IFUSP/P-99

PROPERTIES OF THE GRIFFIN-HILL-WHEELER INTEGRAL EQUATION

BY

A.F.R. de Toledo Piza, E.J.V. de Passos, D.Galetti*, M.C. Nemes* and M.M. Watanabe*

Instituto de Física da

Universidade de São Paulo, S. P.

* Supported by FAPESP.

ABSTRACT

We discuss a method for constructing an explicit orthonormal representation for the collective subspace associated with the generator coordinate method, assuming that the overlap $\langle\alpha|\alpha'\rangle$ is a Hilbert-Schmidt kernel in the space of weight functions. We show that the equivalence between the diagonalization of the many-body hamiltonian in the collective subspace and the solution of the Griffin-Hill-Wheeler equation is a dynamical question which cannot be answered by kinematical considerations alone. The treatment gives a simple picture of well known misbehaviours of the generator coordinate weight functions. An application is made to the Lipkin model.

I. INTRODUCTION

C.W.Wong has put forth in reference (1) the point of view that the solution of the Griffin-Hill-Wheeler[2] (GHW) integral equation can be identified with the diagonalization of the hamiltonian in a subspace of the many-body Hilbert space, the collective subspace S.

In specific cases one finds difficulties in the implementation of this correspondence[1-3] and in what follows we will give a brief outline of the nature of these difficulties.

In the generator coordinate method of GHW[2] one considers many-body wave functions generated by the ansatz

$$|f\rangle = \int f(\underline{\alpha})\, |\underline{\alpha}\rangle\, d\alpha \tag{1}$$

where $f(\underline{\alpha})$, the weight function, is a function defined in the Hilbert space of the complex valued, square integrable functions. The generator states $|\underline{\alpha}\rangle$ are a family of many-body wave functions parametrized by the label $\underline{\alpha}$, the generator coordinate. In the method of GHW the function $f(\underline{\alpha})$ is determined by the variational method,

$$\underline{\delta}\ \frac{\langle f | H | f \rangle}{\langle f | f \rangle} = 0 \tag{2}$$

which leads to the GHW integral equation

$$\int \left[\langle \underline{\alpha} | H | \underline{\alpha}' \rangle - E \langle \underline{\alpha} | \underline{\alpha}' \rangle \right] f(\underline{\alpha}')\, d\underline{\alpha}' = 0 \tag{3}$$

C.W.Wong[1] has investigated the formal correspondence between the solutions of Eq. (3) and the solutions found in the diagonalization of the hamiltonian in a subspace of the many-body Hilbert space, the collective subspace S.

$$(P_s H P_s - E) P_s |f\rangle = 0 \tag{4}$$

Where P_s is the projection operator onto the collective subspace S.

In the work by Wong, the formal correspondence between the two equations is shown in terms of an assumed biorthogonal expansion of P_S involving the generator states:

$$P_s = \int d\underline{\alpha}\, |\tilde{\underline{\alpha}}\rangle\langle\underline{\alpha}| = \int d\underline{\alpha}\, |\underline{\alpha}\rangle\langle\tilde{\underline{\alpha}}| \tag{5}$$

where

$$\langle\tilde{\underline{\alpha}}'|\underline{\alpha}\rangle = \underline{\delta}(\underline{\alpha}' - \underline{\alpha}) .$$

In this case Eq. (4) reduces to Eq. (3) if we make the identification of the weight function with the projection of $|f\rangle$ on the biorthogonal base[1].

$$f(\underline{\alpha}) = \langle\tilde{\underline{\alpha}}|f\rangle$$

In specific models like the "gaussian overlap" approximation[2] one finds difficulties in the implementation of this correspondence[1-3]. Wong shows that in this model ,Eq. (4) can have solutions which are not obtained in the solution of the GHW integral equation, Eq. (3). However it is well known that if in the variational principle, Eq. (2), one varies $|f\rangle$ in a closed subspace of the many-body Hilbert space, the solution of Eq. (2) is equivalent to the diagonalization of the hamiltonian in the above subspace. So the difference found by Wong leads us to the conjecture that the subspace generated by the ansatz (1) is different from the collective subspace S. In order to shed some light on this problem we investigate the properties of the GHW integral equation when the overlap $\langle \underline{\alpha} | \underline{\alpha}' \rangle$ is a Hilbert-Schmidt kernel in the space of square integrable functions. This is not a purely academic case. First, projection of particle number and angular momentum fall in this category. Second, we can consider weight function spaces with an appropriate measure with respect to which the kernel at hand will be a Hilbert-Schmidt kernel. Third, in the numerical handling of the GHW integral equation[4] we always replace Eq. (3) by

$$\sum_{j=1}^{N} \left[\langle \underline{\alpha}_i | H | \underline{\alpha}_j \rangle - E \langle \underline{\alpha}_i | \underline{\alpha}_j \rangle \right] f(\underline{\alpha}_j) = 0$$

where the $\underline{\alpha}_i$'s are a finite, discrete set of points and in this case the kernel $\langle \underline{\alpha}_i | \underline{\alpha}_j \rangle$ is trivially a Hilbert-Schmidt kernel. Our treatment generalizes this case to that of a continuous label $\underline{\alpha}$.

Besides, the interpretation of our treatment lead us to expect that the main qualitative features of more general cases are already present in it.

For kernels having this property we give a method to construct an explicit orthonormal representation for the collective subspace S. This will allow us to understand the misbehaviours described by Wong[1] and to relate them to properties of the biorthogonal expansion (5). It has often been suggested[5] that problems could arise in connection with zero eigenvalues of the overlap kernel $\langle \underline{\alpha} | \underline{\alpha}' \rangle$. Out treatment shows clearly how ever that these problems can be trivially avoided by means of a suitable restriction of the allowed weight functions. Problems of a more fundamental nature, on the other hand, appear in connection with situations in which a sequence of non-zero eigenvalues approaches zero as a limit point. Our work is organized as follows: in section II, we construct a representation for the collective subspace S when the overlap $<\underline{\alpha}|\underline{\alpha}'>$ is a Hilbert-Schmidt kernel in the space of square integrable functions. Furthermore we discuss in detail under what conditions Eq. (3) and Eq. (4) are equivalent. As an application, in section III we consider the generator coordinate method applied to the Lipkin model which is a particular case where the diagonalization of the hamiltonian in the collective subspace S is equivalent to the solution of the GHW integral equation. In section IV we present some concluding remarks.

II. THE OVERLAP $<\underline{\alpha}|\underline{\alpha}'>$ AS A HILBERT-SCHMIDT KERNEL

A. A representation for the collective subspace

Consider the case where the overlap $<\underline{\alpha}|\underline{\alpha}'>$ is a Hilbert-Schmidt kernel in the space of square integrable functions, i.e.

$$\int d\underline{\alpha}\, d\underline{\alpha}' \, |\langle\underline{\alpha}|\underline{\alpha}'\rangle|^2 < \infty \tag{6}$$

In this case, there is a decomposition of the kernel in orthonormal eigenfunctions[6]

$$\langle \underline{\alpha} | \underline{\alpha}' \rangle = \sum_{n,\lambda_n \neq 0}' \underline{\lambda}_n u_n(\underline{\alpha}) u_n^*(\underline{\alpha}') \tag{7}$$

where the $u_n(\underline{\alpha})$ satisfy the eigenvalue equation

$$\int \langle \underline{\alpha} | \underline{\alpha}' \rangle u_n(\underline{\alpha}') d\underline{\alpha}' = \underline{\lambda}_n u_n(\underline{\alpha}) \tag{8}$$

If we include the eigenfunctions of zero eigenvalue, the functions $u_n(\underline{\alpha})$ form an orthonormal base in the space of functions[6]

$$\sum_n u_n(\underline{\alpha}) u_n^*(\underline{\alpha}') = \underline{\delta}(\underline{\alpha} - \underline{\alpha}')$$

$$\int u_n^*(\underline{\alpha}) u_{n'}(\underline{\alpha}) d\underline{\alpha} = \underline{\delta}_{nn'}$$

An important property of the Hilbert-Schmidt kernels is that if the eigenfunctions of eigenvalue different from zero span a space of infinite dimension, $\underline{\lambda}_n$ has a limit point for $\underline{\lambda}=0$ and this is the only possible limit point[6]. In what follows we will denote the subspace of the function space spanned by the eigenfunctions of zero eigenvalue, the null space, by L_o and, the subspace spanned by the eigenfunctions of eigenvalue different from zero, its orthogonal complement, by $L_\perp$.

It is shown in the appendix A, that due to the properties of the spectrum of the Hilbert-Schmidt kernels, the many-body vectors produced by the GHW ansatz

$$| f \rangle = \int d\underline{\alpha} \, | \underline{\alpha} \rangle f(\underline{\alpha}) \tag{9}$$

form a linear subspace of the many-body Hilbert space which is however not closed except in the special case when $L_\perp$ has finite dimension. In order to remedy this undesirable feature we note first that since the functions $\{u_n(\underline{\alpha})\}$form a base in the space of functions, the vectors

$$|n\rangle = \int u_n(\underline{\alpha})\, |\underline{\alpha}\rangle \, d\underline{\alpha}$$

form a complete set in the subspace (in general not closed) generated by the ansatz (9).

The norm of the states $|n\rangle$ is equal to

$$\langle n | n' \rangle = \underline{\lambda}_n \, \delta_{nn'} \qquad ($$

The consequences of Eq. (10) are the following:

a) The eigenfunctions of the overlap kernel $\langle\underline{\alpha}|\underline{\alpha}'\rangle$ with ze eigenvalue give rise to vectors of zero norm in the many body Hilbert space

$$\int d\underline{\alpha}\, u_n(\underline{\alpha})\, |\underline{\alpha}\rangle = 0 \qquad \text{if } \underline{\lambda}_n = 0$$

b) The $\underline{\lambda}_n$ are semi-positive definite, since they are norm vectors defined in the many-body Hilbert space.

c) The vectors

$$|\bar{n}\rangle = \frac{1}{(\underline{\lambda}_n)^{1/2}} |n\rangle = \frac{1}{(\underline{\lambda}_n)^{1/2}} \int d\underline{\alpha}\, u_n(\underline{\alpha}) |\underline{\alpha}\rangle$$

for $\underline{\lambda}_n \neq 0$, are orthonormal vectors

It is also shown in the appendix A that the clo subspace of the many-body Hilbert space generated by the normal states $\{|\bar{n}\rangle\}$ is in fact the closure of the linear produced by means of the GHW ansatz. This subspace, defined by means of the projection operator

$$P_s = \sum_{n,\lambda_n \neq 0} |\bar{n}\rangle\langle\bar{n}| \ ,$$

is what we call the collective subspace. When this space is of finite dimension, the norm of all vectors $|n\rangle$, Eq.(10), has a lower bound (the smallest non-vanishing eigenvalue) and as a result of this the linear space generated by Eq. (9) is closed and equal to the collective subspace. It is in fact the existence of eigenfunctions of the overlap kernel with arbitrarily small eigenvalue which gives rise to dificulties in the general case. One important point that emerges from the above considerations is that if we do not restrict the weight functions to belong to the subspace $L_\perp$, the correspondence between the weight functions and the vectors of the many-body Hilbert space is not unique. It becomes unique only if $f(\underline{\alpha})$ belongs to $L_\perp$. In what follows we will restrict $f(\underline{\alpha})$ to $L_\perp$.

The existence of eigenfunctions of the kernel $\langle\underline{\alpha}|\underline{\alpha}'\rangle$ having zero eigenvalue is a manifestation of the fact that the generator states are not linearly independent. This is a direct consequence of Eq. (10). Also using Eq. (11) we have

$$|\underline{\alpha}\rangle = \sum_{n,\underline{\lambda}_n \neq 0} (\underline{\lambda}_n)^{1/2} u_n^*(\underline{\alpha}) |\bar{n}\rangle \tag{12}$$

which can be written as

$$|\underline{\alpha}\rangle = \int d\underline{\alpha}' \ P_\perp(\underline{\alpha}', \underline{\alpha}) |\underline{\alpha}'\rangle$$

where

$$P_\perp(\underline{\alpha}',\underline{\alpha}) = \sum_{n,\lambda_n \neq 0} u_n(\underline{\alpha}')\, u_n^*(\underline{\alpha}) \tag{13}$$

is the projection operator onto the subspace of the weight function space spanned by the eigenfunctions of the kernel $\langle\underline{\alpha}|\underline{\alpha}'\rangle$ with non-vanishing eigenvalue, $L_\perp$. This shows that the generator states being linearly dependent, they are reproduced by a kernel which is the restriction of a delta function to the subspace $L_\perp$.

B. Biorthogonal bases in the collective subspace

In order to determine vectors $|\tilde{\underline{\alpha}}\rangle$ having biorthogonal properties with respect to the generator states $|\underline{\alpha}\rangle$[1], we consider the equation

$$\langle\tilde{\underline{\alpha}}|\underline{\alpha}'\rangle = P_\perp(\underline{\alpha},\underline{\alpha}') \tag{14}$$

This is in fact sufficient in view of the linear dependence of the generator states as discussed in the preceding subsection. Using Eq. (13), Eq. (14) can be written as

$$\sum_n \langle\tilde{\underline{\alpha}}|\bar{n}\rangle\langle\bar{n}|\underline{\alpha}'\rangle = \sum_n u_n(\underline{\alpha})\, u_n^*(\underline{\alpha}')$$

Eq. (11) now shows us that

$$\langle\underline{\alpha}|\bar{n}\rangle = (\lambda_n)^{1/2}\, u_n(\underline{\alpha}) \tag{15}$$

Using Eq. (15) it is very easy to verify that the solution for $\langle\tilde{\underline{\alpha}}|n\rangle$ is

$$\langle\tilde{\underline{\alpha}}|\bar{n}\rangle = \frac{u_n(\underline{\alpha})}{(\underline{\lambda}_n)^{1/2}} \tag{16}$$

Using Eq. (15) and Eq. (16) we can show formally that

$$\int d\alpha\ |\tilde{\underline{\alpha}}\rangle\langle\underline{\alpha}| = \int d\alpha\ |\underline{\alpha}\rangle\langle\tilde{\underline{\alpha}}| = \sum_{n,\lambda_n \neq 0} |\bar{n}\rangle\langle\bar{n}| \tag{17}$$

The biorthogonal states $|\tilde{\underline{\alpha}}\rangle$ defined by the Eq. (16), however, do not in general belong to the many-body Hilbert space. To see this consider

$$\langle\tilde{\underline{\alpha}}|\underline{\alpha}'\rangle = \sum_{n,\underline{\lambda}_n \neq 0} \frac{u_n(\underline{\alpha})\, u_n^*(\underline{\alpha}')}{\underline{\lambda}_n} \tag{18}$$

Eq. (18) shows that in general $|\tilde{\underline{\alpha}}\rangle$ does not have a finite norm since, although $u_n(\tilde{\underline{\alpha}})$ is a normalized wave function, λ_n has a limit point for $\underline{\lambda}=0$. The only case where $|\tilde{\underline{\alpha}}\rangle$ is guaranteed to have a finite norm is when $L_\perp$ has a finite dimension.

C. The equivalence between the solutions of the GHW integral equation and the diagonalization of the hamiltonian in the collective subspace.

Even though the $|\tilde{\underline{\alpha}}\rangle$'s in general do not have a finite norm they may still be useful if $\langle\tilde{\underline{\alpha}}|f\rangle$ is a well defined function in the space of square integrable functions, i.e.

$$\int \left| \langle \tilde{\underline{\alpha}} | f \rangle \right|^2 d\alpha = \sum_{n,\, \lambda_n \neq 0} \frac{\left| \langle \tilde{n} | f \rangle \right|^2}{\lambda_n} < \infty \tag{19}$$

In the case when L_1 has a finite dimension, $|\tilde{\underline{\alpha}}\rangle$ has a finite norm and Eq. (19) is satisfied. In this case the ansatz (9) generates a closed subspace of the many-body Hilbert space which is identical to the collective subspace. Furthermore the biorthogonal representations (17) is a well defined representation for the collective subspace. The consequence of all this is that the diagonalization of the hamiltonian in the collective subspace is equivalent to the solution of the GHW integral equation if we make the identification of the weight function $f(\underline{\alpha})$ with $\langle \tilde{\underline{\alpha}} | f \rangle$,

$$f(\underline{\alpha}) = \langle \tilde{\underline{\alpha}} | f \rangle$$

In the case where the subspace L_1 has infinite dimension the $|\tilde{\underline{\alpha}}\rangle$'s do not have finite norm and we see easily that not all vectors defined in the collective subspace, which are such that

$$|f\rangle = \sum_{n,\, \lambda_n \neq 0} f_n |\tilde{n}\rangle \quad , \quad \sum_n |f_n|^2 < \infty$$

satisfy Eq. (19).

This means of course that there are vectors in the collective subspace that cannot be expressed by the ansatz (9) with a normalized weight function. In this case the biorthogonal bases cannot be used for a general vector in the collective subspace. This has the consequence that the Eq. (4) can have solutions which cannot be obtained by the solution of the GHW integral equation. These are the states which belong to the collective subspace but not to the linear space generated by the GHW ansatz (which is in this case strictly smaller than the collective subspace). It should be stressed that the misbehaviour of the weight function for certain vectors of the collective subspace is solely a consequence of the representation in terms of the biorthogonal sets and therefore has a purely kinematical origin.

We see also that there can be no "a priori" criterion to decide whether the solutions found in the diagonalization of some hamiltonian in the collective subspace can be obtained by solving the GHW integral equation. This is a dynamical question which cannot be answered without the explicit use of the specific hamiltonian. The discussion above shows that even when the GHW integral equation is not well defined, the diagonalization of the hamiltonian in the collective subspace is always a well defined procedure.

III. GENERATOR COORDINATE METHOD IN THE LIPKIN MODEL

As an example of the approach discussed in the preceding section we study the Lipkin model, which is a case where $L_\perp$ has a finite dimension and the two equations (3) and (4) are equivalent.

The Lipkin model is extensively studied in the literature[7,8] and in what follows we will only review its most essential features.

We have N fermions distributed in two N-fold degenerate levels separated by the energy ε. Each level is characterized by a quantum number σ, which is equal to +1(-1) for the higher (lower) level and a quantum number p associated with the degeneracy in each level.

The hamiltonian of the model is given by

$$H = \frac{\varepsilon}{2}\sum_{p,\sigma} a^{+}_{p\sigma} a_{p\sigma} + \frac{V}{2}\sum_{\substack{p,\sigma \\ p',\sigma'}} a^{+}_{p\sigma} a^{+}_{p'\sigma} a_{p'-\sigma} a_{p-\sigma} \tag{20}$$

Defining the quasi-spin operators

$$J_{+} = \sum_{p} a^{+}_{p+1} a_{p-1} = J_{-}^{+}$$

$$J_{z} = \frac{1}{2}\sum_{p,\sigma} \sigma a^{+}_{p\sigma} a_{p\sigma} \tag{21}$$

we can easily see that they satisfy the following commutation relations

$$[J_{+}, J_{-}] = 2J_{z} \quad , \quad [J_{z}, J_{\pm}] = \pm J_{\pm} \tag{22}$$

Using the quasi-spin operators (21) the hamiltonian (20) can be written as

$$H = \varepsilon J_{z} + \frac{V}{2}\left(J_{+}^{2} + J_{-}^{2}\right) \tag{23}$$

The Eq. (23) shows us that $(H,J^{2})=0$. Therefore Eq. (23) can be diagonalized within each multiplet. In particular the ground state belongs to the multiplet $J = \frac{1}{2}N$ which can be seen by noticing that the unperturbed ground state

$$|0\rangle = a^{+}_{p_1 -} a^{+}_{p_2 -} \cdots a^{+}_{p_N -} | \rangle \tag{24}$$

is an eigenstate of J^2 and J_z with eigenvalues $\frac{1}{2}N(\frac{1}{2}N+1)$ and $-\frac{1}{2}N$ respectively[7]. In order to use the generator coordinate method we choose for the family of generator states a family of normalized Slater determinants belonging to the multiplet $J=\frac{1}{2}N$ and parametrized in the following way[8]:

$$|\phi\rangle = \cos^N\frac{\phi}{2} \;\exp\left(tg\frac{\phi}{2} J_+\right)|0\rangle \tag{25}$$

The overlap of two generator states is equal to

$$\langle\phi'|\phi\rangle = \cos^N\frac{(\phi'-\phi)}{2} \tag{26}$$

The overlap (26) is a periodic function with period 2π. We therefore choose for the space of functions the space of square integrable periodic functions and restrict them to the interval $(-\pi, \pi)$. In this space the overlap $\langle\phi'|\phi\rangle$ is a Hilbert-Schmidt kernel,

$$\int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\phi' \left|\langle\phi'|\phi\rangle\right|^2 = \frac{4\pi^2(2N)!}{2^{2N}(N!)^2} < \infty \tag{27}$$

The eigenfunctions of the kernel $\langle\phi'|\phi\rangle$ satisfy the equation

$$\int_{-\pi}^{\pi} \langle\phi|\phi'\rangle\, u_n(\phi')\, d\phi' = \lambda_n\, u_n(\phi)$$

which has the solution

$$u_n(\phi) = \frac{1}{(2\pi)^{1/2}}\, e^{in\phi}$$

$$\lambda_n = \int_{-\pi}^{\pi} \cos^N \frac{\phi}{2}\, e^{in\phi}\, d\phi = \begin{cases} \dfrac{2\pi\, N!}{2^N \left(\frac{N}{2}+n\right)! \left(\frac{N}{2}-n\right)!} & -\frac{N}{2} \leq n \leq \frac{N}{2} \\ 0 & n > \frac{1}{2} N \\ 0 & n < -\frac{1}{2} N \end{cases} \tag{28}$$

The many-body states

$$\int_{-\pi}^{\pi} \frac{1}{(2\pi)^{1/2}}\, e^{in\phi}\, |\phi\rangle\, d\phi = 0 \qquad n > \left|\frac{N}{2}\right|$$

have zero norm and the states

$$|\bar{n}\rangle = \frac{1}{(2\pi\,\lambda_n)^{1/2}} \int_{-\pi}^{\pi} e^{in\phi}\, |\phi\rangle\, d\phi \qquad -\frac{N}{2} \leq n \leq \frac{N}{2}$$

are orthonormal vectors in the many-body Hilbert space of the model and form a base for the collective subspace S.

$$P_S = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} |\bar{n}\rangle\langle\bar{n}|$$

So we see that the Lipkin model is a case in which L_1, and therefore also the collective subspace, have finite dimension. In particular the set $|\bar{n}\rangle$ is related to the usual J_z representation by means of an unitary transformation and therefore the collective subspace is the same as that generated by the standart multiplet $J=\frac{1}{2}N$, $-\frac{1}{2}N < J_z < \frac{1}{2}N$. The existence of eigenfunctions of the kernel with zero eigenvalue implies that the $|\phi\rangle$'s are not linearly independent, the linear dependence being expressed by

$$|\underline{\phi}\rangle = \int_{-\pi}^{\pi} d\underline{\phi}' \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} e^{in(\underline{\phi}'-\underline{\phi})} |\underline{\phi}'\rangle$$

The states biorthogonal to the states $|\underline{\phi}\rangle$, on the other hand, obey the equation

$$\langle\tilde{\underline{\phi}}|\underline{\phi}'\rangle = P_c(\underline{\phi},\underline{\phi}') = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \frac{e^{in(\underline{\phi}-\underline{\phi}')}}{2\pi}$$

and are thus given by

$$|\tilde{\underline{\phi}}\rangle = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \frac{e^{-in\underline{\phi}}}{(2\pi\lambda_m)^{1/2}} |\bar{n}\rangle \qquad (29)$$

As discussed in section III (see Eq. (29)), $|\tilde{\underline{\phi}}\rangle$ has a finite norm in this case and we can construct a representation for P_s in terms of the biorthogonal states.

$$P_s = \int d\underline{\phi} |\tilde{\underline{\phi}}\rangle\langle\underline{\phi}| = \int d\underline{\phi} |\underline{\phi}\rangle\langle\breve{\underline{\phi}}|$$

The diagonalization of the hamiltonian in the collective subspace is therefore equivalent to the solution of the GHW integral equation. This tell us that in the Lipkin model the generator coordinate method with the generator states given by Eq. (25) gives the exact solutions.

IV. <u>CONCLUSIONS</u>

The preceding discussion lead us to the following conclusions:

a) In the case where the subspace generated by the eigenfunctions of the kernel $\langle \underline{\alpha} | \underline{\alpha}' \rangle$ with eigenvalue different from zero has finite dimension, the GHW ansatz (9) generates a closed subspace of the many-body Hilbert space. Furthermore, the subspace generated by Eq. (9) is identical to the collective subspace S and there is a well-defined representations for S in terms of states biorthogonal to the generator states. In this case the diagonalization of the hamiltonian in the collective subspace is equivalent to the solution of the GHW integral equation.

b) In the case where the subspace generated by the eigenfunctions of the kernel $\langle \underline{\alpha} | \underline{\alpha}' \rangle$ of eigenvalue different from zero has infinite dimension the subspace generated by the ansatz (9) is not closed. Its closure is the collective subspace S. In this case it can happen that the diagonalization of the hamiltonian in S leads to state vectors which cannot be obtained by solving the GHW integral equation. To understand this behaviour in the case of infinite dimension one may, following Ref. 9 , first consider the case where S has finite dimension. In this case Eqs. (3) and (4) are equivalent. When the dimension of S increases there are solutions of Eq. (4) such that the norm of the weight function $f(\underline{\alpha})$,

$$f(\underline{\alpha}) = \langle \tilde{\underline{\alpha}} | f \rangle$$

increases without bound even though $|f\rangle$ has a finite norm. In the limit when the dimension of S becomes infinite the norm of $f(\underline{\alpha})$ diverges. These are the solutions which cannot be found by solving the GHW integral equation. Note however that the divergence of $f(\underline{\alpha})$ does not mean that the corresponding many-body vector also diverges.

It is important to notice that the question whether the two equations have the same set of solutions cannot be decided without the explicit use of the hamiltonian. This fact tells us that there is no guarantee "a priori" that the GHW integral equation can have solutions with a square integrable $f(\underset{\sim}{\alpha})$. This difficulty can be avoided by noticing that the diagonalization of H in S is always a well defined procedure.

Finally we have shown how to construct an explicit orthonormal representation for the collective subspace S.This allows us, in principle, to solve Eq. (4) and to investigate the definition of appropriate collective dynamical variables.

The generalization of the present treatment to general overlap kernels is under investigation and will be subject of a separate publication.

APPENDIX A

We show that a) the linear space generated by the GHW ansatz, Eq (9) is in general not a closed subspace, and that b) the closure of this space is the collective subspace S. This can be done easily by considering a normalized many-body vector in S written as

$$|g\rangle = \sum_{n=1,\,\lambda_n \neq 0}^{\infty} g_n \, |\bar{n}\rangle \quad , \quad \sum_{n,\,\lambda_n \neq 0} |g_n|^2 = 1 \tag{A.1}$$

We also introduce the sequence of vectors

$$|g^N\rangle = \sum_{n=1,\,\lambda_n \neq 0}^{N} g_n \, |\bar{n}\rangle \tag{A.2}$$

which clearly converges to $|g\rangle$ as $N \to \infty$. It is easy to check that each of the vectors $|g^N\rangle$ can be generated by means of the GHW ansatz with the square integrable weight function $g^{(N)}(\underset{\sim}{\alpha})$ given by

$$g^{(N)}(\underset{\sim}{\alpha}) = \sum_{n=1,\,\lambda_n \neq 0}^{N} \frac{g_n \, u_n(\underset{\sim}{\alpha})}{(\lambda_n)^{1/2}} \tag{A.3}$$

i.e.

$$|g^N\rangle = \int g^{(N)}(\underset{\sim}{\alpha}) \, |\underset{\sim}{\alpha}\rangle \, d\underset{\sim}{\alpha} \tag{A.4}$$

The desired results now emerge when we consider that the eigenvalues λ_n have zero as a limit point. In fact, there are vectors satisfying Eq. (A.1) and that are such that the corresponding sequence of functions $g^{(N)}(\underline{\alpha})$, Eq. (A.3) diverges in the weight function space as $N \to \infty$. The norm of these functions is, in fact

$$\int \left| g^{(N)}(\underline{\alpha}) \right|^2 d\underline{\alpha} = \sum_{n=1,\, \underline{\lambda}_n \neq 0}^{\infty} \frac{|g_n|^2}{\underline{\lambda}_n} \tag{A.5}$$

the convergence of which is not guaranteed by (A.1) in view of the decrease of the $\underline{\lambda}_n$ for large n.

We see thus that, by means of Eqs. (A.2), (A.3) and (A.4) we can generate a sequence of vectors each of which has a well defined weight function that will converge, in the collective subspace, to any pre-assigned vector $|g\rangle$. This proves b). The corresponding sequence of weight functions, however, will not in general converge in the weight function space L_1, so that not every vector in S can be associated with a weight function. In this case, the Cauchy sequence (A.2), which lies in the linear space generated by the GHW ansatz, Eq. (9), does not converge in this space, thus proving a).

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