IFUSP/P-100

PEIERLS' ARGUMENT FOR SOME LATTICE FIELD MODELS

by

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(*) Financial support by Fundação de Amparo à Pesquisa do Estado de São Paulo.

ABSTRACT

We show how Peierls' Argument can be applied to a class of "Euclidean field theories" in the lattice with an even interaction polynomial $P(\phi)$, to prove the existence of phase transition. The class includes polynomials such that $P(\phi) + 1/2\ (2d+m_0^2)\phi^2$ has two "deep and widely" separated minima.

I - Introduction and notation

The purpose of this note is to show that some even Euclidean theories $P(\phi)$ in a lattice have the symmetry $\phi \to -\phi$ broken under suitable conditions on the polynomial P. We restrict our analysis to fixed lattice spacing δ. We use essentially Peierls' Argument as adapted to the continuous spin case by Bortz and Griffiths [1]. Our analysis generalizes a result due to Nelson [2] which combines the existence of a phase transition in the continuous spin Ising Model (the spin varying in the interval $[-c,+c]$) with Griffiths' inequalities to conclude phase transition for the $(a\phi^4 - b\phi^2)$ Euclidean lattice model, with suitable a and b.

We take for simplicity $\delta = 1$ and $d=2$ although our result is true in any dimension $d \geq 2$. This result follows from the corresponding one for d=2 by the usual application of Griffiths' inequalities [3].

The Euclidean action in a lattice with $\delta = 1$, d=2 and "volume" Λ is given by: [4]

$$H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi}) = H^{\Lambda}_{0,\alpha_0}(\underset{\sim}{\phi}) + \mathcal{P}(\underset{\sim}{\phi}) \qquad (1)$$

$$H^{\Lambda}_{0,\alpha_0}(\underset{\sim}{\phi}) = \frac{1}{2}(4+m_0^2)\sum_{i\subset\Lambda}\phi(i)^2 - \frac{1}{2}\sum_{\substack{\langle i,j\rangle \\ i,j\subset\Lambda}}\phi(i)\phi(j) - \frac{1}{2}\sum_{i\subset\partial\Lambda}\alpha_0\phi(i) \qquad (2)$$

$$\mathcal{P}(\underset{\sim}{\phi}) = \sum_{i\subset\Lambda} P(\phi(i)) \qquad (3)$$

Here, $\underset{\sim}{\phi} = \{\phi(i)\}_{i\in\mathbb{Z}^2\cap\Lambda}$ denotes a configuration in $\mathbb{R}^{|\Lambda|}$ ($|\Lambda|$ is the number of lattice points inside Λ), $\langle i,j\rangle$ denotes sum over nearest neighbour and $\partial\Lambda$ denotes the points in the boundary of Λ. The last term in (2) fixes a "boundary condition" (it is roughly speaking equivalent to fix the value of $\phi(j)$ "outside" Λ in $\phi(j) = \alpha_0$).

TheSchwinger functions are given in the usual way:

$$S_{\Lambda}(i_1, \ldots, i_n) = \langle \phi(i_1)\ldots\phi(i_n)\rangle_{\Lambda} = \frac{\int \phi(i_1)\ldots\phi(i_n)\, e^{-H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi})}\, d^{\Lambda}\phi}{\int e^{-H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi})}\, d^{\Lambda}\phi} \qquad (4)$$

where $d^{\Lambda}\phi = \prod_{i\subset\Lambda} d\phi(i)$.

We can group together the local terms in $H_{\alpha_0}^{\Lambda}(\underset{\sim}{\phi})$ in order to have:

$$H_{\alpha_0}^{\Lambda}(\underset{\sim}{\phi}) = I(\underset{\sim}{\phi}) + \sum_{i \subset \Lambda} Q(\phi(i)) \qquad (5)$$

$$Q(\phi) \quad P(\phi) + \frac{1}{2}(4 + m_0^2)\,\phi^2 \qquad (6)$$

$$I(\underset{\sim}{\phi}) = -\frac{1}{2} \sum_{\substack{\langle i,j \rangle \\ i,j \subset \Lambda}} \phi(i)\,\phi(j) - \frac{1}{2} \sum_{i \subset \partial\Lambda} \alpha_0\, \phi(i) \qquad (7)$$

$I(\phi)$ is the usual Ising interaction between nearest neighbours (where we include the nearest neighbour interaction with the outside of Λ, given by the second term in (7)).

In the next section we show that if $Q(\phi)$ has two "deep and widely separated" absolute minima, and $\alpha_0 > 0$ is suitably chosen, then $\langle \phi(i) \rangle = \lim_{\Lambda \to \infty} \langle \phi(i) \rangle_\Lambda \geq \varepsilon > 0$, that is, the $\phi \to -\phi$ symmetry is broken.

In section III we give examples of theories satisfying the conditions on $Q(\phi)$. It will turn out from the discussion that the allowed polynomials include a class which cannot be treated by the method of Nelson [2].

Part of these results appeared as a M. Sc. Thesis [5].

II - Peierls' Argument

We begin by stating the assuptions on $Q(\phi)$.

- Condition C: $Q(\phi)$ is even, has two absolute minima in $\phi = \pm\alpha_0$ and furthermore there exists a number $k > 1$ such that

$$Q\left(\phi \pm \frac{2\alpha_0}{k}\right) \leq Q(\phi) \qquad \text{if} \quad |\phi| \leq \frac{\alpha_0}{k} .$$

Next we state:

- Theorem: If $Q(\phi)$ satisfies Condition C with $\frac{\alpha_0}{k} > 0$ and sufficiently large, then exists $\varepsilon > 0$ such that $\langle \phi(i) \rangle \geq \varepsilon$.

In the proof, we follow closely Bortz and Griffiths [1]. We begin by saying that in a configuration $\underset{\sim}{\phi}$ the site j is in class n=1, 2 or 3 if:

$$\begin{aligned} n=1, &\quad \phi(j) \geq \frac{\alpha_0}{k} \\ n=2, &\quad -\frac{\alpha_0}{k} < \phi(j) < \frac{\alpha_0}{k} \\ n=3, &\quad \phi(j) \leq -\frac{\alpha_0}{k} \end{aligned} \qquad (8)$$

We now imagine, as in the usual Peierls' Argument, lines drawn between pairs of adjascent sites if one of them is in class 1 and the other is in class 2 or 3. Since the value of $\phi(j)$ outside Λ is kept in $\phi(j)=\alpha_0$, we obtain in this way closed polygons, which we call borders, and we will consider only "outer borders" that is, borders that do not fall inside any other border.

Let p be the probability that the site j is in the interior of some outer border. If we show that p can be made sufficiently small, then the theorem will be proved. To do this we will bound P_B, the probability of ocurrence of a particular outer border B. Namely, we will show that

$$P_B \leq a\, m^{\ell} e^{-\gamma \ell} \qquad (9)$$

where ℓ is the lenght of B and a, m, γ are constants. If we show that γ can be made sufficiently large, then (9) implies that p can be made sufficiently small, since given a site j the number of closed polygons of lenght $\ell\,(\geq 4)$ that contain this site is bounded by $\ell^2 3^{\ell}$, and we would have:

$$p = \sum_{B} P_B$$ (with the sum extended over all possible outer borders B that contain site j)

$$\leq \sum_{\ell=4}^{\infty} \sum_{B'_\ell} P_{B'_\ell}$$ (where B'_ℓ are the possible outer borders containing site j with a fixed lenght ℓ)

$$\leq \sum_{\ell=4}^{\infty} a\, m^{\ell} \ell^2 3^{\ell} e^{-\gamma \ell} \leq \sum_{\ell=4}^{\infty} e^{-\gamma' \ell} \leq \varepsilon'$$

if γ is sufficiently large.

We must then establish inequality (9) for

$$P_B = \int_{\mathcal{B}} e^{-H^{\Lambda}_{\alpha_0}(\phi)}\, d^{\Lambda}\phi \Big/ \int_{\mathbb{R}^{|\Lambda|}} e^{-H^{\Lambda}_{\alpha_0}(\phi)}\, d^{\Lambda}\phi \qquad (10)$$

where $\mathcal{B}$ is the set of configurations in $\mathbb{R}^{|\Lambda|}$ in which the border B occurs as an outer border.

In order to do this, we will construct a map $\mathcal{T}$ from $\mathcal{B}$ into $\mathbb{R}^{|\Lambda|}$ with the following properties:

i) If $\underset{\sim}{\phi} \in \mathcal{B}$, $H^{\Lambda}_{\alpha_0}(\mathcal{T}\underset{\sim}{\phi}) \leqslant H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi}) - \gamma \ell$

ii) The region $\mathcal{B}$ is the union of disjoint regions $\mathcal{B}_i$ and $\mathcal{T}$ maps each $\mathcal{B}_i$ one to one onto a region $\mathcal{C}_i \subset \mathbb{R}^{|\Lambda|}$ with the same Lebesgue measure as $\mathcal{B}_i$. The $\mathcal{C}_i$ will not in general be disjoint, but the number of $\mathcal{C}_i$ that intercept at any point of $\mathbb{R}^{|\Lambda|}$ is bounded by am^{ℓ}.

From these two properties we would have:

$$\int e^{-H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi})} d^{\Lambda}\phi = \sum_i \int_{\mathcal{B}_i} e^{-H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi})} d^{\Lambda}\phi \leqslant$$

$$\leqslant \sum_i e^{-\gamma \ell} \int_{\mathcal{C}_i} e^{-H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi})} d^{\Lambda}\phi$$

$$\leqslant am^{\ell} e^{-\gamma \ell} \int_{\mathbb{R}^{|\Lambda|}} e^{-H^{\Lambda}_{\alpha_0}(\underset{\sim}{\phi})} d^{\Lambda}\phi$$

and (9) would be proved.

Next we analyse the construction of $\mathcal{T}$. As in Bortz--Griffiths, circunference site denotes a site in the interior of B that has least one nearest neighbour in the exterior of B. Let $\mathcal{T}_1$ be the map that to each circunference site in class 2 associates $\phi(j) \longrightarrow \phi(j) - \frac{2\alpha_0}{k}$, while keeping invariant all $\phi(j)$ if j is a circunference site in class 3 or if j is not a circunference site.

Let $\mathcal{T}_r$ be the usual reflection of Peierls' Argument that changes $\phi(j) \longrightarrow -\phi(j)$ if j is a site inside B.

If $\mathcal{T}_1$ causes $I(\underset{\sim}{\phi})$ to decrease or to increase up to a limit given by:

$$I(\mathcal{T}_1 \underset{\sim}{\phi}) - I(\underset{\sim}{\phi}) \leqslant \left(\frac{\alpha_0}{k}\right)^2 \ell \qquad (11)$$

we define $\mathcal{T} = \mathcal{T}_r \circ \mathcal{T}_1$. Since in the configuration $\mathcal{T}_1 \underset{\sim}{\phi}$ every circunference site is in class 3 (and is therefore nearest neighbour of an exterior site in class 1) the application of $\mathcal{J}_r$ to $\mathcal{J}_1 \underset{\sim}{\phi}$ has the effect:

$$I(\mathcal{J}_r \mathcal{J}_1 \underset{\sim}{\phi}) - I(\mathcal{J}_1 \underset{\sim}{\phi}) \leq -2\left(\frac{\alpha_o}{k}\right)^2 \ell \qquad (12)$$

Hence,

$$I(\mathcal{J}\underset{\sim}{\phi}) - I(\underset{\sim}{\phi}) \leq -\left(\frac{\alpha_o}{k}\right)^2 \ell \qquad (13)$$

The reflection does not change $Q(\phi)$ and, by condition C,

$$Q(\mathcal{J}_1 \phi) - Q(\phi) \leq 0 \qquad (14)$$

From (13) and (14) we conclude that if $\underset{\sim}{\phi}$ does not violate (11), we have:

$$H^{\wedge}_{\alpha_o}(\mathcal{J}\underset{\sim}{\phi}) - H^{\wedge}_{\alpha_o}(\underset{\sim}{\phi}) \leq -\left(\frac{\alpha_o}{k}\right)^2 \ell \qquad (15)$$

If a configuration $\underset{\sim}{\phi}$ violates (11), then $\mathcal{J}\underset{\sim}{\phi}$ is defined to be $\mathcal{J}_2 \underset{\sim}{\phi}$, where $\mathcal{J}_2$ associates $\phi(j) \longrightarrow \phi(j) + \frac{2\alpha_o}{k}$ if j is a circunference site in class 2, and leaves unchanged all the other sites.

In this case we will have:

$$I(\mathcal{J}\underset{\sim}{\phi}) = I(\mathcal{J}_2 \underset{\sim}{\phi}) \leq I(\underset{\sim}{\phi}) - \left(\frac{\alpha_o}{k}\right)^2 \ell \qquad (16)$$

This results follows from the same discussion as in Bortz-Griffiths: the change in $I(\phi)$ caused by the application of $\mathcal{J}_1$ or $\mathcal{J}_2$ has two sources. First there are pairs of sites one of which is a circunference site in class 2 and the other is not a circunference site or is a circunference site in class 3. The change in $I(\phi)$ due to these pairs is equal in modulus but opposite in sign for the two transformations $\mathcal{J}_1$ and $\mathcal{J}_2$. A second contribuition comes from pairs of sites both of which are circunference sites in class 2. The application of $\mathcal{J}_1$ or $\mathcal{J}_2$ to these pairs invariably decreases $I(\phi)$. Hence, if for some configuration $\underset{\sim}{\phi}$ (11) is violated, then (16) must be satisfied.

In this case, again by our condition on $Q(\phi)$ we would have:

$$H^{\wedge}_{\alpha_o}(\mathcal{J}_2 \underset{\sim}{\phi}) - H^{\wedge}_{\alpha_o}(\underset{\sim}{\phi}) \leq -\left(\frac{\alpha_o}{k}\right)^2 \ell \qquad (17)$$

We have then condition i) satisfied.

The maps $\mathcal{T}_1$ and $\mathcal{T}_2$ are not one to one. But once we have fixed the class of the circunference sites, both $\mathcal{T}_1$ and $\mathcal{T}_2$ are one to one. Since there are ℓ circunference sites, there exists 2^{ℓ} distinct regions $\mathcal{B}_i$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are one to one. So the maximum number of $\mathcal{C}_i$ that can intercept at any point of $\mathbb{R}^{|\Lambda|}$ is bounded by $2\times 2^{\ell}$. So condition ii) is fulfilled with a=m=2.

This establishes (9) with $\gamma = \left(\frac{\alpha_0}{k}\right)^2$

The theorem then follows taking $\frac{\alpha_0}{k}$ sufficiently large.

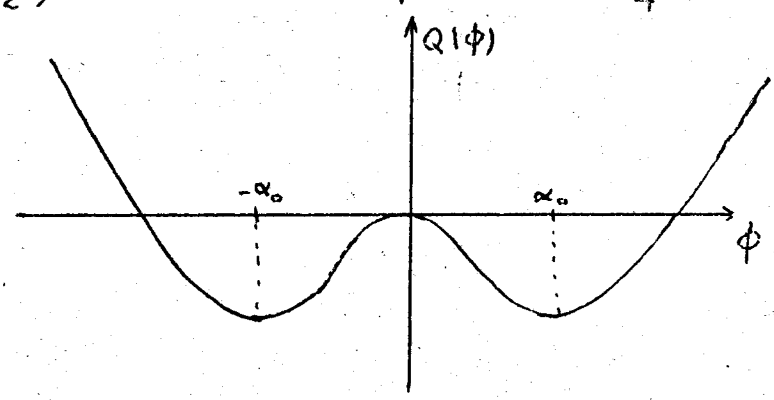
III - Examples

In this section we give two examples of theories satisfying condition C.

1) $Q(\phi) = a\phi^4 - b\phi^2 \quad , \quad a>0 \; , \; b>0 \, .$

Here $\alpha_0 = \sqrt{\frac{b}{2a}}$, and we can choose k=4, independent of a and b, for

$$Q\left(\phi \pm \frac{\alpha_0}{2}\right) \leq Q(\phi) \qquad \text{if} \quad |\phi| \leq \frac{\alpha_0}{4}$$



If we choose b/2a sufficiently large, we have all the conditions on Q satisfied. This theory has been discussed in [2].

2) $Q(\phi) = a\phi^6 - b\phi^4 + c\phi^2 \quad , \quad a, b, c > 0 \, .$

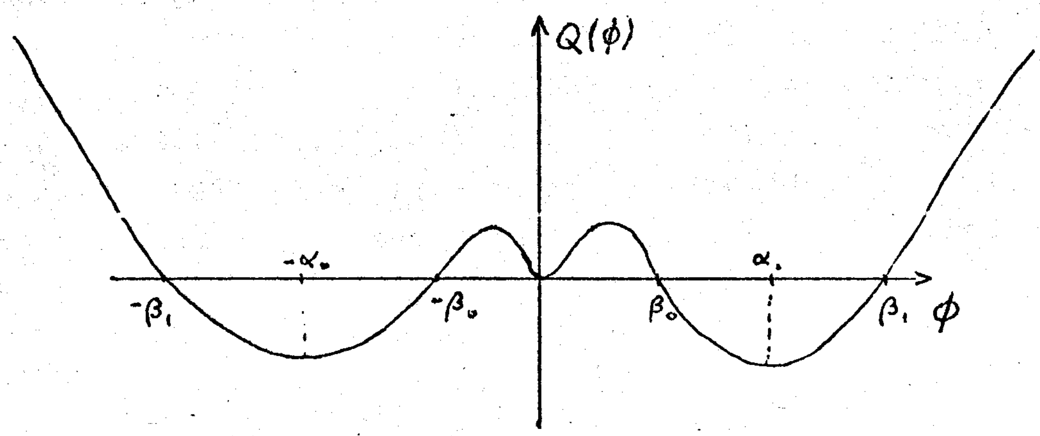
Here we have two absolute minima if

$$b^2 - 4ac > 0$$

In this case, the zeroes of $Q(\phi)$ are located at

$$\phi = 0 \quad , \quad \phi = \pm\beta_0 \quad , \quad \phi = \pm\beta_1$$

$$\beta_0 = \sqrt{\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}} \quad , \quad \beta_1 = \sqrt{\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}}$$



If we choose b/2a very large, and then a and c such that $\left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{1}{4}\left(\frac{b}{2a}\right)^2$, we will have that

$$\beta_1 = 3\beta_0 = 3\sqrt{\frac{b}{4a}}$$

In this case, α_0 will be as large as we please (since $\alpha_0 > \beta_0$) and $k = \frac{\alpha_0}{\beta_0} < 3$ can be chosen such that all the conditions on Q are satisfied:

$$Q\left(\phi \pm \frac{2\alpha_0}{k}\right) = Q(\phi \pm 2\beta_0) \leq Q(\phi) \quad \text{if} \quad |\phi| \leq \beta_0$$

This theory cannot be discussed as in ref. [2], since in this case we cannot apply Griffiths' inequalities.

IV - Acknowledgments:

We would like to thank Dr. W. Wrezinsky and Cezar A. Bonato for stimulating discussions and suggestions on the subject. One of us (R.A.N. Silva) would like to thank the FAPESP for financial support.

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