Quantization of the Damped Harmonic Oscillator Revisited

M.C. Baldiotti*, R. Fresneda†, and D.M. Gitman‡

May 21, 2010

Instituto de Física, Universidade de São Paulo,
Caixa Postal 66318-CEP, 05315-970 São Paulo, S.P., Brasil

Abstract

We return to the description of the damped harmonic oscillator by means of a closed quantum theory with a general assessment of previous works, in particular the Bateman-Caldirola-Kanai model and a new model recently proposed by one of the authors. We show the local equivalence between the two models and argue that latter has better high energy behavior and is naturally connected to existing open-quantum-systems approaches.

1 Introduction

The problem of constructing a quantum theory for the damped harmonic oscillator as well as for similar dissipative systems, e.g. radiating point-like charge, has attracted attention for already more than 50 years. In spite of the success of interaction-with-reservoir approaches, we feel there is still room for some formal developments in the direction of a closed theory approach. In this article we analyze a novel Lagrangian model for the damped harmonic oscillator, which was recently proposed in [1] as a particular case of a general procedure for finding action functionals for non-Lagrangian equations of motion. Even though we find it is locally equivalent to the renowned Bateman-Caldirola-Kanai (BCK) model [2, 3, 4] (see Section 2 for a revision), a complete global equivalence is absent, and we believe it has some formal advantages over its predecessor, regarding the high energy behavior of solutions to the Schrödinger equation. Notwithstanding these formal discrepancies between the two models, we show that they share a very close physical interpretation with regard to their asymptotic behavior in time and to their physical functions.

Ever since the proposition of the BCK model, there have been divided opinions as to whether it really describes the damped harmonic oscillator or dissipation. There are those who dispute it as a possible dissipative model [5, 6, 7], and those, in addition to the original authors, who maintain it accounts for some form of dissipation [8, 9]. As we explain further on, the BCK theory is not defined globally in time, and many of the pathologies usually appointed to the quantum theory can be seen as an artifact of its infinite time indefinition. Nonetheless, regardless of these disputes,

*baldiott@fma.if.usp.br
†fresneda@gmail.com
‡gitman@dfn.if.usp.br

there has been fruitful applications of the BCK theory as a model of dissipation, at least in the case of the canonical description of the Fabry-Pérot cavity [10].

The underlying understanding of dissipative systems is that they are physically part of a larger system, and dissipation is a result of a non-elastic interaction between the reservoir and the subsystem. Thus, quantization can conceivably follow two different approaches: the first one takes the classical equations of motion of the system and applies to them formal quantization methods, trying to overcome the difficulties related to the "non-Lagrangian" nature of the system. Without attempting to exhaust all the literature on this matter, we cite, for instance, the use of canonical quantization of classical actions [11, 12], Fermi quantization [13], path-integral quantization [14], doubling of the degrees of freedom [15, 16], group theory methods [17], complex classical coordinates [18], propagator methods [19], non-linear Schrödinger equation [20], and finally a constrained dynamics approach [21]. The second approach aims at constructing a quantum theory of the subsystem by averaging over the reservoir, see e.g. [5]. It is immediately clear that two different outcomes must follow from these approaches. Indeed, in the first case, we consider the system as being closed, and thus naturally only pure states can result, and they represent physical states without any further restrictions. In the second case, the subsystem's states will be necessarily described by a statistical operator, such that only mixed states are physically sensible. In spite of the general understanding that the second approach seems to be more physical and probably should produce a more adequate quantum theory for dissipative systems, in this article we wish once again to analyze the first, formal, approach. Its comparison to the second one will be given in a future work. Our analysis will be particular to the damped harmonic oscillator as an example of a dissipative system, and it will be devoted to the comparison of two different canonical quantizations.

In Section 2 we revise the BCK theory, emphasizing problems which were not considered before. In Section 3 we present the first-order theory, define some useful physical quantities, and we also construct the coherent and squeezed states of the first-order theory in order to obtain the classical limit. In Section 4, we prove the local equivalence of the theories presented in Sections 2 and 3, and discuss possible divergences related to their global behavior. In Section 5 we present some final remarks and discuss our results as well as open problems.

2 BCK Theory

One of the peculiarities of dissipative systems, which hindered early quantization attempts, was the non-Lagrangian nature of the classical equations of motion. In the particular case of the damped harmonic oscillator of constant frequency ω and friction coefficient $\alpha > 0$, the second-order equation of motion

$$\ddot{q} + 2\alpha\dot{q} + \omega^2 q = 0 \tag{1}$$

cannot be directly obtained as the Euler-Lagrange (EL) equation of any Lagrangian, since it fails to satisfy the Helmholtz conditions [22]. Nevertheless, there is an equivalent second-order equation for which a variational principle can be found, namely,

$$e^{2\alpha t}\left(\ddot{q} + 2\alpha\dot{q} + \omega^2 q\right) = 0\,. \tag{2}$$

The exponential factor is known as the integrating multiplier, and it is enough to make the above equation satisfy the Helmholtz conditions [23]. The fact that a Lagrangian can always be found for the one-dimensional problem such that its EL equation is equivalent to a given second-order equation was established by Darboux [24].

As was already mentioned, the equation (1) is traditionally considered to be non-Lagrangian, albeit the existence of a questionable action functional [6, 7] which reproduces the equivalent equations of motion (2). In this respect, we have to mention that an action principle for the equation of motion (2) was first proposed by Bateman [2] in terms of the Lagrangian

$$L_B = \frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right) e^{2\alpha t}\,. \tag{3}$$

If Bateman had constructed the corresponding Hamiltonian formulation, he would have discovered that the corresponding Hamiltonian theory is canonical without constraints and with Hamiltonian

$$H_{BCK}\left(q,p\right) = \frac{1}{2}\left[e^{-2\alpha t}p^2 + \omega^2 e^{2\alpha t}q^2\right]\,. \tag{4}$$

This Hamiltonian was proposed independently by Caldirola and Kanai [3, 4] to describe the damped harmonic oscillator in the framework of quantum mechanics. Consequently, we write the subscript BCK (Bateman-Caldirola-Kanai) to label the Hamiltonian.

Formal canonical quantization of the Lagrangian action (3) is straightforward,

$$[\hat{q},\hat{p}] = i\,,\ [\hat{q},\hat{q}] = [\hat{p},\hat{p}] = 0,\ \hat{H}_{BCK} = H_{BCK}\left(\hat{q},\hat{p}\right)\,, \tag{5}$$

and coincides with Caldirola and Kanai's quantum theory. Solutions to the Schrödinger equation with Hamiltonian $\hat{H}_{BCK}$ have been found in the form

$$\psi_n^{BCK}\left(q,t\right) = \left(2^n n!\right)^{-1/2}\left(\frac{\tilde{\omega}}{\pi}\right)^{1/4}\exp\left(-iE_n t + \frac{\alpha t}{2} - \left(\tilde{\omega} + i\alpha\right)\frac{q^2}{2}e^{2\alpha t}\right)H_n\left(\sqrt{\tilde{\omega}}qe^{\alpha t}\right)\,, \tag{6}$$

where $E_n = \tilde{\omega}\left(n + 1/2\right)$, $\tilde{\omega} = \sqrt{\omega^2 - \alpha^2}$ and H_n are Hermite polynomials. These are the familiar pseudostationary states [25, 26, 11] or loss-energy states [8] which are also eigenstates of the Hamiltonian $\hat{H}_{BCK} + \frac{\alpha}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)$ with eigenvalues E_n. Even though $\left|\psi_n^{BCK}\right|^2$ depends on time, the total probability $\int dq\left|\psi_n^{BCK}\right|^2 = 1$ is time-independent, as can be seen by the transformation of variables $q \mapsto q' = qe^{\alpha t}$. Moreover, the mean value of the BCK Hamiltonian in the pseudostationary states is constant,

$$\left\langle\psi_n^{BCK}\right|\hat{H}_{BCK}\left|\psi_n^{CK}\right\rangle = \left\langle\psi_n^{BCK}\right|E_n - \frac{\alpha}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\left|\psi_n^{BCK}\right\rangle = \frac{\omega^2}{\tilde{\omega}}\left(n + \frac{1}{2}\right)\,, \tag{7}$$

which is a reflection of the fact that in the classical theory defined by (3) the average of the Hamiltonian over the period of one oscillation is constant. On the other hand, mean values of the mechanical energy $E = \frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right)$ decay exponentially with time, $\left\langle E\right\rangle_n = e^{-2\alpha t}\left\langle H_{BCK}\right\rangle_n$ [25, 11]. Coherent states for the BCK theory are given in [9], for which the uncertainty relations are

$$\Delta q\Delta p = \frac{\omega}{2\tilde{\omega}} \geq \frac{1}{2}\,. \tag{8}$$

Now we draw attention to the high-energy behavior of pseudostationary states. In the appendix we consider asymptotic (in n) pseudostationary functions, and for these one has the limiting eigenvalue equation

$$\hat{H}_{BCK}\psi_n^{BCK} = \left[E_n - \frac{\alpha}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right]\psi_n^{BCK} = \left(E_n + \frac{i\alpha}{2}\right)\psi_n^{BCK} + O\left(n^{-1/4}\right)\,. \tag{9}$$

The appearance of imaginary eigenvalues is actually explained by taking into account the domain of the operator $\hat{q}\hat{p} + \hat{p}\hat{q}$, which, as is shown in the appendix, does not include the asymptotic part of ψ_n^{BCK}. In this connection, it should be noted that the above is in contrast to [8, 27], where it is claimed that the pseudostationary states are eigenstates of $\hat{H}_{BCK}$ with eigenvalues $(\tilde{\omega} + i\alpha)(n + 1/2)$.

One overlooked aspect of the BCK theory is that EL equation (2) obtained from the Lagrangian (3) is only equivalent to the equation of motion of the damped harmonic oscillator (1) for finite times. The equation (2) is indicative that the theory described by (3) is not globally defined, i.e., it is not defined for infinite times. In effect, the manifold difficulties which appear in connection to the $t \to \infty$ limit, such as the violation of the Heisenberg uncertainty principle and the vanishing of the ground state energy for infinite times, can be seen as consequence of inadvertently assuming equations (1) and (2) are equivalent for all values of the time parameter.

3 First-order action

3.1 Action, hamiltonization, and quantization

Next we consider the canonical quantization of the damped harmonic oscillator based on the alternative action proposed in [1]. The idea was to reduce the second-order equations (1) to the first-order system

$$\dot{x} = y, \;\; \dot{y} = -\omega^2 x - 2\alpha y \,, \tag{10}$$

for which, according to the general theory [28, 29], the action functional has the form

$$S = \frac{1}{2}\int dt \left[y\dot{x} - x\dot{y} - \left(y^2 + 2\alpha xy + \omega^2 x^2 \right) \right] e^{2\alpha t} \,. \tag{11}$$

The EL equations of motion derived from (11) are locally equivalent to (10),

$$\frac{\delta S}{\delta x} = \left(\dot{y} + 2\alpha y + \omega^2 x \right) e^{2\alpha t} \,, \;\; \frac{\delta S}{\delta y} = \left(\dot{x} - y \right) e^{2\alpha t} \,.$$

Note that, as in the case of the BCK theory, the theory fails to describe the damped harmonic oscillator as the time approaches infinity, and thus one should expect problems in the quantum theory such as violation of the uncertainty principle for infinite times, for instance.

This action describes a singular system with second-class constraints, and furthermore, these constraints are time-dependent (we follow the terminology of the book [30]). Even though the constraints are explicitly time-dependent, it is still possible to write the Hamiltonian formalism with the help of Dirac brackets and perform the canonical quantization, as is explained in [30].

In order to do this, one must extend the initial phase space of canonical variables $\eta = (x, y, p_x, p_y)$ by the inclusion of the time t and its associated momentum ε. As a result, the Poisson brackets between functions defined on the extended phase space is

$$\{F, G\} = \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial p_x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial p_y} + \frac{\partial F}{\partial t}\frac{\partial G}{\partial \varepsilon} \right) - F \leftrightarrow G \,.$$

With the above definition, the first-order equations of motion (10) are equivalently written in terms of Dirac brackets,

$$\dot{\eta} = \left\{ \eta, H(x, y, t) + \epsilon \right\}_{D(\phi)} \,, \;\; \phi_x = \phi_y = 0 \,, \tag{12}$$

where the Hamiltonian $H(x,y,t)$ and the constraints ϕ are

$$H(x,y,t) = \frac{1}{2}\left(y^2 + 2\alpha xy + \omega^2 x^2\right) e^{2\alpha t}\,, \tag{13}$$
$$\phi_x = p_x - \frac{1}{2} y e^{2\alpha t}\,,\ \phi_y = p_y + \frac{1}{2} x e^{2\alpha t}\,.$$

The nonzero commutation relations between the independent variables are

$$\{x,y\}_{D(\phi)} = e^{-2\alpha t}\,. \tag{14}$$

Quantization of this system follows the general method described in [30] for theories with time-dependent constraints. One introduces time-dependent operators $\hat{\eta}(t)$ which satisfy the differential equations $d\hat{\eta}/dt = i\{\eta,\epsilon\}_{D(\phi)}\Big|_{\eta=\hat{\eta}}$ with initial conditions subject to an analog of the Dirac quantization,

$$[\hat{\eta}(0),\hat{\eta}(0)] = i\{\eta,\eta\}_{D(\phi_0)}\Big|_{\eta=\hat{\eta}(0)}\,,$$
$$\hat{p}_x(0) - \frac{1}{2}\hat{y}(0) = \hat{p}_y(0) + \frac{1}{2}\hat{x}(0) = 0\,.$$

The above operatorial constraints allows us to work only in terms of the independent operators $\hat{x}$ and $\hat{y}$. Let us define $\hat{x}(0) \equiv \hat{q}$ and $\hat{y}(0) \equiv \hat{p}$, so that the above quantum brackets have the familiar form

$$[\hat{q},\hat{p}] = i\,,\ [\hat{q},\hat{q}] = [\hat{p},\hat{p}] = 0,\quad \hat{x}(0) \equiv \hat{q},\quad \hat{y}(0) \equiv \hat{p}\,. \tag{15}$$

The differential equations for $\hat{x}$ and $\hat{y}$ can be easily integrated to

$$\hat{x}(t) = e^{-\alpha t}\hat{q}\,,\ \hat{x}(0) \equiv \hat{q}\,,\ \hat{y}(t) = e^{-\alpha t}\hat{p}\,,\ \hat{y}(0) \equiv \hat{p}\,. \tag{16}$$

The quantum Hamiltonian is obtained from the classical Hamiltonian (13) as a function of the operators $\hat{x}(t)$ and $\hat{y}(t)$ (16): it does not depend on time at all,

$$\hat{H} = H(\hat{x}(t),\hat{y}(t),t) = \frac{1}{2}\left[\hat{p}^2 + \alpha(\hat{q}\hat{p} + \hat{p}\hat{q}) + \omega^2\hat{q}^2\right]\,, \tag{17}$$

where we have used Weyl (symmetric) ordering for the mixed product $2\alpha xy$. The Hamiltonian $\hat{H}$ governs the time-evolution of the state vector in the Schrödinger picture, and it has appeared in a number of different contexts: in [31, 10] with regard to the electromagnetic field in a resonant cavity; in the quantization of the complex symplectic theory [32]; and in [33] in connection to the Lindblad theory of open quantum systems for the damped harmonic oscillator.

Since the Hamiltonian is time-independent, the evolution operator is given simply by $U(t) = e^{-i\hat{H}t}$, with $\hat{H}$ given by (17). The Heisenberg operators $\check{x}$ and $\check{y}$ corresponding to the classical variables x and y are

$$\check{x} = U^{-1}\hat{x}(t)\,U = e^{-\alpha t}\left(\cos\tilde{\omega}t + \frac{\alpha}{\tilde{\omega}}\sin\tilde{\omega}t\right)\hat{q} + \frac{1}{\tilde{\omega}}e^{-\alpha t}\sin(\tilde{\omega}t)\,\hat{p}\,,$$
$$\check{y} = U^{-1}\hat{y}(t)\,U = e^{-\alpha t}\left(\cos\tilde{\omega}t - \frac{\alpha}{\tilde{\omega}}\sin\tilde{\omega}t\right)\hat{p} - \frac{\omega^2}{\tilde{\omega}}e^{-\alpha t}\sin(\tilde{\omega}t)\,\hat{q}\,,$$
$$\check{x}(0) \equiv \hat{q}\,,\ \check{y}(0) \equiv \hat{p}\,,\ [\hat{q},\hat{p}] = i\,,\ [\hat{q},\hat{q}] = [\hat{p},\hat{p}] = 0\,.$$

From the above expressions, one also finds

$$\frac{d\check{x}}{dt} = \check{y}\,,\ \frac{d\check{y}}{dt} = -\omega^2\check{x} - 2\alpha\check{y}\,,$$

which coincide in form with the classical equations,

$$\frac{d\check{x}}{dt} = \{x, H+\epsilon\}_{D(\phi)}\Big|_{\eta=\check{\eta}}\,,\ \frac{d\check{y}}{dt} = \{y, H+\epsilon\}_{D(\phi)}\Big|_{\eta=\check{\eta}}\,. \tag{18}$$

Thus, Heisenberg equations (18) reproduce the classical equations of motion, and therefore mean values of x and y follow classical trajectories.

Moreover, the only nonzero commutator becomes

$$[\check{x}, \check{y}] = ie^{-2\alpha t}\,,$$

which matches the classical Dirac bracket (14). Thus, the resulting quantum theory at least obeys the correspondence principle.

One can easily find solutions of the Schrödinger equation by making a (time-independent) unitary transformation $\hat{H}_{\tilde{\omega}} = \hat{S}^{-1}\hat{H}\hat{S}$,

$$\hat{S} = \exp\left(-\frac{i\alpha}{2}\hat{q}^2\right)\,,\ \hat{S}^{-1}\hat{p}\hat{S} = \hat{p} - \alpha\hat{q}\,, \tag{19}$$

one obtains

$$\hat{H}_{\tilde{\omega}} = \frac{1}{2}\left(\hat{p}^2 + \tilde{\omega}^2\hat{q}^2\right),\quad \tilde{\omega} = \sqrt{\omega^2 - \alpha^2}\,. \tag{20}$$

The Hamiltonian (20) has the familiar stationary states

$$\begin{aligned}
&\psi_n^{\tilde{\omega}}(t,q) = \exp\left(-itE_n\right)\psi_n^{\tilde{\omega}}(q)\,,\\
&\psi_n^{\tilde{\omega}}(q) = \left(2^n n!\right)^{-1/2}\left(\frac{\tilde{\omega}}{\pi}\right)^{1/4}\exp\left(-\frac{\tilde{\omega}}{2}q^2\right)H_n\left(\sqrt{\tilde{\omega}}q\right)\,,\\
&\hat{H}_{\tilde{\omega}}\psi_n^{\tilde{\omega}} = E_n\psi_n^{\tilde{\omega}}\,,\ E_n = \tilde{\omega}\left(n+\frac{1}{2}\right)\,.
\end{aligned}$$

Therefore, the wave functions

$$\psi_n(q) = \hat{S}\psi_n^{\tilde{\omega}}(q) = \left(2^n n!\right)^{-1/2}\left(\frac{\tilde{\omega}}{\pi}\right)^{1/4}\exp\left(-\left(\tilde{\omega}+i\alpha\right)\frac{q^2}{2}\right)H_n\left(\sqrt{\tilde{\omega}}q\right) \tag{21}$$

are eigenfunctions of the Hamiltonian $\hat{H}$, and the solutions to the corresponding Schrödinger equation are

$$\begin{aligned}
&\psi_n(t,q) = \hat{S}\psi_n^{\tilde{\omega}}(t,q) = \exp\left(-itE_n\right)\psi_n(q)\,,\\
&\hat{H}\psi_n(q) = E_n\psi(q)\,.
\end{aligned}$$

One can also check directly that (21) are eigenfunctions of $\hat{H}$ with eigenvalue E_n by using properties of Hermite functions.

We now define some useful quantities to be used in Subsection 4.2 for the purpose of establishing the physical equivalence between the approaches presented here. Let us write the classical Lagrangian energy as

$$\mathcal{E}_L \equiv \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{y}}\dot{y} - L = \frac{1}{2}\left(y^2 + 2\alpha xy + \omega^2 x^2\right) e^{2\alpha t}\,.$$

The corresponding Weyl-ordered Schrödinger operator for the Lagrangian energy is

$$\hat{\mathcal{E}}_L = \mathcal{E}_L\left(\eta\right)\big|_{\eta=\hat{\eta}} = \frac{1}{2}\left[\hat{p}^2 + \alpha\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right) + \omega^2\hat{q}^2\right] = \hat{H}\,, \tag{22}$$

which coincides with the Hamiltonian and thus with its Heisenberg representation, and is therefore conserved. Likewise, we define the following conserved "energy" in the BCK approach,

$$\begin{aligned}
&\mathcal{E} = \frac{1}{2}\left(\dot{q}^2 + 2\alpha q\dot{q} + \omega^2 q^2\right) e^{2\alpha t} = H_{BCK} + \alpha qp\,,\\
&\frac{d}{dt}\mathcal{E} = \left(\dot{q} + \alpha q\right) e^{2\alpha t}\frac{\delta S}{\delta q}\,,
\end{aligned}$$

which is constant on-shell. Its image as a Weyl-ordered Schrödinger operator is

$$\hat{\mathcal{E}} = \hat{H}_{BCK} + \frac{\alpha}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\,,\ \hat{\mathcal{E}}\psi_n^{BCK} = E_n\psi_n^{BCK}\,. \tag{23}$$

Now consider the mechanical energy in the theory with Lagrangian (11) at $\alpha = 0$:

$$E_M = \frac{1}{2}\left(y^2 + \omega^2 x^2\right)\,.$$

The corresponding operator is equal to

$$\hat{E}_M = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right) e^{-2\alpha t} = e^{-2\alpha t}\hat{H} - e^{-2\alpha t}\frac{\alpha}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\,. \tag{24}$$

The mean value of $\hat{E}_M$ in the energy eigenstates is

$$\left\langle\psi_n\right|\hat{E}_M\left|\psi_n\right\rangle = e^{-2\alpha t}\frac{\omega^2}{\tilde{\omega}}\left(n + \frac{1}{2}\right)\,.$$

Finally, we consider the observable defined by the mechanical energy in the BCK description,

$$E = \frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right) = \frac{1}{2}\left(e^{-2\alpha t}p^2 + \omega^2 e^{2\alpha t}q^2\right) e^{-2\alpha t} = H_{BCK}e^{-2\alpha t}\,. \tag{25}$$

Thus, by (7),

$$\left\langle\psi_n^{BCK}\right|\hat{E}\left|\psi_n^{BCK}\right\rangle = e^{-2\alpha t}\frac{\omega^2}{\tilde{\omega}}\left(n + \frac{1}{2}\right) = \left\langle\psi_n\right|\hat{E}_M\left|\psi_n\right\rangle\,.$$

3.2 Semiclassical description

3.2.1 Coherent states

Finally, we obtain semiclassical states for the damped harmonic oscillator from the coherent states of the simple harmonic oscillator using the unitary transformation $\hat{S}$ (19). To this end, we introduce first creation and annihilation operators $\hat{a}^+$ and $\hat{a}$ and the corresponding coherent states $|z\rangle$,

$$\begin{aligned} &\hat{a} = \frac{1}{\sqrt{2\tilde{\omega}}}\left(\tilde{\omega}\hat{q} + i\hat{p}\right)\ ,\ \hat{a}^+ = \frac{1}{\sqrt{2\tilde{\omega}}}\left(\tilde{\omega}\hat{q} - i\hat{p}\right)\ ,\ \left[\hat{a}, \hat{a}^+\right] = 1\,, \\ &|z\rangle = D\left(z\right)|0\rangle\ ,\ D\left(z\right) = \exp\left(z\hat{a}^+ - \bar{z}\hat{a}\right)\ ,\ a\,|z\rangle = z\,|z\rangle\ . \end{aligned} \tag{26}$$

In terms of these creation and annihilation operators, the Hamiltonian (20) is

$$\hat{H}_{\tilde{\omega}} = \tilde{\omega}\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)\ .$$

Thus, the coherent states for the Hamiltonian $\hat{H}$ are $\hat{S}\,|z\rangle$ and the mean values of $\check{x}$ and $\check{y}$ in these coherent states are

$$\begin{aligned} \langle x\rangle &\equiv \langle z|\,\hat{S}^{-1}\check{x}\hat{S}\,|z\rangle = \frac{1}{\sqrt{2\tilde{\omega}}}e^{-\alpha t}\left(ze^{-i\tilde{\omega}t} + \bar{z}e^{i\tilde{\omega}t}\right)\ , \\ \langle y\rangle &= i\sqrt{\frac{\tilde{\omega}}{2}}e^{-\alpha t}\left(\bar{z}e^{i\tilde{\omega}t} - ze^{-i\tilde{\omega}t}\right) - \alpha\,\langle x\rangle\ . \end{aligned}$$

One can now easily verify that the mean values of the coordinates x and y follow the classical trajectories,

$$\frac{d}{dt}\langle x\rangle = \langle y\rangle\ ,\ \frac{d}{dt}\langle y\rangle = -\omega^2\,\langle x\rangle - 2\alpha\,\langle y\rangle\ .$$

The pathological behavior of the first-order theory with regard to the limit $t \to \infty$ can be seen here in the computation of the uncertainty relation

$$\Delta x\Delta y = \frac{1}{2}e^{-2\alpha t}\frac{\omega}{\tilde{\omega}}\,. \tag{27}$$

The unphysical result of the above violation of the uncertainty principle is an indication that the first-order theory is not defined for all values of the time parameter, as was pointed out in connection to the classical equations of motion and to the commutation relation between $\hat{x}$ and $\hat{y}$. Another indication of the failure of the theory at infinite times is in the observation that the radius of the trajectory of the mean values vanishes:

$$\rho\left(z = re^{i\theta}\right) = \sqrt{\langle x\rangle^2 + \langle y\rangle^2} = \sqrt{\frac{2}{\tilde{\omega}}}e^{-\alpha t}r\sqrt{1 + \alpha^2\cos\left(\tilde{\omega}t - \theta\right) + \alpha\sin 2\left(\tilde{\omega}t - \theta\right)}\,.$$

3.2.2 Squeezed-state

One can also consider the family of conserved creation and annihilation operators

$$\begin{aligned} &\hat{b}\left(t\right) = \cosh\xi e^{i\tilde{\omega}t}\hat{a} + \sinh\xi e^{-i\tilde{\omega}t}\hat{a}^\dagger\ ,\ \hat{b}^\dagger\left(t\right) = \cosh\xi e^{-i\tilde{\omega}t}\hat{a}^\dagger + \sinh\xi e^{i\tilde{\omega}t}\hat{a}\,, \\ &\left[\hat{b}\left(t\right), \hat{b}^\dagger\left(t\right)\right] = 1\,,\ \frac{d}{dt}\hat{b} = \frac{d}{dt}\hat{b}^\dagger = 0\,, \end{aligned}$$

and construct squeezed coherent states $|z,\xi\rangle = \exp\left(z\hat{b}^{\dagger} - \bar{z}\hat{b}\right)|0\rangle$ [34]. For $\xi = 0$ one arrives at the previous coherent states,

$$\langle z,0|\,\hat{x}\,|z,0\rangle = \langle z|\,\hat{S}^{-1}\check{x}\hat{S}\,|z\rangle\ ,\ \langle z,0|\,\hat{y}\,|z,0\rangle = \langle z|\,\hat{S}^{-1}\check{y}\hat{S}\,|z\rangle\ .$$

For arbitrary values of ξ and with $\hat{x}$ and $\hat{y}$ given by (16), one has for the mean values of coordinates

$$\langle z,\xi|\,\hat{x}\,|z,\xi\rangle = \frac{e^{-\alpha t}}{\sqrt{2\tilde{\omega}}}\left[\left(e^{-i\tilde{\omega}t}z + e^{i\tilde{\omega}t}\bar{z}\right)\cosh\xi - \left(e^{i\tilde{\omega}t}z + e^{-i\tilde{\omega}t}\bar{z}\right)\sinh\xi\right]\ ,$$

$$\langle z,\xi|\,\hat{y}\,|z,\xi\rangle = i\sqrt{\frac{\tilde{\omega}}{2}}e^{-\alpha t}\left[\left(e^{i\tilde{\omega}t}\bar{z} - e^{-i\tilde{\omega}t}z\right)\cosh\xi - \left(e^{i\tilde{\omega}t}z - e^{-i\tilde{\omega}t}\bar{z}\right)\sinh\xi\right] - \alpha\,\langle z,\xi|\,\hat{x}\,|z,\xi\rangle\ .$$

The main interest in squeezed states is that they allow one to change the uncertainty in either direction x or y by adjusting the parameter ξ. For example, the uncertainty in x can be written for arbitrary values of ξ as

$$(\Delta x)^2 = \frac{1}{2\tilde{\omega}}e^{-2\alpha t}\left[(\cosh\xi + \sinh\xi)^2 - 4\cosh\xi\sinh\xi\cos^2\tilde{\omega}t\right] \geqslant 0\,,$$

and it reduces to the preceding coherent states calculations for $\xi = 0$.

4 Comparison of the BCK system and the first-order system

4.1 BCK model as a transformation of the first-order system

Here we show how one can obtain the classical and quantum description of the BCK damped harmonic oscillator as a canonical transformation of the first-order approach (3). At the classical level, both systems are transformed one into the other by means of the following time-dependent canonical transformation[1]

$$q = \frac{1}{2}\left(x - 2e^{-2\alpha t}p_y\right)\ ,\ p = p_x + \frac{1}{2}e^{2\alpha t}y\,,$$
$$\Omega^1 = \frac{1}{2}y - p_x e^{-2\alpha t}\ ,\ \Omega_2 = p_y + \frac{1}{2}xe^{2\alpha t}\,,$$

where (q,p) and $\left(\Omega^1,\Omega_2\right)$ are new pairs of canonical variables. In these new variables, the equation of motion become Hamiltonian:

$$\dot{q} = \{q, H_{BCK}\}\ ,\ \dot{p} = \{p, H_{BCK}\}\ ,\ \Omega = 0\,,$$
$$H_{BCK} = \frac{1}{2}\left[e^{-2\alpha t}p^2 + \omega^2 e^{2\alpha t}q^2\right]\ ,$$

where the Hamiltonian $H_{BCK}\,(q,p,t)$ is the canonically transformed Hamiltonian $H\,(x,y,t)$ (13) on the equivalent constraint surface $\Omega = 0$.

[1]The generating function for these canonical transformations depending on the new and old momenta is

$$F\,(p_x,p_y,p,\Omega_2,t) = -2e^{-2\alpha t}\,(\Omega_2 - p_y)\,p_x - 2e^{-2\alpha t}pp_y + e^{-2\alpha t}p\Omega_2\,.$$

It is useful to write the following relation between old coordinates x and y and the new variables q and p:

$$\begin{aligned} x &= q + O(\Omega) , \\ y &= e^{-2\alpha t} p + O(\Omega) . \end{aligned} \tag{28}$$

This relation shows that x is physically equivalent to q, while y is physically equivalent to $e^{-2\alpha t}p$, since they coincide on the constraint surface $\Omega = 0$.

The quantum theory can be readily obtained by a quantum time-dependent canonical transformation[2]

$$\hat{D} = \exp\left[\frac{i\alpha t}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right] , \tag{29}$$

which is suggested from the classical generating function and the relationship between old and new variables. The effect of $\hat{D}$ on the canonical variables is to make the dilation

$$\hat{D}^{-1}\hat{q}\hat{D} = e^{-\alpha t}\hat{q} , \ \hat{D}^{-1}\hat{p}\hat{D} = e^{\alpha t}\hat{p} .$$

The dilation operator $\hat{D}$ has been discussed in a more general setting in [35] in connection with the description of open systems by time-dependent Hamiltonians. There, as is the case here, the dilation operator simplifies the calculation of the evolution operator.

The dilaton operator transforms the BCK Hamiltonian into the first-order Hamiltonian $\hat{H}$,

$$\hat{H} = \hat{D}^{-1}\hat{H}_{BCK}\hat{D} - i\hat{D}^{-1}\frac{\partial \hat{D}}{\partial t} = \frac{1}{2}\left[\hat{p}^2 + \alpha\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right) + \omega^2\hat{q}^2\right] .$$

Since $\psi_n(q,t)$ satisfy the Schrödinger equation with $\hat{H}$, it follows that $\hat{D}\psi_n(q,t)$ satisfy the Schrödinger equation with $\hat{H}_{BCK}$. The wave functions $\hat{D}\psi_n(q,t)$ are indeed the pseudostationary states, as can be seen by direct application of $\hat{D}$. Thus, we write $\psi_n^{BCK}(q,t) \equiv \hat{D}\psi_n(q,t)$. Similarly, it also follows that $U = \hat{D}\exp\left(-it\hat{H}\right)$ satisfies the Schrödinger equation with Hamiltonian $\hat{H}_{BCK}$,

$$U(t) = \hat{D}\exp\left(-i\hat{H}t\right) , \ i\frac{\partial U}{\partial t} = \hat{H}_{BCK}U , \ U(0) = I .$$

One should not be to eager to jump to the conclusion that the two theories here presented, the BCK theory and the first-order theory, are physically equivalent solely on the grounds of the time-dependent canonical transformation $\hat{D}$. The existence of this transformation *per se* is insufficient to prove physical equivalence, since in principle, and at least locally, one can always construct a time-dependent canonical transformation between any two given theories, classical or quantum. For instance, given two quantum theories T_1 and T_2, and corresponding evolution operators U_1 and U_2, one can always write a general solution $\psi_2(t)$ of the Schrödinger equation of T_2 in terms of the general solution $\psi_1(t)$ of T_1 by means of the transformation $\psi_2(t) = U_2U_1^{-1}\psi_1(t)$. Besides the necessary ingredient of the unitary transformation relating the two theories, it is imperative to show that the physical observables pertaining both theories are also unitarily equivalent in order to prove physical equivalence. A proof which we postpone to the next Section.

[2] A proof that $\hat{x}\hat{p} + \hat{p}\hat{x}$ is self-adjoint is provided in the appendix.

4.2 Physical Equivalence

In this Section we show that the two approaches presented in this article are physically equivalent. In order to sum up the previous results in a coherent whole, let us present both quantum theories anew.

Starting with the BCK theory, it is defined by the Hamiltonian $\hat{H}_{BCK}$ (5) written in terms of the canonically conjugated operators $\hat{q}$ and $\hat{p}$ (5) with the usual realization in terms of multiplication and derivation operators in the Hilbert space $\mathcal{H}_{BCK}$ of square-integrable states $\psi_{BCK}(q)$ with measure

$$\langle \psi_{BCK} \, | \psi_{BCK} \rangle_{\mathcal{H}_{BCK}} = \int_{-\infty}^{+\infty} dq \bar{\psi}_{BCK}(q) \, \psi_{BCK}(q) \, .$$

The first-order theory is defined by the Hamiltonian $\hat{H}$ (17) and canonically conjugated operators $\hat{q}$ and $\hat{p}$ (15) realized as the usual multiplication and derivation operators in the Hilbert space $\mathcal{H}$ of square-integrable states $\psi(q)$ with measure

$$\langle \psi \, | \psi \rangle_{\mathcal{H}} = \int_{-\infty}^{+\infty} dq \bar{\psi}(q) \, \psi(q) \, .$$

We have seen that the time-dependent unitary transformation $\hat{D}$ (29) maps the two Hilbert spaces,

$$\hat{D} : \mathcal{H} \to \mathcal{H}_{BCK} \, , \; \psi \mapsto \psi_{BCK} = \hat{D}\psi \, ,$$

and is a canonical transformation,

$$\hat{H}_{BCK} = \hat{D}\hat{H}\hat{D}^{-1} + i\frac{\partial \hat{D}}{\partial t}\hat{D}^{-1} \, .$$

Therefore, for every $\psi(t)$ solution of the Schrödinger equation of of the first-order theory, $\hat{D}\psi(t)$ is a solution of the BCK Schrödinger equation. Furthermore, one has $\langle \psi_{BCK} \, | \psi_{BCK} \rangle_{\mathcal{H}_{BCK}} = \langle \psi \, | \psi \rangle_{\mathcal{H}}$, as can be seen by noting that $\hat{D}\psi(q) = e^{\alpha t/2}\psi(qe^{\alpha t})$ and

$$\langle \psi_{BCK} \, | \psi_{BCK} \rangle_{\mathcal{H}_{BCK}} = \int_{-\infty}^{+\infty} dq e^{\alpha t} \bar{\psi}\left(qe^{\alpha t}\right) \psi\left(qe^{\alpha t}\right) = \int_{-\infty}^{+\infty} dq \bar{\psi}(q) \, \psi(q) = \langle \psi \, | \psi \rangle_{\mathcal{H}} \, .$$

Finally, to complete the proof of the physical equivalence, it remains to show that any two physical observables $\hat{\mathcal{O}}_{BCK}$ and $\hat{\mathcal{O}}$ are D-equivalent, that is, $\hat{\mathcal{O}}_{BCK} = \hat{D}\hat{\mathcal{O}}\hat{D}^{-1}$. One can check that this is indeed the case for the physical observables previously considered, such as the mechanical energies (24,25)

$$\hat{E} = \hat{D}\hat{E}_M\hat{D}^{-1} \, ,$$

and the conserved energies (22,23),

$$\hat{\mathcal{E}} = \hat{D}\hat{\mathcal{E}}_L\hat{D}^{-1} \, .$$

As a result of the equivalence, one can easily obtain the BCK coherent states presented in [9] by merely transforming the coherent states (26) given in the context of the first-order theory. Taking into account the relation $y = e^{-2\alpha t}p + \{\Omega\}$ between the physical variables of the two theories (28), one can see why the uncertainty relations we obtain decay exponentially with time (27), while those

(8) calculated in [9] are constant. This has a simple explanation in terms of the physical equivalence of y and $e^{-2\alpha t}p$: y is physically equivalent to the physical momentum of the BCK oscillator, and it is in terms of the physical momentum that the Heisenberg uncertainty relations are violated (see Dekker [32] for a comprehensive review).

5 Conclusion

In this article we have proved that the classical and quantum description of the damped harmonic oscillator by the BCK time-dependent Hamiltonian [2, 3, 4] is locally equivalent to the first order approach given in terms of a constrained system [21]. This equivalence allowed us to easily obtain the evolution operator for the BCK oscillator and many other results in a simpler manner, due to the time-independence of the quantum first-order Hamiltonian. As has been pointed out (see Dekker [32] for a comprehensive review), the BCK oscillator has a pathological behavior for infinite times, since the (mechanical) energy mean values and the Heisenberg uncertainty relation - between the coordinate and the physical momentum - go to zero as time approaches infinity, so that even the ground state's energy eventually vanishes. Despite these shortcomings, the quantum theory has a well-defined classical limit, and for time values less than $\frac{1}{2\alpha}\ln\frac{\omega}{\tilde{\omega}}$ the Heisenberg uncertainty principle is not violated. It is our understanding that this unphysical behavior is a result of extending the proposed theories beyond their validity. The fact that both theories are not globally defined in time has its roots already at the classical level, as a consequence of the non-Lagrangian nature of the equations of motion of the damped harmonic oscillator.

Finally, we recall the intriguing behavior of the asymptotic pseudostationary states, which so far has escaped notice from all works dedicated to the BCK oscillator. At first glance eq. (9) implies that the BCK Hamiltonian loses self-adjointness as n approaches infinity. Fortunately, these asymptotic states are not in the domain of the BCK Hamiltonian and thus pose no threat whatsoever. On the other hand, there is no such constraint on the domain of the first-order Hamiltonian $\hat{H}$, where there is no upper bound to the energy eigenstates. This inconsistency can spoil the physical equivalence of the two theories at high energies, but it is not altogether unexpected. As the energy grows, that is to say, as n grows, the wave functions spread farther out in space and become highly non-local. Any canonical transformation, on the contrary, is only locally valid, and thus our time-dependent dilation transformation $\hat{D}$ (29) cannot guarantee there are no global issues in establishing the physical equivalence of the two theories (or any two theories, for that matter). This is not unexpected, since in general the non-local properties of the wave functions can spoil any equivalence that the two theories might have on the level of the Heisenberg equations.

Darboux's method for solving the inverse problem of the calculus of variations for second-order equations of motion of one-dimensional systems produces theories which, despite being classically equivalent, do not possess the expected quantum properties. Thus, whereas every classical one-dimensional system can be "Lagrangianized", in quantum mechanics one still encounters one-dimensional systems which can still be called "non-Lagrangian". For such systems, quantum anomalies resulting from limiting processes (in our case, the limit $t \to \infty$) can be identified already at the classical level by an asymptotic inequivalence between the original second-order equation and the EL equation resulting from the Lagrangianization process.

We note that the quantum Hamiltonian (17) naturally appears in the master equation for open quantum systems in the description of the damped harmonic oscillator in the Lindblad theory [33] and Dekker's complex symplectic approach [36]. This relation has yet to be clarified, and it should

prove useful in the comparison of the BCK theory and the first-order theory along the lines of a subsystem plus reservoir approach.

Acknowledgements: M.C.B. thanks FAPESP; D.M.G. thanks FAPESP and CNPq for permanent support.

A Appendix

Large n limit of $\hat{A}\psi_n^{BCK}$ Let us compute the action of $\hat{A} = \hat{q}\hat{p} + \hat{p}\hat{q}$ on the functions ψ_n^{BCK} (6),

$$\begin{aligned}(\hat{q}\hat{p} + \hat{p}\hat{q})\,\psi_n^{BCK} &= -i\left(2q\frac{\partial}{\partial q} + 1\right)\psi_n^{BCK} \\ &= i\sqrt{(n+2)\,(n+1)}\psi_{n+2}^{CK} - i\sqrt{n\,(n-1)}\psi_{n-2}^{BCK} \\ &\quad -\frac{\alpha}{\tilde{\omega}}\left[\sqrt{(n+2)\,(n+1)}\psi_{n+2}^{CK} + (2n+1)\,\psi_n^{CK} + \sqrt{n\,(n-1)}\psi_{n-2}^{BCK}\right]. \quad (30)\end{aligned}$$

The asymptotic form of the pseudostationary states for large values of n is [37]

$$\begin{aligned}\psi_{2n}^{BCK}(q,t) &= \left(\frac{\tilde{\omega}}{\pi}\right)^{1/4}\exp\left(-i\alpha\frac{q^2e^{2\alpha t}}{2}\right)(-1)^n\sqrt{\frac{(2n-1)!!}{2n!!}}\cos\left(\sqrt{(4n+1)\,\tilde{\omega}}e^{\alpha t}q\right) + O\left(n^{-1/4}\right) \\ &= \psi_{2n}^{asym} + O\left(n^{-1/4}\right), \\ \psi_{2n+1}^{BCK}(q,t) &= \left(\frac{\tilde{\omega}}{\pi}\right)^{1/4}\exp\left(-i\alpha\frac{q^2e^{2\alpha t}}{2}\right)(-1)^n\sqrt{\frac{(2n-1)!!}{2n!!}}\sin\left(\sqrt{(4n+3)\,\tilde{\omega}}e^{\alpha t}q\right) + O\left(n^{-1/4}\right) \\ &= \psi_{2n+1}^{asym} + O\left(n^{-1/4}\right). \quad (31)\end{aligned}$$

Taking into account that $\cos\sqrt{4n+\alpha}x - \cos\sqrt{4n+1}x = O\left(n^{-1/4}\right)$ for any real α, and substituting the above in the right-hand side of (30), one has

$$(\hat{q}\hat{p} + \hat{p}\hat{q})\,\psi_n^{BCK} = -i\psi_n^{asym} + O\left(n^{-1/4}\right).$$

On the other hand $(\hat{q}\hat{p} + \hat{p}\hat{q})\,\psi_n^{BCK} = (\hat{q}\hat{p} + \hat{p}\hat{q})\,\psi_n^{asym} + O\left(n^{-1/4}\right)$, so

$$(\hat{q}\hat{p} + \hat{p}\hat{q})\,\psi_n^{asym} = -i\psi_n^{asym} + O\left(n^{-1/4}\right),$$

or to the same approximation,

$$(\hat{q}\hat{p} + \hat{p}\hat{q})\,\psi_n^{BCK} = -i\psi_n^{BCK} + O\left(n^{-1/4}\right).$$

Proposition: The asymptotic pseudostationary functions ψ_n^{asym} are not in the domain of $\hat{A} = \hat{q}\hat{p} + \hat{p}\hat{q}$. We first find the domain of $\hat{A}$ for which it is symmetric. Consider, for $\phi, \psi \in D\left(\hat{A}\right)$:

$$\begin{aligned}\left\langle\phi, \hat{A}\psi\right\rangle - \left\langle\hat{A}\phi, \psi\right\rangle &= -2i\int_{-\infty}^{\infty} dq\frac{d}{dq}\left(q\bar{\phi}(q)\,\psi(q)\right) \\ &= -2i\lim_{q\to\infty} q\left(\bar{\phi}(q)\,\psi(q) + \bar{\phi}(-q)\,\psi(-q)\right).\end{aligned}$$

Therefore, functions such that $\lim_{x\to\pm\infty} q\psi(q) \neq 0$ are not in the domains of $\hat{A}$. On the other hand, we know that for large values of n the pseudostationary functions have the asymptotic form (31). Thus, clearly $\lim_{q\to\pm\infty} q\psi_n^{asym}(q) \neq 0$ and $\psi_n^{asym} \neq D\left(\hat{A}\right)$. Note that the closure of $\hat{A}$ is not affected by the exclusion of the asymptotic functions, since they can't be the limit of any sequence.

Proposition: $\hat{A}$ is self-adjoint. It suffices to show that the equation $\hat{A}^*\psi = \pm i\psi$ does not have solutions in $D\left(\hat{A}^*\right)$ [38]. The solutions are of the form

$$\psi_\pm = x^{\lambda_\pm}\,,\ \lambda_\pm = \mp\frac{1}{2}\,,$$

which are not square-integrable in the interval $(-\infty, \infty)$. Therefore, the corresponding deficiency indices are $(0, 0)$ and $\hat{A}$ is essentially self-adjoint.

References

[1] D.M. Gitman and V.G. Kupriyanov. Canonical quantization of so-called non-Lagrangian systems. *The European Physical Journal C*, 50(3):691–700, February 2007.

[2] H. Bateman. On Dissipative Systems and Related Variational Principles. *Physical Review*, 38(4):815–819, 1931.

[3] Piero Caldirola. Forze non conservative nella meccanica quantistica. *Il Nuovo Cimento*, 18(9):393–400, 1941.

[4] E. Kanai. On the Quantization of the Dissipative Systems. *Progress of Theoretical Physics*, 3(4):440–442, 1950.

[5] I. Senitzky. Dissipation in Quantum Mechanics. The Harmonic Oscillator. *Physical Review*, 119(2):670–679, 1960.

[6] John R. Ray. Lagrangians and systems they describe - how not to treat dissipation in quantum mechanics. *American Journal of Physics*, 47(7):626, 1979.

[7] Daniel M. Greenberger. A critique of the major approaches to damping in quantum theory. *Journal of Mathematical Physics*, 20(5):762, 1979.

[8] V. V. Dodonov and V. I. Man'ko. Loss energy states of nonstationary quantum systems. *Il Nuovo Cimento B*, 44(2):265–274, 1978.

[9] V. V. Dodonov and V. I. Man'ko. Coherent states and the resonance of a quantum damped oscillator. *Physical Review A*, 20(2):550–560, August 1979.

[10] R. K. Colegrave and M. Sebawe Abdalla. A Canonical Description of the Fabry-pérot Cavity. *Journal of Modern Optics*, 28(4):495–501, April 1981.

[11] Rainer W. Hasse. On the quantum mechanical treatment of dissipative systems. *Journal of Mathematical Physics*, 16(10):2005, 1975.

[12] Susumu Okubo. Canonical quantization of some dissipative systems and nonuniqueness of Lagrangians. *Physical Review A*, 23(6):2776–2784, June 1981.

[13] D C Latimer. Quantizing the damped harmonic oscillator. *Journal of Physics A: Mathematical and General*, 38(9):2021–2027, 2005.

[14] Denis Kochan. Functional integral for non-Lagrangian systems. *Physical Review A*, 81(2):14, 2010.

[15] M BLASONE. Bateman's dual system revisited: quantization, geometric phase and relation with the ground-state energy of the linear harmonic oscillator. *Annals of Physics*, 312(2):354–397, 2004.

[16] E CELEGHINI, M RASETTI, and G VITIELLO. Quantum dissipation. *Annals of Physics*, 215(1):156–170, 1992.

[17] J M Cervero and J Villarroel. On the quantum theory of the damped harmonic oscillator. *Journal of Physics A: Mathematical and General*, 17(15):2963, October 1984.

[18] H. Dekker. Quantization of the linearly damped harmonic oscillator. *Physical Review A*, 16(5):2126–2134, November 1977.

[19] Yoel Tikochinsky. Exact propagators for quadratic Hamiltonians. *Journal of Mathematical Physics*, 19(4):888, 1978.

[20] M. D. Kostin. On the Schrodinger-Langevin Equation. *The Journal of Chemical Physics*, 57(9):3589, 1972.

[21] D.M. Gitman and V.G. Kupriyanov. Canonical quantization of so-called non-Lagrangian systems. *The European Physical Journal C*, 50(3):691–700, 2007.

R FEYNMAN and F VERNON. The theory of a general quantum system interacting with a linear dissipative system. *Annals of Physics*, 24(1-2):118–173, October 1963.

H. Haken. Cooperative phenomena in systems far from thermal equilibrium and in nonphysical systems. *Reviews of Modern Physics*, 47(1):67–121, January 1975.

A O Caldeira and A J Leggett. Path integral approach to quantum Brownian motion. *Physica*, 121A:587–616, 1983.

I. Pedrosa and B. Baseia. Coherent states and dissipative systems. *Physical Review D*, 30(4):765–769, August 1984.

D. Walls and G. Milburn. Effect of dissipation on quantum coherence. *Physical Review A*, 31(4):2403–2408, April 1985.

Hermann Grabert. Quantum Brownian motion: The functional integral approach. *Physics Reports*, 168(3):115–207, October 1988.

L.H. Yu and Chang-Pu Sun. Evolution of the wave function in a dissipative system. *Physical Review A*, 49(1):592–595, January 1994.

Izumi Joichi, Shigeki Matsumoto, and Motohiko Yoshimura. Quantum Dissipation in Open Harmonic Systems. *Progress of Theoretical Physics*, 98(1):9–28, July 1997.

[22] Hermann von Helmholtz. Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung. *Journal für die reine angewandte Mathematik*, 100:137–166, 1887.

[23] W. Starlet. The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics. *Journal of Physics A: Mathematical and General*, 15(5):1503, 1982.

[24] G. Darboux. *Leçons sur la Théorie Générale des Surfaces vol 3*. Gauthier-Villars, Paris, 1894.

[25] Edward H. Kerner. NOTE ON THE FORCED AND DAMPED OSCILLATOR IN QUANTUM MECHANICS. *Canadian Journal of Physics*, 36:371–377, 1958.

[26] Bopp. F. Zur Quantenmechanik des gedampften harmonishen Oszillators. *ZEITSCHRIFT FUR ANGEWANDTE PHYSIK*, 14:699, 1962.

[27] V. Dodonov and V. Man'ko. Coherent states and the resonance of a quantum damped oscillator. *Physical Review A*, 20(2):550–560, 1979.

[28] D M Gitman and V G Kupriyanov. The action principle for a system of differential equations. *Journal of Physics A: Mathematical and Theoretical*, 40(33):10071–10081, 2007.

[29] DM Gitman and VG Kupriyanov. Action principle for so-called non-Lagrangian systems. In *5th International Conference on Mathematical Methods in Physics (IC 2006)*, page 16, Rio de Janeiro, 2006. PoS.

[30] D M Gitman and I V Tyutin. *Quantization of fields with constraints*. Springer-Verlag, 1991.

[31] K W H Stevens. The Wave Mechanical Damped Harmonic Oscillator. *Proceedings of the Physical Society*, 72(6):1027–1036, December 1958.

[32] H Dekker. Classical and quantum mechanics of the damped harmonic oscillator. *Physics Reports*, 80(1):1–110, December 1981.

[33] a. Isar. Uncertainty, Entropy and Decoherence of the Damped Harmonic Oscillator in the Lindblad Theory of Open Quantum Systems. *Fortschritte der Physik*, 47(7-8):855–879, September 1999.

[34] VV Dodonov. *Parametric excitation and generation of nonclassical states in linear media*, chapter 3, pages 153–218. Number 2. CRC Press, 1st edition, 2002.

[35] E. Onofri. Time-dependent scale-gauge transformations and the Adiabatic Principle. *Lettere Al Nuovo Cimento Series 2*, 21(14):489–492, April 1978.

[36] H DEKKER. On the phase space quantization of the linearly damped harmonic oscillator1. *Physica A: Statistical and Theoretical Physics*, 95(2):311–323, February 1979.

[37] V.I. Smirnov. *A Course of Higher Mathematics, vol.3 part 2*. Pergamon Press, 1964.

[38] M. Reed and B. Simon. *I: Functional Analysis, Volume 1 (Methods of Modern Mathematical Physics)*. Academic Press Inc, 1981.