (and may be extedned to the fifth order at the expense of tedious but otherwise straightforward calculations). Modulo some cohomology issues which we do not consider here, the method gives an explicit and physics-friendly description of the star products.

1 Introduction

The modern history of deformation quantization started with the paper [1], see [2] for an overview. An important ingredient of the deformation quantization program is a construction of a star product. One takes the algebra $\mathcal{A}$ of smooth functions on a Poisson manifold equipped with a Poisson structure ω and deformes $\mathcal{A}$ to $\mathcal{A}_\omega$ by introducing a star product in such a way that the star commutator of any two functions mimics in the leading order of the ω-expansion the Poisson bracket between these functions. In 1997 Kontsevich [3] demonstrated the existence of a star product on any smooth Poisson manifold and presented a formula which, in principle, allowed to calculate such a product. However, if one needs the star product beyond the second order of the expansion the formulae of Kontsevich are not very useful since there is no regular method to calculate the integrals involved. Besides, there is another (psichological) reason to look for a more simple formulation of the star product. The machinery of the Formality Theorem is a bit too heavy to be easily diggested by a considerable part of the physics community. Also note that the Formality Theorem actually gives much more than just a star product. Therefore, when the hard part of the job is already done, one can start looking for a formulation of star products in a more physical language admitting a simpler computational algorithm for higher orders of the expansion of star products.

In this paper we develop an approach based on the symmetric ordering of the operators $\hat{x}^j$ which represent deformed Cartesian coordinates on $\mathbb{R}^N$. We require that $\hat{x}^j$ satisfy the main commutation relation $[\hat{x}^j, \hat{x}^k] = 2\alpha\hat{\omega}^{jk}(\hat{x})$ (with α being a formal deformation parameter) and construct a 1-polydifferential representation for $\hat{x}^j$. The main advantage of this approach is the existence of an iterative procedure in α such that the order $(n-1)$ in $\hat{\omega}$ allows to calculate the order n in $\hat{x}$, which in turn defines the order n of $\hat{\omega}$. Besides, we operate with object which are familiar from quantum mechanics. Some elements of this construction can be found scattered in the literature. Behr and Sykora [4] used the Weyl ordered operator products to construct generic star products on $\mathbb{R}^N$, but without constructing a polydifferential representation for $\hat{x}$. Technical difficulties did not allow them to go beyond the second order of the expansion. Particular realization of generators as formal power series is a very natural tool in the case of linear Poisson structure (Lie algebras), which was effectivly used to analyze star products [5–7].

Computations of the third order star product were done in [8] by using the Hochschild cohomologies and in [9] by changing variables in the Moyal formula. Our results to this order are in perfect agreement with [8]. Precise relations of our star product to that of [9] are less clear to us, but at least we do not see any contradictions.

2 The Weyl map and the star product

Consider a set of noncommuting operators $\hat{x}^j$, $j = 1, \ldots, N$ and a function f on $\mathbb{R}^N$ which can be expanded in the Taylor series around zero,

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}_{i_1 \ldots i_n} x^{i_1} \ldots x^{i_n} \tag{1}$$

with x^i being Cartesian coordinates on $\mathbb{R}^N$. We associate to this function a symmetrically (Weyl) ordered operator function $\hat{f}(\hat{x})$ according to the rule

$$\hat{f}(\hat{x}) = \sum_{n=0}^{\infty} f^{(n)}_{i_1 \ldots i_n} \sum_{P_n} \frac{1}{n!} P_n(\hat{x}^{i_1} \ldots \hat{x}^{i_n}), \tag{2}$$

where P_n are all permutation of n elements. We shall also use another notation,

$$\hat{f} \equiv W(f). \tag{3}$$

There is another, more convenient form of the Weyl ordering. Let $\tilde{f}(p)$ be a Fourier transform of f,

$$\tilde{f}(p) = \int d^N x\, f(x)\, e^{ip_j x^j}, \tag{4}$$

then

$$\hat{f}(\hat{x}) = \int \frac{d^N p}{(2\pi)^N} \tilde{f}(p)\, e^{-ip_m \hat{x}^m}. \tag{5}$$

(More on various orderings and corresponding integral representations see in [10]).

Consider a skew-symmetric matrix-valued function $\omega^{ij}(x)$. Let us now impose a commutation relation on the operators $\hat{x}$:

$$[\hat{x}^i, \hat{x}^j] = 2\alpha\hat{\omega}^{ij}(\hat{x}), \tag{6}$$

where α is a formal expansion parameter. In the physics literature $\alpha = i\hbar/2$. The commutation relation (6) yields a consistency condition

$$[\hat{x}^i, \hat{\omega}^{jk}(\hat{x})] + \text{cycl.}(ijk) = 0. \tag{7}$$

We define a star product as

$$W(f \star g) = W(f) \cdot W(g). \tag{8}$$

This product is associative due to associativity of the operator products. We also like to keep the unity of the algebra of functions undeformed, yielding $f \star 1 = f$, or

$$W(f) \cdot 1 = f. \tag{9}$$

The equations (8) and (9) yield the following formula

$$(f \star g)(x) = W(f)\, g(x) = \hat{f}(\hat{x})\, g(x)\ . \tag{10}$$

where the right hand side means an action of a polydifferential operator on a function.

Note that neither x^j nor unit function are Schwarz class. They do not belong to the algebra we are going to deform, but rather to the algebra of multipliers (see [11]). Of course, it is important that they stay in the multiplier algebra also after the deformation. However, since we shall work with formal expansions only, these subtleties will be ignored.

There is an obvious symmetry of this construction which changes the sign on α, $\alpha \to -\alpha$, and reverses the order of the operators. (This is nothing else than the complex conjugation if one remembers that $\alpha = i\hbar/2$ is in fact imaginary). This symmetry interchages f and g in (8). A consequence of this symmetry is that even orders in α are symmetric with respect to interchanging f and g, while the odd orders are antisymmetric. Note, that only the deformations of $\mathcal{A}$ which have this property are usually regarded as deformation quantizations.

We shall look for a representation of the operators $\hat{x}$ as polydifferential operators in the form of an α-expansion,

$$\hat{x}^i = x^i + \sum_{n=1}^{\infty} \Gamma^{i(n)}(\alpha, x)(\alpha\partial)^n , \tag{11}$$

where

$$\Gamma^{i(n)}(\alpha, x) = \Gamma^{ii_1\dots i_n}(\alpha, x) \tag{12}$$

and each Γ, in turn, is expanded in α

$$\Gamma^{i(n)}(\alpha, x) = \sum_{k=0}^{\infty} \alpha^k \Gamma_k^{i(n)}(x) . \tag{13}$$

The matrix-valued function $\omega^{ij}(x)$ is also expanded in a power series in α, and the α^0 term is a Poisson structure, i.e., ω_0^{ij} satisfies the Jacobi identity

$$\omega_0^{kl}\partial_l\omega_0^{ij} + \omega_0^{jl}\partial_l\omega_0^{ki} + \omega_0^{il}\partial_l\omega_0^{kj} = 0 . \tag{14}$$

Since the coefficient functions $\Gamma^{i(n)}$ are contracted in (11) with partial derivatives only the part of $\Gamma^{i(n)}$ which is symmetric in the last n indices contributes. From now on we assume that $\Gamma^{i(n)}$ *is* symmetric in the last n indices.

The star product will be constructed by using an iterative procedure. One start with a zeroth order ω which may be an arbitrary Poisson structure and solves (6) to the order α to obtain the leading order of the coefficient functions Γ and of the operators $\hat{x}^j$. Then, by using these operators, one constructs a Weyl ordered $\hat{\omega}^{ij}(\hat{x})$ in terms of ω_0, substitutes it to the consistency condition (7) and finds corrections to the ω_0. This corrections measure the failure of ω_0 to be a Poisson bivector. (Untill rather high orders in α the condition (7) is satisfied automatically and no corrections to ω appear). The one repeats the procedure with this new ω in the next order in α.

3 General properties of the expansion

3.1 Expansion of $\hat{x}$

Consider the restrictions on the coefficient functions Γ imposed by the condition (9). This condition is equivalent to the requirement

$$W(x^{i_1}\dots x^{i_n}) \cdot 1 = x^{i_1}\dots x^{i_n} \tag{15}$$

for all monomials of the coordinates and in all orders in α. Note, that the right hand side of (15) does not depend on α. Consequently, all contributions to the left hand side higher than zeroth order in α must vanish. It is easy to see, that this, in turn, is equivalent to vanishing of totally symmetrized parts of all Γ,

$$\Gamma^{(ii_1\ldots i_k)} = 0 \ . \tag{16}$$

Or, the contraction of each $\Gamma^{(n)}$ with $n+1$ commuting vectors p_i vanishes,

$$p_i p_{i_1} \ldots p_{i_k} \Gamma^{ii_1\ldots i_k} = 0 \ , \tag{17}$$

For brevity, by somewhat stretching the terminology, we shall call this condition the tracelessness condition.

Let us now turn to the consequences of the commutation relation (6). As we shall see below, the relation (6) allows to express

$$\Gamma_k^{[ii_1]i_2\ldots i_p} \equiv \Gamma_k^{ii_1i_2\ldots i_p} - \Gamma_k^{i_1ii_2\ldots i_p} \tag{18}$$

with $k+p=n$ in terms of the lower order Γ's. It is convenient to expand the right hand side of (6) as

$$\hat{\omega}^{ij}(\hat{x}) = \sum_{n=0} \hat{\omega}_n^{ij} \ . \tag{19}$$

where $\hat{\omega}_n$ is of the order α^n. We also introduce corresponding expansions for $\hat{x}$ as

$$\hat{x}^j = \hat{x}_n^j + O(\alpha^{n+1}) \ , \qquad \hat{x}_{n+1}^j = \hat{x}_n^j + \alpha^{n+1}\gamma_{n+1}^j \ . \tag{20}$$

Suppose, one has already calculated the expansion of $\hat{x}$ up to the order α^n, i.e

$$[\hat{x}_n^i, \hat{x}_n^j] = 2\alpha\hat{\omega}_{n-1} + o(\alpha^n). \tag{21}$$

Next we check the consistency condition (7) in the order α^n:

$$\left[\hat{x}_n^k, \hat{\omega}_n^{ij}\right] + \left[\hat{x}_n^j, \hat{\omega}_n^{ki}\right] + \left[\hat{x}_n^i, \hat{\omega}_n^{jk}\right] = o\left(\alpha^n\right) \ . \tag{22}$$

To the lowest orders this equation is satisfied automatically for any Poisson bivector ω^{ij}. In the higher orders it does not, and (22) must be considered as a condition of non-Poisson corrections to ω^{ij}. Solving this equation, and consequncies of the corrections is the most nontrivial part of our procedure. This part will be considered separately in sec. 3.2. For the time being we assume that corresponding corrections are constructed and (22) is satisfied.

In order to construct the next, $(n+1)$th order in the decomposition we have to solve the following equation

$$\left[\hat{x}_{n+1}^i, \hat{x}_{n+1}^j\right] = 2\alpha\hat{\omega}_n^{ij} + o\left(\alpha^{n+1}\right) \ . \tag{23}$$

Or,

$$\left[\hat{x}_n^i + \alpha^{n-1}\gamma_{n+1}^i, \hat{x}_n^j + \alpha^{n+1}\gamma_{n+1}^j\right] = 2\alpha\hat{\omega}_n^{ij} + o\left(\alpha^{n+1}\right)$$

which implies

$$\left[\hat{x}^i_n, \hat{x}^j_n\right] + \alpha^{n+1}\gamma^{[ij]}_{n+1} = 2\alpha\hat{\omega}^{ij}_n + o\left(\alpha^{n+1}\right), \tag{24}$$

where

$$\gamma^{[ij]}_{n+1} \equiv \sum_{p+k=n+1} p\Gamma^{[ij]i_2\ldots i_p}_k \partial_{i_2}\ldots\partial_{i_p}. \tag{25}$$

Therefore, the antisymmetric part of Γ_{n+1} is determined from the equation

$$\alpha^{n+1}\gamma^{[ij]}_{n+1} = G^{ij}_{n+1} + o\left(\alpha^{n+1}\right), \tag{26}$$

where

$$G^{ij}_{n+1} = 2\alpha\hat{\omega}^{ij}_n - \left[\hat{x}^i_n, \hat{x}^j_n\right] + o\left(\alpha^{n+1}\right). \tag{27}$$

From this equation G^{ij}_{n+1} is defined up to terms $o\left(\alpha^{n+1}\right)$. We do not include any higher order terms in G^{ij}_{n+1}, so that $G^{ij}_{n+1} \sim \alpha^{n+1}$. The operators G^{ij}_{n+1} can be expanded as

$$G^{ij}_{n+1} = \sum_p G^{iji_2\ldots i_p}_{n+1}\partial_{i_2}\ldots\partial_{i_p}. \tag{28}$$

The coefficient functions in this expansion are antisymmetric in i, j and symmetric in $i_2, \ldots, i_p$ by the construction, and they also have the following property.

Lemma 1 *The functions $G^{iji_2\ldots i_p}_{n+1}$ obey the cyclic condition*

$$G^{iji_2\ldots i_p}_{n+1} + \mathrm{cycl.}(iji_2) = 0. \tag{29}$$

Proof. From (22) one has

$$\left[\hat{x}^k_n, 2\alpha\hat{\omega}^{ij}_n\right] + \mathrm{cycl.}(kij) = o\left(\alpha^{n+1}\right), \tag{30}$$

or, using (27),

$$\left[\hat{x}^k_n, G^{ij}_{n+1} + \left[\hat{x}^i_n, \hat{x}^j_n\right]\right] + \mathrm{cycl.}(kij) = o\left(\alpha^{n+1}\right). \tag{31}$$

The equation

$$\left[\hat{x}^k_n, \left[\hat{x}^i_n, \hat{x}^j_n\right]\right] + \mathrm{cycl.}(kij) = 0 \tag{32}$$

holds true at all orders of α, including the order α^{n+1}. Therefore, from (31) one obtains

$$\left[x^k_n, G^{ij}_{n+1}\right] + cycl = 0. \tag{33}$$

Next one substitutes the expansion (28) in (33), calculates the commutator, and uses the symmetry of G_{n+1} in the last $p-1$ indices to complete the proof. ∎

Because of the symmetry of $G^{iji_2\ldots i_p}_{n+1}$ in the last $p-1$ indices, the cyclic conditions holds for permuations of (i, j, i_k) for any $k = 2, \ldots, p$.

Now we are able to construct $\Gamma^{ij\ldots k}_p$ and thus the operator $\hat{x}$ to the order α^{n+1} by symmetrising the functions $G^{iji_1\ldots i_k}_{n+1}$. The equations which should be satisfied by the functions Γ in order to solve the commutation relation (23) read

$$G^{ji_1\ldots i_p}_{n+1} = \alpha^{n+1}p\Gamma^{[ji_1]\ldots i_p}_k, \qquad k+p = n+1. \tag{34}$$

A solution to these equations is given by the following Lemma.

Lemma 2 *The tensors*

$$\Gamma_k^{ji_1\ldots i_p} = \frac{\alpha^{-(n+1)}}{p(p+1)} \left(G_{n+1}^{ji_1i_2\ldots i_p} + G_{n+1}^{ji_2i_1i_3\ldots i_p} + \cdots + G_{n+1}^{ji_pi_1i_2\ldots i_{p-1}} \right) \tag{35}$$

are symmetric in the last p indices, satisfy the equation (34) and the tracelessness conditon (16).

Proof. The symmetry follows by the construction, and the tracelessnes is a consequence of the antisymmetry of $G_{n+1}^{ji_1i_2\ldots i_p}$ in the first two indices. To prove the remaining assertion, let us consider antisymmetrized in j and i_1 combinations of the tensors entering the right hand side of (34).

$$G_{n+1}^{ji_1i_2\ldots i_p} - G_{n+1}^{i_1ji_2\ldots i_p} = 2G_{n+1}^{ji_1i_2\ldots i_p}$$

due to the antisymmetry of G in the first two indices.

$$G_{n+1}^{ji_2i_1\ldots i_p} - G_{n+1}^{i_1i_2j\ldots i_p} = G_{n+1}^{ji_1i_2\ldots i_p}$$

due to the cyclic condition in j, i_1, i_2. Remaining $p-2$ combinations are treated similarly, and the assertion follows immediately. ∎

This Lemma implies that the tensors (35) are indeed the coefficient functions of the expansion of $\hat{x}^j$ we are looking for.

For the notational convenience we define $\star_k$ which is a part of the star product having the order α^k.

$$f \star g = \sum_{k=0}^{n} f \star_k g + O(\alpha^{n+1}). \tag{36}$$

To evaluate the star product and $\hat{\omega}(\hat{x})$ to a given order of α we shall need an effective tool to calculate an α expansion of the Weyl-ordered operator $\hat{f}(\hat{x})$ for any given f and a given expansion of $\hat{x}$. To this end we shall use the integral representation (5) and the Duhamel formula

$$e^{A+B} = e^A + \int_0^1 e^{(A+B)s} B e^{(1-s)A} ds \ . \tag{37}$$

Here $A + B = -ip_i\hat{x}^i$, $A = -ip_ix^i$ and $B = -ip_i(\hat{x}^i - x^i)$. Therefore, B is of the order α^1, and A is of the order α^0, but, because of the property (17), each commutator is at least one order higher, $[B, A] = O(\alpha^2)$, $[[B, A], A] =$

$O(\alpha^3)$, $[[[B, A], A], A] = O(\alpha^4)$. By using these rules, one easily finds,

$$\begin{aligned} e^{A+B} = \; & e^A \Big(1 + B + \frac{1}{2}[B, A] + \frac{1}{2}B^2 \\ & + \frac{1}{6}[[B, A], A] + \frac{1}{3}[B, A]B + \frac{1}{6}B[B, A] + \frac{1}{6}B^3 \\ & + \frac{1}{24}[[[B, A], A], A] + \frac{1}{8}[[B, A], A]B + \frac{1}{8}[B, A]^2 \\ & + \frac{1}{24}B[[B, A], A] + \frac{1}{8}[B, A]B^2 + \frac{1}{12}B[B, A]B \\ & + \frac{1}{24}B^2[B, A] + \frac{1}{24}B^4\Big) + O(\alpha^5) \; . \end{aligned} \tag{38}$$

3.2 Corrections to ω

Here we give a short overview of what happens if corrections to ω are needed. A more detailed discussion can be found in sec. 4.4 below where we deal with a particular case of the third order star product. Suppose we have completed our construction at some order n, i.e., we have operators $\hat{x}^j_n$ fulfilling the relation (21). Next, we take the function ω_{n-1} and construct the corresponding Weyl ordered operator $\hat{\omega}^{ij}(\hat{x}_n)$ and the star product $\tilde{\star}_n$, where the twiddle means that we still have to check the consistency condition (22) at the order n. Let us assume for simplicity that at all lower orders no corrections to ω appeared, i.e., that all lower order consistency conditions were satisfied by $\omega^{ij} = \omega^{ij}_0$. Let us denote $\tilde{\star} \equiv \star_1 + \cdots + \star_{n-1} + \tilde{\star}_n$ and compute

$$x^i \tilde{\star} \omega^{jk} - \omega^{jk} \tilde{\star} x^i + \mathrm{cycl}(ijk) \, . \tag{39}$$

If the expression (39) is $o(\alpha^n)$ for $\omega = \omega_0$, no corrections are needed at this order as well. If the expression (39) contains some α^n terms (lower order terms vanish due to the lower order consistency conditions), we have to correct ω and, consequently, the star product. It is easy to see, that no correction to ω at the order α^n will do the job. We must correct ω in the order α^{n-1}, so that $\omega = \omega_0 + \alpha^{n-1}\omega_{n-1}$. This looks very dangerous for the whole approach, since we are going to correct the order we have already constructed. However, this is not so. Let us consider more closely what is going on. The only effect the correction to ω at the order α^{n-1} has on the operators $\hat{x}_n$ is that now the part Γ^{ij}_{n-1} is non-zero,

$$\Gamma^{ij}_{n-1} = \omega^{ij}_{n-1} . \tag{40}$$

The star product is modified in the n-th order only, $\tilde{\star}_n \longrightarrow \star_n$,

$$f \star_n g = f \tilde{\star}_n g + \alpha^n \omega^{ij}_{n-1} \partial_i f \partial_j g . \tag{41}$$

This formula is related to the well known ambiguity in the star products (see, e.g., [8]). If the star product $\tilde{\star}$ is associative up to α^{n+1} terms, then the star product $\star$ is also associative up to α^{n+1} independently of the choice of ω_{n-1}^{ij}. Therefore, both products are legitimate star products. The only diffrerence is that $\tilde{\star}$ cannot be extended to the next order *with our methods*, while $\star$ can. If we need a star product to the order n only, we can as well ignore all corrections coming from the nth and higher order consistency conditions.

Another problem is to find actually an expression for ω_{n-1}^{ij} which solves the consistency condition. Here we shall not attempt to present any method of solving the consistency condition for ω_{n-1}^{ij} or even analyse the existence of a solution for the following reasons. The equation for ω_{n-1}^{ij} appearing in our approach are practically identical to that coming from the Hochschild cohomologies [8]. Studying relations between our operator algebra approach and Hochschiled cohomologies is an interesting problem on its own right, and we are going to address it in a separate work. For practical purposes, to calculate the star product up to the fifth order, one only needs to solve a single non-trivial consistency condition at the third order. This can be done directly, see sec. (4.4). Besides, third order consistency conditions have been already analysed in [8] in a different formalism.

4 Calculation of the star product

4.1 First order star product

As a warm up, let us calculate the zeroth and first orders of the α-expansion of the star product. In the zeroth order, the condition (6) reads $[\hat{x}^i, \hat{x}^j] = 0$, so that we have an undeformed commutative algebra, $\hat{x}^j = x^j$, and, as expected,

$$f \star_0 g = f \cdot g. \tag{42}$$

To the first order in α the expansion (11) reads

$$\hat{x}^i = x^i + \alpha \Gamma_0^{ij}(x)\,\partial_j + O\left(\alpha^2\right) . \tag{43}$$

By substituting this expansion in (6) and keeping only the terms which are linear in α, we obtain

$$\Gamma_0^{[ij]} = \Gamma_0^{ij}(x) - \Gamma_0^{ji}(x) = 2\omega^{ij}(x) \ , \tag{44}$$

which implies

$$\Gamma_0^{ij}(x) = \omega^{ij}(x) + S_0^{ij}(x) \ . \tag{45}$$

Note, that to this order in α the Weyl ordering is trivial, $\hat{\omega}(\hat{x}) = \omega(x)$. The symmetric part S_0^{ij} is eliminated by the tracelessness condition (16), $S_0^{ij} = 0$, and

$$\Gamma_0^{ij}(x) = \omega^{ij}(x) \ . \tag{46}$$

The Duhamel formula (37) gives

$$\begin{aligned} e^{-ip_m\hat{x}^m} &= e^A + e^A B + O\left(\alpha^2\right) \\ &= e^{-ip_m x^m} - i\alpha e^{-ip_m x^m} p_j \omega^{jk}(x)\,\partial_k + O\left(\alpha^2\right) . \end{aligned} \tag{47}$$

By using the equation (10) we immediately obtain

$$f \star_1 g = \alpha\omega^{ij}\partial_i f\, \partial_j g, \tag{48}$$

and the expansion

$$\hat{\omega}^{ij}(\hat{x}) = \omega^{ij}(x) + \alpha\partial_l\omega^{ij}(x)\,\omega^{lk}(x)\,\partial_k + O\left(\alpha^2\right) \ , \tag{49}$$

which will be used below to calculate the next order star product.

The consistency condition (7) to this order

$$(x^k \star_1 \omega^{ij} - \omega^{ij} \star_1 x^k) + \text{cycl.}(kil) = 0 \tag{50}$$

yields the Jacobi identity (14), i.e., $\omega = \omega_0$ is a Poisson bivector. We shall drop the subscript 0 from the notations whenever this cannot lead to a confusion.

4.2 Second order star product

As is seen from the equation (49), at the order α^1 the operator $\hat{\omega}^{ij}$ is a first order differential operator with a vanishing zeroth order part. Consequently, γ_2^i does not have a first order part, and we may write

$$\hat{x}_2^i = x^i + \alpha\omega^{ij}(x)\,\partial_j + \alpha^2\Gamma_0^{ijk}(x)\,\partial_j\partial_k. \tag{51}$$

The commutation relation (21) with $n = 1$ yields

$$2\alpha^2\Gamma_0^{[il]k} = G_2^{ilk} = 2\alpha^2\left(\omega^{jk}\partial_j\omega^{il} + \frac{1}{2}\omega^{lj}\partial_j\omega^{ik} - \frac{1}{2}\omega^{ij}\partial_j\omega^{lk}\right). \tag{52}$$

The cyclic condition

$$G_2^{ilk} + G_2^{kil} + G_2^{lki} = 0 \tag{53}$$

can easily be checked. It is equivalent to the Jacobi identity on ω^{ij}. With the help of this identity we can rewrite (52) as

$$G_2^{ilk} = \alpha^2\omega^{jk}\partial_j\omega^{il} \ . \tag{54}$$

According to the general prescription of Lemma 2,

$$\alpha^2\Gamma_0^{ilk} = \frac{1}{6}\left(G_2^{ilk} + G_2^{ikl}\right),$$

or,

$$\Gamma_0^{ilk} = \frac{1}{6}\omega^{jk}\partial_j\omega^{il} + \frac{1}{6}\omega^{jl}\partial_j\omega^{ik} \ . \tag{55}$$

Obviously, Γ_0^{ilk} obeys the tracelessness condition (16). Eq. (52) can be checked directly by using Jacobi identity.

Now we are ready calculate star product up to the second order using the formula (10). First we calculate

$$[B, A] = -\frac{\alpha^2}{3} p_i p_k \omega^{ji} \partial_j \omega^{kl} \partial_l + O\left(\alpha^3\right).$$

The decomposition (38) yields

$$\begin{aligned} e^{-ip_i\hat{x}^i} &= e^{-ip_m x^m} - i\alpha e^{-ip_m x^m} p_i \omega^{ij}(x)\,\partial_j \\ &-\frac{\alpha^2}{2} e^{-ip_m x^m} p_i p_k \omega^{ij}\omega^{kl}\partial_j\partial_l - \frac{\alpha^2}{3} e^{-ip_m x^m} p_i p_k \omega^{ij}\partial_j \omega^{kl}\partial_l \\ &+\frac{i\alpha^2}{6} e^{-ip_m x^m} p_k \left(\omega^{jk}\partial_j\omega^{il} + \omega^{jl}\partial_j\omega^{ik}\right)\partial_i\partial_l + O\left(\alpha^3\right). \end{aligned} \tag{56}$$

From eq. (5), we have

$$\begin{aligned} f(\hat{x}) &= f(x) + \alpha\omega^{ij}\partial_i f\partial_j \\ &+\frac{\alpha^2}{2}\omega^{ij}\omega^{kl}\partial_i\partial_k f\partial_j\partial_l + \frac{\alpha^2}{3}\omega^{ij}\partial_j\omega^{kl}\partial_i\partial_k f\partial_l \\ &-\frac{\alpha^2}{3}\left(\omega^{jk}\partial_j\omega^{il} + \omega^{jl}\partial_j\omega^{ik}\right)\partial_k f\partial_i\partial_l + O\left(\alpha^3\right). \end{aligned} \tag{57}$$

And then, from (10) we obtain

$$\begin{aligned} (f\star g)(x) &= \hat{f}(\hat{x})\,g(x) = fg + \alpha\partial_i f\omega^{ij}\partial_j g \\ &+\frac{\alpha^2}{2}\omega^{ij}\omega^{kl}\partial_i\partial_k f\partial_j\partial_l g + \frac{\alpha^2}{3}\omega^{ij}\partial_j\omega^{kl}\left(\partial_i\partial_k f\partial_l g - \partial_k f\partial_i\partial_l g\right) + O\left(\alpha^3\right). \end{aligned} \tag{58}$$

This expression coincides with the Kontsevich formula [3]. The same expression was rederived from the Weyl-ordered operator products by Behr and Sykora [4]. Since the product (58) coincides with known ones, there is no need to check the consistency conditions (7) at this order. The consistency condition also follows from a more strong statement

$$f \star_2 g - g \star_2 f = 0, \tag{59}$$

which is a consequence of the symmetry we described in sec. 2 and may be easily verified from (58). No correction to ω appears, meaning that to this order $\omega^{ij} = \omega_0^{ij}$ is a Poisson structure.

As a preparation to the next order calculations we also write an expansion for $\hat{\omega}$:

$$\begin{aligned} \hat{\omega}^{ij}(\hat{x}) &= \omega^{ij} + \alpha\omega^{kl}\partial_k\omega^{ij}\partial_l \\ &+\frac{\alpha^2}{2}\omega^{nk}\omega^{ml}\partial_n\partial_m\omega^{ij}\partial_k\partial_l + \frac{\alpha^2}{3}\omega^{nk}\partial_k\omega^{ml}\partial_n\partial_m\omega^{ij}\partial_l \\ &-\frac{\alpha^2}{3}\left(\omega^{nk}\partial_n\omega^{lm} + \omega^{nl}\partial_n\omega^{km}\right)\partial_m\omega^{ij}\partial_k\partial_l + O\left(\alpha^3\right). \end{aligned} \tag{60}$$

4.3 Third order star product

The operator $\hat{\omega}$ in the order α^2, eq. (60), as well as in the previous order, does not contain a part which is a zeroth order differential operator. Consequently, γ_3^j does not contain zeroth or first order terms, and we may write

$$
\begin{aligned}
&\hat{x}_2^i + \gamma_3^i = \hat{x}_3^i = x^i + \alpha\Gamma_0^{ij}(x)\,\partial_j + \alpha^2\Gamma_0^{ijk}(x)\,\partial_j\partial_k + \\
&\alpha^3\left(\Gamma_1^{ijk}(x)\,\partial_j\partial_k + \Gamma_0^{ijkl}(x)\,\partial_j\partial_k\partial_l\right) + O\left(\alpha^4\right) \,. \qquad (61)
\end{aligned}
$$

The function Γ_0^{ijk} is known, while Γ_1^{ijk} and Γ_0^{ijkl} have to be defined. By comparing (60) to the commutator of (61) we obtain (cf. (27))

$$
G_3^{ijk} = \frac{\alpha^3}{3}\omega^{nl}\left(2\partial_l\omega^{mk}\partial_n\partial_m\omega^{ij} + \partial_l\omega^{mj}\partial_n\partial_m\omega^{ik} + \partial_l\omega^{mi}\partial_n\partial_m\omega^{kj}\right) \qquad (62)
$$

and

$$
\begin{aligned}
G_3^{ijkl} = \alpha^3 \Big(&\omega^{nk}\omega^{ml}\partial_n\partial_m\omega^{ij} + \frac{1}{3}\omega^{kn}\partial_n\omega^{lm}\partial_m\omega^{ij} \\
&+\frac{1}{3}\omega^{ln}\partial_n\omega^{km}\partial_m\omega^{ij} + \omega^{jm}\partial_m\Gamma_0^{ikl} - \omega^{im}\partial_m\Gamma_0^{jkl} \qquad (63)\\
&+ \Gamma_0^{jkm}\partial_m\omega^{il} + \Gamma_0^{jlm}\partial_m\omega^{ik} - \Gamma_0^{ikm}\partial_m\omega^{jl} - \Gamma_0^{ilm}\partial_m\omega^{jk}\Big) \,.
\end{aligned}
$$

The cyclic identities (Lemma 1) can be verified directly as consequencies of the Jacobi identity. According to Lemma 2, we have

$$
\Gamma_1^{ijk} = \frac{1}{6}\omega^{nl}\partial_l\omega^{mk}\partial_n\partial_m\omega^{ij} + \frac{1}{6}\omega^{nl}\partial_l\omega^{mj}\partial_n\partial_m\omega^{ik}. \qquad (64)
$$

and

$$
\alpha^3\Gamma_0^{ijkl} = \frac{1}{12}\left(G_3^{ijkl} + G_3^{ikjl} + G_3^{ilkj}\right) \,. \qquad (65)
$$

With the help of the commutators (85), the Duhamel formula (38), and the expressions (10), (62) - (65) we compute the third order star product.

$$
\begin{aligned}
f\star_3 g \;=\; \alpha^3 \Big[&\frac{1}{3}\omega^{nl}\partial_l\omega^{mk}\partial_n\partial_m\omega^{ij}\left(\partial_i f\partial_j\partial_k g - \partial_i g\partial_j\partial_k f\right) \\
&+\frac{1}{6}\omega^{nk}\partial_n\omega^{jm}\partial_m\omega^{il}\left(\partial_i\partial_j f\partial_k\partial_l g - \partial_i\partial_j g\partial_k\partial_l f\right) \\
&+\frac{1}{3}\omega^{ln}\partial_l\omega^{jm}\omega^{ik}\left(\partial_i\partial_j f\partial_k\partial_n\partial_m g - \partial_i\partial_j g\partial_k\partial_n\partial_m f\right) \qquad (66)\\
&+\frac{1}{6}\omega^{jl}\omega^{im}\omega^{kn}\partial_i\partial_j\partial_k f\partial_l\partial_n\partial_m g \\
&+\frac{1}{6}\omega^{nk}\omega^{ml}\partial_n\partial_m\omega^{ij}\left(\partial_i f\partial_j\partial_k\partial_l g - \partial_i g\partial_j\partial_k\partial_l f\right)\Big] \,.
\end{aligned}
$$

We put a twiddle over the star to stress that (66) does not include corrections to ω required by the consistency condition (22) at the third order. However, this is a fully legitimate star product. It only cannot be extended to the fourht order in our procedure. Eq. (66) is in agreement with [8] and does fullfill the requirements following from associativity [9]. It does not coincide with a particular star product constructed in [9] by changing coordinates in the Moyal formula, but the difference presumably resides in the ambiguity discussed in sec. 3.2 plus a gauge transformation.

4.4 Corrections to the third order star product

Let us study the third order consistency conditions for the star product we have obtained in the previous subsection. We have to calculate (39) for $n = 3$ and check whether α^3 terms vanish. This boils down to the condition

$$\begin{aligned} & x^a \tilde{\star}_3 \omega^{bc} - \omega^{bc} \tilde{\star}_3 x^a + \mathrm{cycl}(abc) \\ & = \alpha^3 \left(\frac{2}{3} \omega^{nl} \partial_l \omega^{mk} \partial_n \partial_m \omega^{aj} \partial_j \partial_k \omega^{bc} + \frac{1}{3} \omega^{nk} \omega^{ml} \partial_n \partial_m \omega^{aj} \partial_j \partial_k \partial_l \omega^{bc} \right) \\ & \qquad + \mathrm{cycl}(abc) = 0 \ . \end{aligned} \tag{67}$$

The condition (67) is not satisfied generically for $\omega^{bc} = \omega_0^{bc}$, i.e. it does not follow from the Jacobi identity [8]. Therefore, a correction to ω_0 is needed, and, according to sec. 3.2, this has to be an α^2 correctio.

$$\omega^{bc}(x) = \omega_0^{bc}(x) + \alpha^2 \omega_2^{bc}(x) + \ldots \tag{68}$$

This leads to a non-zero Γ_2^{ij} according to the equation

$$\Gamma_2^{ij} - \Gamma_2^{ji} = 2\omega_2^{ij} \ , \tag{69}$$

which, together with the tracelssness condition, yields

$$\Gamma_2^{ij} = \omega_2^{ij} \ . \tag{70}$$

The third order star product is changed, $\tilde{\star}_3 \to \star_3$, as

$$f \star_3 g = f \tilde{\star}_3 g + \alpha^3 \partial_i f \omega_2^{ij} \partial_j g \ . \tag{71}$$

ω_2^{ij} has to be defined from the consistency condition which reads

$$\begin{aligned} & x^a \star_3 \omega_0^{bc} - \omega_0^{bc} \star_3 x^a + \alpha^2 x^a \star_1 \omega_2^{bc} - \alpha^2 \omega_2^{bc} \star_1 x^a + \mathrm{cycl.}(abc) \\ & = \alpha^3 \left[2\omega_0^{ad} \partial_d \omega_2^{bc} + 2\omega_2^{ad} \partial_d \omega_0^{bc} + \frac{2}{3} \omega_0^{nl} \partial_l \omega_0^{mk} \partial_n \partial_m \omega_0^{aj} \partial_j \partial_k \omega_0^{bc} \right. \\ & \left. + \frac{1}{3} \omega_0^{nk} \omega_0^{ml} \partial_n \partial_m \omega_0^{aj} \partial_j \partial_k \partial_l \omega_0^{bc} \right] + \mathrm{cycl.}(abc) = 0 \ . \end{aligned} \tag{72}$$

ω_2^{jk} which solves this equation must be a sum of monomials each containing three ω_0 and four derivatives. The most general ansatz is

$$\begin{aligned}\omega_2^{bc} &= c_1\partial_m\omega_0^{nl}\partial_n\omega_0^{mk}\partial_l\partial_k\omega_0^{bc} + c_2\partial_k\omega_0^{bm}\partial_l\omega_0^{cn}\partial_n\partial_m\omega_0^{kl} \\ &+c_3\partial_n\partial_k\omega_0^{bm}\partial_m\partial_l\omega_0^{cn}\omega_0^{kl}\ .\end{aligned} \tag{73}$$

Working hard we find that $c_1 = -\frac{1}{12}$, $c_2 = 0$ and $c_3 = \frac{1}{6}$. The same solution of the equation (72) was obtained in [8].

Now we are able to obtain the third order operator valued function $\hat{\omega}_3^{ij}$ by collecting correspoding orders in α in the Weyl ordered expression $W(\omega_0^{ij}+\alpha^2\omega_2^{ij})$. Explicitely, it reads

$$\begin{aligned}\alpha^{-3}\hat{\omega}_3^{ij} &= \left(\left(\Gamma_1^{ijk} + \frac{1}{2}\Gamma_0^{jmn}\partial_m\partial_n\omega_0^{ik}\right)\partial_a\partial_b\omega_0^{ij} + \left(\Gamma_0^{ijlk} + \frac{2}{3}\Gamma_0^{ijm}\partial_m\omega_0^{lk} + \right.\right.\\ &\left.\frac{1}{3}\omega_0^{im}\partial_m\Gamma_0^{jlk} + \frac{1}{6}\omega_0^{im}\partial_m\omega_0^{jn}\partial_n\omega_0^{lk} + \frac{1}{6}\omega_0^{im}\omega_0^{jn}\partial_m\partial_n\omega_0^{lk}\right)\partial_a\partial_b\partial_l\omega_0^{ij} \\ &\left.+\omega_0^{lk}\partial_l\omega_2^{ij} + \omega_2^{lk}\partial_l\omega_0^{ij}\right)\partial_k \qquad (74)\\ &+\left(\Gamma_1^{akl}\partial_a\omega_0^{ij} + \left(\frac{3}{2}\Gamma_0^{abkl} + \frac{1}{2}\omega_0^{am}\partial_m\Gamma_0^{bkl} + \Gamma_0^{bmk}\partial_m\omega_0^{al}\right)\partial_a\partial_b\omega_0^{ij}\right.\\ &\left.+\left(\Gamma_0^{abk}\omega_0^{ml} + \frac{1}{2}\omega_0^{bn}\partial_n\omega_0^{ak}\omega_0^{ml}\right)\partial_a\partial_b\partial_m\omega_0^{ij}\right)\partial_k\partial_l \\ &+\left(\Gamma_0^{bml}\omega_0^{ak}\partial_a\partial_b\omega_0^{ij} + \frac{1}{6}\omega_0^{ak}\omega_0^{bl}\omega_0^{nm}\partial_a\partial_b\partial_n\omega_0^{ij}\right.\\ &\left.+\Gamma_0^{aklm}\partial_a\omega_0^{ij}\right)\partial_k\partial_l\partial_m\ .\end{aligned} \tag{75}$$

4.5 Fourth order star product

First we write $\hat{x}_4^i$:

$$\hat{x}_4^i = \hat{x}_3^i + \gamma_4^i, \tag{76}$$

where

$$\gamma_4^i = \alpha^4\left(\Gamma_2^{ijk}(x)\,\partial_j\partial_k + \Gamma_1^{ijkl}(x)\,\partial_j\partial_k\partial_l + \Gamma_0^{ijklm}(x)\,\partial_j\partial_k\partial_l\partial_m\right). \tag{77}$$

Now comparing (74) to the commutator of (76) one has

$$\begin{aligned}\alpha^{-4}G_4^{ijklm} &= 2\Gamma_0^{bml}\omega_0^{ak}\partial_a\partial_b\omega_0^{ij} + \frac{1}{3}\omega_0^{ak}\omega_0^{bl}\omega_0^{nm}\partial_a\partial_b\partial_n\omega_0^{ij} + 2\Gamma_0^{aklm}\partial_a\omega_0^{ij} \\ &+3\Gamma_0^{jlmn}\partial_n\omega_0^{ik} - 3\Gamma_0^{ilmn}\partial_n\omega_0^{jk} - 2\Gamma_0^{imn}\partial_n\Gamma_0^{jkl} + 2\Gamma_0^{jmn}\partial_n\Gamma_0^{ikl} \\ &+\omega_0^{in}\partial_n\Gamma_0^{jklm} - \omega_0^{jn}\partial_n\Gamma_0^{iklm}\ ,\end{aligned} \tag{78}$$

$$
\begin{aligned}
\alpha^{-4}G_4^{ijkl} &= \left(3\Gamma_0^{abkl} + 3\omega_0^{am}\partial_m\Gamma_0^{bkl} + 2\Gamma_0^{bmk}\partial_m\omega_0^{al}\right)\partial_a\partial_b\omega_0^{ij} \\
&+2\Gamma_1^{akl}\partial_a\omega_0^{ij} + \left(2\Gamma_0^{abk}\omega_0^{ml} + \omega_0^{bn}\partial_n\omega_0^{ak}\omega_0^{ml}\right)\partial_a\partial_b\partial_m\omega_0^{ij} \\
&+3\Gamma_0^{jlmn}\partial_m\partial_n\omega_0^{ik} - 3\Gamma_0^{ilmn}\partial_m\partial_n\omega_0^{jk} - 2\Gamma_1^{ilm}\partial_m\omega_0^{jk} \\
&+2\Gamma_1^{jlm}\partial_m\omega_0^{ik} + \omega_0^{in}\partial_n\Gamma_1^{jkl} - \omega_0^{jn}\partial_n\Gamma_1^{ikl} \\
&-\Gamma_0^{imn}\partial_m\partial_n\Gamma_0^{jkl} + \Gamma_0^{jmn}\partial_m\partial_n\Gamma_0^{ikl} \ ,
\end{aligned} \tag{79}
$$

and

$$
\begin{aligned}
\alpha^{-4}G_4^{ijk} &= \left(2\Gamma_1^{ijk} + \Gamma_0^{jmn}\partial_m\partial_n\omega_0^{ik}\right)\partial_a\partial_b\omega_0^{ij} + \left(2\Gamma_0^{ijlk} + \frac{4}{3}\Gamma_0^{ijm}\partial_m\omega_0^{lk}\right. \\
&+\frac{2}{3}\omega_0^{im}\partial_m\Gamma_0^{jlk} + \frac{1}{3}\omega_0^{im}\partial_m\omega_0^{jn}\partial_n\omega_0^{lk} \\
&\left.+\frac{1}{3}\omega_0^{im}\omega_0^{jn}\partial_m\partial_n\omega_0^{lk}\right)\partial_a\partial_b\partial_l\omega_0^{ij} + 2\omega_0^{lk}\partial_l\omega_2^{ij} + 2\omega_2^{lk}\partial_l\omega_0^{ij} \\
&-\omega_0^{il}\partial_l\omega_2^{jk} + \omega_0^{jl}\partial_l\omega_2^{ik} - \omega_2^{il}\partial_l\omega_0^{jk} + \omega_2^{jl}\partial_l\omega_0^{ik} \\
&-\Gamma_1^{ilm}\partial_l\partial_m\omega_0^{jk} + \Gamma_1^{jlm}\partial_l\partial_m\omega_0^{ik} - \Gamma_0^{ilmn}\partial_l\partial_m\partial_n\omega_0^{jk} \\
&+\Gamma_0^{jlmn}\partial_l\partial_m\partial_n\omega_0^{ik} \ .
\end{aligned} \tag{80}
$$

Using the Lemma 2, one obtains

$$
\alpha^4\Gamma_0^{ijklm} = \frac{1}{20}\left(G_4^{ijklm} + G_4^{ikjlm} + G_4^{iljkm} + G_4^{imjkl}\right) \ , \tag{81}
$$

$$
\alpha^4\Gamma_0^{ijkl} = \frac{1}{12}\left(G_4^{ijkl} + G_4^{ikjl} + G_4^{ilkj}\right) \tag{82}
$$

and

$$
\alpha^4\Gamma_2^{ijk} = \frac{1}{6}\left(G_4^{ijk} + G_4^{ikj}\right) \ . \tag{83}
$$

With the help of the commutators (85), the Duhamel formula (38), and the expressions (10), (78) - (83) one computes the fourth order star product $f \star_4 g$.

At the fourth order, as in all even orders,

$$
f \star_4 g - g \star_4 f = 0, \tag{84}
$$

so that the consistency condition is satisfied automatically, and ω must not be corrected.

5 Discussion and conclusions

If, for some reason, the consistency condition (7) is satisfied automatically at all orders (which is true, e.g., for deformatios on $\mathbb{R}^2$ or for some polynomial algebras) our formulae provide a star product at any finite order in α, and, in principle, one can even try to guess or derive a non-perturbative expression for the star product. It would also be interesting to relate our iteratition formulae to the Ward identities of the path integral of Cattaneo and Felder [12] with the help of the methods [13, 14] developed for calculation of such integrals.

Combining our approach with other approaches may also bring interesting results. It is very natural to consider linear Poisson structures, where simpler results are possible and a considerable progress has already been made by the methods quite similar to the ones proposed above [5–7]. For the reason we already mentioned, our formalism simplifies on two dimensional Poisson manifolds, where a coherent state formalism was used to construct star products [15]. Finally, it is interesting and imporatant to give our construction a geometric flavor along the lines of [16], [17] and [18].

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A Some useful formulae

In order to use the Duhamel expansion (38) one has to calculate repeated commutators of $A = -ip_m x^m$ and $B = -ip_j(\hat{x}^j - x^j)$. Denote $B_k = -ip_j \Gamma^{ji_1 \dots i_k} \partial_{i_1} \dots \partial_{i_k}$. Then

$$
\begin{aligned}
&[B_k, A] = k(-ip_j)(-ip_{i_1})\Gamma^{ji_1 i_2 \dots i_k} \partial_{i_2} \dots \partial_{i_k}, \quad k \geq 2, \\
&[[B_k, A], A] = k(k-1)(-ip_j)(-ip_{i_1})(-ip_{i_2})\Gamma^{ji_1 i_2 \dots i_k} \partial_{i_3} \dots \partial_{i_k}, \quad k \geq 3, \\
&[[[B_k, A], A], A] \\
&\quad = k(k-1)(k-2)(-ip_j)(-ip_{i_1})(-ip_{i_2})(-ip_{i_3})\Gamma^{ji_1 i_2 \dots i_k} \partial_{i_4} \dots \partial_{i_k}, \quad k \geq 4.
\end{aligned}
\tag{85}
$$

If k does not satisfy the inequalities given above, the right hand side vanishes. It is convenient to keep the momenta in the combinations $(-ip_m)$ since after taking the integral (5) they are replaced by partial derivatives ∂_m acting on f.

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