IFUSP/P-101

SEMICLASSICAL QUANTIZATION OF A FIELD THEORETICAL MODEL IN ANY NUMBER OF SPATIAL DIMENSIONS

by

I. VENTURA(*) and G.C. MARQUES(**)

Instituto de Física, Universidade de São Paulo,
C.P. 20.516 - São Paulo, SP, Brazil

SEMICLASSICAL QUANTIZATION OF A FIELD THEORETICAL MODEL IN ANY NUMBER OF SPATIAL DIMENSIONS

by

I. VENTURA(*) and G. C. MARQUES(**)

Instituto de Física, Universidade de São Paulo,
C.P. 20.516 - São Paulo, SP, Brazil

ABSTRACT

The semiclassical canonical quantization of the non relativistic Logarithmic Theory is done in any number of spatial dimension d. Instead of the usual expansion of the Hamiltonian about the classical fields, we propose an alternative route to semiclassical quantization by making an expansion around the charge operator. When we take the value d=0 in the energy expression the exact zero-dimensional spectrum is obtained. The mechanism of confinement characteristic of these Logarithmic Theories is also discussed.

(*) Supported in part by FAPESP (São Paulo, Brazil).
(**) Supported in part by CNPq (Brazil).

I. INTRODUCTION

Recently a lot of effort has been concentrated into the study of semiclassical methods of quantization in Field Theory[1-9]. By using these non-perturbative instruments the spectra of a large number of 1+1 dimensional models were obtained. Here we succeed in performing the semiclassical quantization of a Theory in any number of spatial dimensions.

The model which we quantize is just the non relativistic Logarithmic Theory proposed by Birula and Mycielski[10].

The relativistic Logarithmic Theory[10-13] has been intensively studied by us. In ref. (11) the stability of its soliton-like solutions was discussed, whereas the set of these solutions was considerably enlarged in ref. (12). In ref. (13) we pointed out the remarkable fact that the Logarithmic Theories exhibits confinement. Unfortunatelly, up to now we have not succeeded in quantizing the relativistic version of the model - the main difficulty coming from the question of renormalizability of theories with logarithmic nonlinearities.

The non relativistic Logarithmic Theory shares the most interesting features of the relativistic one (solitons and confinement), and - at least up to WKB quantization - it does not display any trouble concerning to renormalization. This is the reason for studying it in this paper.

The model is presented in section II. There the stability equation is obtained and solved, leading to a discrete set of stability angles. The mechanism of confinement is discussed in the context of this non relativistic theory.

In section III we compute the static spectrum of the model by using the DHN quantization formula[1,2,7]. The computation is done in any number of spatial dimensions. We show that,

extending our energy formula to dimension zero, the exact zero-dimensional spectrum is obtained.

The canonical quantization of the theory is performed in section IV. There, besides the static spectrum, the kinetic part of the energy is also obtained. Since we use collective variables, our method is similar to those of references (4), (5) and (6). But, an important difference deserves to be pointed out: instead of expanding - as usually - the Hamiltonian about the classical field (soliton), the expansion is done around a leading operator. In the present case the suitable leading operator is just the charge. So, our approach is expected to hold only in the computation of the energies of large charge states. The canonical commutation relations are satisfied in a semiclassical sense.

Conclusions are left to section V, whereas three Appendixes complement some calculations of the text.

II. THE MODEL: SOLITONS, STABILITY and CONFINEMENT

The model that we investigate here is defined by the following Lagrangian density[10]:

$$\mathcal{L}(\varphi^{\dagger},\varphi) = i\,\varphi^{\dagger}\partial_{t}\varphi - U(\varphi^{\dagger},\varphi) \quad , \tag{2.1}$$

where φ represents a complex scalar field and $U(\varphi^{\dagger},\varphi)$ is given by

$$U(\varphi^{\dagger},\varphi) = \frac{1}{2m}\,\partial_{\vec{x}}\varphi^{\dagger}\partial_{\vec{x}}\varphi - \frac{1}{2m\ell^{2}}\,\varphi^{\dagger}\varphi\left[\ln(\varphi^{\dagger}\varphi\, a^{d}) - 1\right] . \tag{2.2}$$

In expression (2.2), ℓ, 1/m and a are dimensional parameters while d stands for the number of spatial dimensions.

The Euler-Lagrange equation resulting from (2.1) is

$$i\,\partial_{t}\varphi + \frac{1}{2m}\,\partial^{2}_{\vec{x}}\varphi + \frac{1}{2m\ell^{2}}\,\ln(\varphi^{\dagger}\varphi\, a^{d})\,\varphi = 0 \quad . \tag{2.3}$$

The energy associated to a certain field $\varphi(\vec{x},t)$ is given by

$$E = \int d^{d}\vec{x}\;\; U(\varphi^{\dagger},\varphi) \quad , \tag{2.4}$$

whereas the other conserved quantity (the charge) will be

$$Q = \int d^{d}\vec{x}\;\; \varphi^{\dagger}(\vec{x},t)\,\varphi(\vec{x},t) \quad . \tag{2.5}$$

We are interested in the following family of classical solutions (solitons)

$$\varphi_{\omega}(\vec{x},t) = A(\omega)\,\exp\!\left(-i\,\omega t - \frac{\vec{x}^{\,2}}{2\ell^{2}}\right) \quad . \tag{2.6}$$

Plugging the field (2.6) into the equation of motion (2.3) we obtain the allowed values of $A(\omega)$:

$$A(\omega) = a^{-d/2} \exp\left(\frac{d}{2} - m\ell^2\omega\right) \quad . \tag{2.7}$$

The solution (2.6) describes an spherically symmetric extended object on its proper rest frame. The energy and the charge associated to this classical particle are given respectively by

$$E_{da}(\omega) = \left(\frac{\sqrt{\pi}\,\ell}{a}\right)^d \left(\omega + \frac{1}{2m\ell^2}\right) \exp\left(d - 2m\ell^2\omega\right) \quad , \tag{2.8}$$

and

$$Q(\omega) = \left(\frac{\sqrt{\pi}\,\ell}{a}\right)^d \exp\left(d - 2m\ell^2\omega\right) \quad . \tag{2.9}$$

A - Stability

In order to study the infinitesimal stability [7,11] of our particle we add to the classical solutions (2.6) a small fluctuation

$$\varphi(\vec{x},t) = \varphi_\omega(\vec{x},t) + e^{-i\omega t}\,\eta(\vec{x},t) \quad . \tag{2.10}$$

Using this representation for $\varphi(\vec{x},t)$ into the equation of motion (2.3), and retaining terms up to first order in $\eta(\vec{x},t)$, we obtain the linearized stability equation:

$$\left[i\partial_t + \frac{1}{2m}\partial_{\vec{x}}^2 - \frac{\vec{x}^2}{2m\ell^4} + \frac{(1+d)}{2m\ell^2}\right]\eta = \frac{1}{2m\ell^2}\,\eta^* \quad . \tag{2.11}$$

Now we may verify, by inspection, that the solutions of eq. (2.11) are of the form

$$\eta_{k_1 \cdots k_d}(\vec{x},t) = A_{k_1 \cdots k_d}\left[\prod_{i=1}^{d} h_{k_i}(x_i/\ell)\right] \cdot$$

$$\cdot\left[\left(\sqrt{K} - \sqrt{K-1}\right) e^{i\gamma_k\left(\frac{\omega t}{2\pi}\right)} - \left(\sqrt{K} + \sqrt{K-1}\right) e^{-i\gamma_k\left(\frac{\omega t}{2\pi}\right)}\right], \tag{2.12}$$

where $A_{k_1 \cdots k_d}$ is a real number, h_k is the normalized k^{th} eingensolution of the unidimensional harmonic oscilator, the integer K is the sum

$$K = \sum_{i=1}^{d} k_i \tag{2.13}$$

and the stability angles[1,2] γ_k are given by

$$\gamma_k = \pm \frac{2\pi}{m\ell^2\omega}\sqrt{K(K-1)} \tag{2.14}$$

We observe that there are[14]

$$D(K,d) = \frac{(K+d-1)!}{K!(d-1)!} \tag{2.15}$$

configurations of the set $\{k_i\}$ that satisfies (2.13). Then, each γ_k has a degeneracy of order D(K,d).

Since in the present case all stability angles are real numbers, we conclude that if a given fluctuation is infinitesimal at t=0, it will remain infinitesimal for any other time. So we say that the solutions of the type (2.6) are stables ones.

The stability angles (2.14) are the basical ingredients

to implement a WKB quantization of the Model[1,2,7]. This quantization will be presented in the next section.

From (2.14) we see that $\gamma_0 = \gamma_1 = 0$. γ_1 -having a degeneracy of degree d (see (2.15)) - is associated to the translational invariance of the Theory; whereas γ_0 - that has no degeneracy - is the zero frequency mode correspondent to gauge invariance.

B - Confinement

Here the mechanism of confinement will be discussed in the context of the nonrelativistic logarithmic theory. The same phenomenum occurs in the relativistic theory, as shown in reference (13). For these logarithmic theories confinement happens to be a consequence of the nonexistence of the weak field limit.

Let us define the theory in a cubic box of volume L^d with periodic boundary conditions L is such that $L \gg \ell$, a, 1/m, and latter we will take the limit $L \to \infty$.

Consider a plane wave solution of equation (2.3)

$$\varphi_{\vec{k},\omega}(\vec{x},t) = A(\vec{k},\omega)\, e^{i(\vec{k}\vec{x} - \omega t)} \qquad (2.16)$$

From direct substitution of (2.14) into (2.3) we obtain the relation

$$\omega = \frac{1}{2m}\left[\vec{k}^2 - \frac{1}{\ell^2}\ln(|A|^2 a^d)\right] \qquad (2.17)$$

The energy associated to this solution will be

$$E(\vec{k},A) = \frac{|A|^2}{2m} L^d \left\{ \vec{k}^2 - \frac{1}{\ell^2}\left[\ln(|A|^2 a^d) - 1 \right] \right\} , \qquad (2.18)$$

whereas the charge is

$$Q(A) = \frac{|A|^2}{2m} L^d \quad . \qquad (2.19)$$

From (2.15), (2.16) and (2.17) we get the following pair of relations involving ω, $\vec{k}$, E and Q

$$\omega(\vec{k},Q) = \frac{1}{2m}\left\{ \vec{k}^2 - \frac{1}{\ell^2}\left[\ln\left(Q\left(\frac{a}{L}\right)^d\right) - 1 \right] \right\} , \qquad (2.20.a)$$

and

$$E(\vec{k},Q) = Q\left\{ \vec{k}^2 - \frac{1}{\ell^2}\left[\ln\left(Q\left(\frac{a}{L}\right)^d\right) - 1 \right] \right\} . \qquad (2.20.b)$$

Now, since $\vec{k}^2$ is allways positive (otherwise $\varphi_{\vec{k},\omega}$ will not be a solution of (2.3)), in the limit $L \to \infty$ we get

$$\omega(\vec{k},Q) = \infty \quad and \quad E(\vec{k},Q) = \infty \quad . \qquad (2.21)$$

This means that in the infinite volume limit we have no plane wave solutions for the classical version of the theory. We have no excitations around the vacuum (φ=0). This is Confinement. It follows from the fact that the Lagrangian (2.1) is non analitical in $|\varphi|^2$ when $|\varphi|^2$=0. It does not occurs in any polinomial Lagrangian.

On the other hand we have seen that there exist excitations around the soliton (2.6). Obeying equation (2.11) these excitations are just linear superpositions of the solutions (2.12). The achievement is obvious: in the logarithmic theories, quantum

fluctuations can manifest themselves only in the presence of a soliton that works as a bag for containing them.

We emphasize that the fluctuations (2.12) goes to zero rapidly when any $|\chi_i|$ grows. This means that they are really confined within the soliton.

An interesting consequence of confinement is the fact that the set of stability angles is a discret one. In theories where a continuum of stability angles exists, the sum over this continuum describes states of a soliton plus a certain ammount of free elementary mesons coexisting[2]. In the present case the continuum does not exists since we have no free mesons.

In the relativistic version of the model the same mechanism occurs[13]. There the equation for the fluctuations is also of the harmonic type. There is a large number of semi - phenomenological papers in the literature where it is tried to confine quarks by means of the harmonic oscilator potential - the trouble is that in those papers the potential is introduced by hand. It is remarkable that in the logarithmic theories we get the same picture starting from a local Lagrangian.

III. QUANTIZATION IN THE MANNER OF DHN

Here, in order to obtain the semiclassical energy spectrum we will use the DHN[1] formula as stated by Coleman[7]. We leave to the next section the deduction of this spectrum by meam of a canonical quantization procedure.

According the DHN method the classical periods are "quantized" in the following way

$$\int_0^T dt \int d^d\vec{x}\, \pi_\omega(\vec{x},t)\,\dot{\varphi}^\circ_\omega(\vec{x},t) + \sum_{\gamma_i>0}\left(n_i + \frac{D(\gamma_i)}{2}\right)\left(T\frac{d\gamma_i}{dT} - \gamma_i\right) = 2\pi N \quad , \tag{3.1}$$

where $\varphi_\omega(\vec{x},t)$ is the classical solution of period $T\,(\omega = 2\pi/T)$ $\pi_\omega(\vec{x},t)$ is its conjugated momentum, $\{\gamma_i\}$ is the set of all stability angles and $D(\gamma_i)$ is the degree of degeneracy of each stability angle. Any quantum state is characterized by an integer N and by the set $\{n_i;\ n_i = 0, 1, 2 \ldots\}$. Solving Eq. (3.1) for the various states we obtain a discrete set of values for the period: $T(N, n_1, n_2, \ldots)$.

The allowed values of the energy are obtained from the expression[7]:

$$E(N, n_1, n_2, \ldots) = E_{cla}[T(N, n_1, n_2)] + \sum_{\gamma_i>0}\left(n_i + \frac{D(\gamma_i)}{2}\right)\frac{d\gamma_i}{dT}\bigg|_{T=T(N,n_1,n_2,\ldots)} \tag{3.2}$$

Here $E_{cla}^{(T)}$ is the classical energy of the soliton whose period

is T. It is given by expression (2.4).

In our case, since $\pi_\omega = i\varphi_\omega^+$ (see Lagrangian (2.1)), the integral in (3.1) must be written as

$$\int_0^T dt \int d^d\vec{x}\; i\,\varphi_\omega^+ \dot{\varphi}_\omega \qquad (3.3)$$

From the fact that the stability angles of this model have a linear dependence on T (see (2.14)) we conclude that the sum $\sum_{\omega_i}$ in Eq. (3.1) is zero. So, our periods are quantized by the following expression

$$\int_0^T dt \int d^d\vec{x}\; i\,\varphi_\omega^+ \dot{\varphi}_\omega = 2\pi N \qquad (3.4)$$

Evaluating explicity the integral of expression (3.4) we get

$$\left(\frac{\sqrt{\pi}\,\ell}{a}\right)^d \exp(d - 2m\ell^2\omega) = N \qquad (3.5)$$

This condition is equivalent to a quantization of the classical charge that is given in (2.9). The equivalence between semi-classical quantization and charge quantization was already pointed out in references (15) and (16).

Using (3.5), the quantized values of ω are obtained:

$$\omega_N = \frac{1}{2m\ell^2}\left[d - \ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln N\right] \qquad (3.6)$$

From (2.8) and (3.6) we conclude that the classical Energy as function of N is given by

$$E_{cla}(N) = \frac{N}{2m\ell^2}\left[d + 1 - \ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln N\right] \qquad (3.7)$$

In expression (2.15) we have the degree of degeneracy of each stability angle, whereas the stability angles themselves are shown in (2.14). Using them, one can writte the sum $\sum_{n_i}$ that appears in (3.2) in the following manner

$$\frac{1}{2m\ell^2}\sum_{k=0}^{\infty}\frac{(k+d-1)!}{k!\,(d-1)!}\sqrt{k(k-1)}' + \\ + \frac{1}{m\ell^2}\sum_{k=0}^{\infty} n_k \sqrt{k(k-1)}' \qquad (3.8)$$

The first sum in the above expression - that is clearly a divergent one - is the vacuum energy (note that it is inde - pendence of the particular state we are looking to), and so it will be dropped from the energy formula. Then, using (3.2), (3.7) and the finite part of (3.8) we get the spectrum

$$E(N; n_1, n_2, \dots) = \\ = \frac{N}{2m\ell^2}\left[d+1-\ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln N\right] + \\ + \frac{1}{m\ell^2}\sum_{k=0}^{\infty} n_k \sqrt{k(k-1)}' \qquad (3.9)$$

This is the WKB spectrum of the nonrelativistic logarithmic model in any number of space dimensions. Since in particular it holds for d=3, this is the first time that - as far as we know - a WKB quantization of a three dimensional model is done.

A - Zero Dimensional Case

In zero dimension, K is always zero, since the set $\{k_1, k_2 \ldots k_d\}$ is an empty one. So, in this case the sum in (3.9) does not exist and the spectrum can be written as

$$E^{(N)}_{d=0} = \frac{N}{2m\ell^2}(1 - \ln N) \tag{3.10}$$

Now are will solve the zero-dimensional version of the logarithmic model in order to compare the exact spectrum with (3.10). Our conclusion will be that (3.10) is the exact spectrum.

When d=0 the Lagrangian is given by

$$L(a^*, a) = i\, a^* \dot{a} + \frac{1}{2m\ell^2}\, a^* a (\ln a^* a - 1) \tag{3.11}$$

From it one concludes that the canonical momentum associated to a is ia^*, whereas the Hamiltonian is

$$H(a^*, a) = \frac{1}{2m\ell^2}\, a^* a\, (1 - \ln a^* a) \tag{3.12}$$

It is well known that after imposing the cannonical commutation relations

$$[a, i a^*] = i \quad \text{or} \quad [a, a^*] = 1 \tag{3.13}$$

the spectrum of a*a is given by

$$Sp(a^* a) = 0, 1, 2, \ldots N \ldots \tag{3.15}$$

So the spectrum of the Hamiltonian (3.12) will be just (3.10).

IV. CANONICAL QUANTIZATION

In this section we perform the canonical quantization of the model. Our procedure has much to do with those of references (4), (5) and (6), even though some important differences must be stressed. In order to point out these differences we shall start this section discussing the main ideas upon which our method is based.

The fields $\varphi(\vec{x}, t)$ and its momentum $\pi(\vec{x}, t) = \dot{\varphi}(\vec{x}, t)$ can - in a large variety of ways - be represented in terms of an infinite set of basical canonically conjugate pairs of operators $\{(a_0, b_0); (a_1, b_1); (a_2, b_2); \ldots\}$. Let us consider the class of states $\{|a_0^n \ldots\ldots>_*\}$ - a_0^n being the n^{th} eigenvalue of a_0 - where the eigenvalues of a_0 are very large when compared with the mean values of the other basical operators (with the exception of b_0, the momentum of a_0), i.e:

$$a_0^n >> \; < a_0^n, \ldots . |a_i| a_0^n, \ldots . > \qquad (4.1a)$$

$$a_0^n >> \; < a_0^n, \ldots . |b_i| a_0^n, \ldots . > \qquad (4.1b)$$

for all $i \neq 0$.

When dealing with the mentioned class of states, it is natural to consider a_0 as a <u>leading operator</u> and to treat the other basical operators as fluctuations about it. So an approximate - and eventually soluble - Hamiltonian can be obtained by an expansion around a_0 retaining only terms up to second order in the non leading operators. The following observations can serve as a brief outline of what we are going to do:

(i) In the present model the leading operator about which the Hamiltonian is expanded is the <u>charge</u> operator. So our approach

is expected to hold only in the description of large charge states.

(ii) The representation that we choose for the fields is designed to satisfy the semiclassical canonical commutation relations

$$\left[\varphi(\vec{x},t),\ \pi(\vec{y},t)\right] = i\,\delta(\vec{x}-\vec{y}) + O\left(\frac{1}{\sqrt{a_0}}\right) \tag{4.2}$$

Then, locality is guaranteed at least in a semiclassical sense.

(iii) Up to quadratic terms in the non leading operators the Hamiltonian decouples into an infinite set of partial Hamiltonians, each of them depending only on a given pair of basical canonically conjugate operators, i.e:

$$H = H_0(a_0) + H_1(a_1, b_1) + H_2(a_2, b_2) + \cdots + \tag{4.3}$$

$$+ \left[\text{terms of order greater than 2 in } a_1,\ b_1;\ a_2,\ b_2;\ \text{etc.}\right]$$

The partial Hamiltonian H_0 is independent of b_0 (the momentum of a_0). Within the spirit of the approximation explained previouly the terms of order greater than two in the fluctuations shall be dropped from the total Hamiltonian. Then the task of getting the spectrum becomes a very easy one because it suffices to sum up the energies of an infinite set of decoupled partial Hamiltonians.

We stress that our approach is different from the usual semiclassical quantization methods, since instead of treating fluctuations around a classical field we do an expansion about a leading operator.

The Hamiltonian and the charge of the non-relativistic Logarithmic Theory are - as we recall - given respectively by

$$H = \int d^d\vec{x}\left\{\frac{1}{2m}\partial_{\vec{x}}\varphi^+\partial_{\vec{x}}\varphi - \frac{1}{2m\ell^2}\varphi^+\varphi\left[\ln(\varphi^+\varphi a^d) - 1\right]\right\} \tag{4.4}$$

and

$$Q = \int d^d\vec{x}\, \varphi^+\varphi \qquad (4.5)$$

Before exhibiting the basical operators and the field representation convenient for our case we must introduce a complete set of functions of the coordinates. Being $\{k_1, k_2, \ldots k_d\}$ a set of d non negative integers, we define

$$F_{\{k_i\}}(\vec{x}) = \prod_{j=1}^{d} h_{k_j}\left(\frac{x_j}{\ell}\right) \quad , \qquad (4.6)$$

where h_k is the k^{th} normalized <u>real</u> wave-function of the unidimensional harmonic oscilator. To some of these functions we give an abbreviated notation:

(a) The first one, where $k_j = 0$ for all j, we name $F_0(\vec{x})$, i.e.

$$F_{\{0,0\ldots 0\}}(\vec{x}) = F_0(\vec{x}) \quad , \qquad (4.7.a)$$

and we recall that, since $F_0(\vec{x})$ is normalized, it will be

$$F_0(\vec{x}) = \left(\frac{1}{\ell\sqrt{\pi}}\right)^{d/2} e^{-\vec{x}^2/2\ell^2} \quad ; \qquad (4.7.b)$$

(b) The functions (4.6) where $k_s = 1$ and $k_j = 0$ for all $j \neq s$ we name $F_s(\vec{x})$, i.e.

$$F_{\{0\ldots k_s=1\ldots 0\}}(\vec{x}) = F_s(\vec{x}) \qquad (4.8)$$

A - <u>The basical operators</u>

We will build fields with the following basical <u>hermitian</u> operators:

(a) the pairs $(A(t), \theta(t))$ obeying the commutation relation

$$[A, \theta] = i \quad ; \tag{4.9}$$

(b) for each spatial direction s we define a pair of canonically conjugate operators $(\hat{z}_s(t), \hat{p}_s(t))$, whose commutation relations are

$$[\hat{z}_s, \hat{p}_{s'}] = i\,\delta_{ss'} \tag{4.10a}$$

$$[\hat{z}_s, \hat{z}_{s'}] = [\hat{p}_s, \hat{p}_{s'}] = 0 \quad , \tag{4.10b}$$

From now on $\hat{\vec{z}}$ $(\hat{\vec{p}})$ will stand for a vector whose s^{th} component is $\hat{z}_s(\hat{p}_s)$. Latter on we will interpret $\hat{\vec{z}}$ as the position of the center of mass of the particle described by a given state (collective coordinate[4-6]), whereas the eigenvalues of $\hat{\vec{p}}$ will be related to the momentum of that particle.

(c) for each set $\{k_i\}$, such that $K = \sum_{i=1}^{d} k_i > 1$ we also define a pair of canonically conjugate operators $\left(a_{\{k_i\}}(t), b_{\{k_i\}}(t)\right)$. Their commutation relations are

$$[a_{\{k_i\}}, b_{\{k'_i\}}] = i\,\delta_{\{k_i\}\{k'_i\}} \tag{4.11a}$$

$$= i\,\delta_{k_1 k'_1}\,\delta_{k_2 k'_2} \cdots \delta_{k_d k'_d} ,$$

$$[a_{\{k_i\}}, a_{\{k'_i\}}] = [b_{\{k_i\}}, b_{\{k'_i\}}] = 0 \quad . \tag{4.11b}$$

The remainder commutation relations among the basical operators are the following

$$[A, \hat{z}_s] = [A, \hat{p}_s] = [A, a_{\{k_i\}}] = [A, b_{\{k_i\}}] = 0 \tag{4.12a}$$

$$[\theta, \hat{z}_s] = [\theta, \hat{p}_s] = [\theta, a_{\{k_i\}}] = [\theta, b_{\{k_i\}}] = 0 \tag{4.12b}$$

and

$$\left[\hat{\vec{z}}_s, a_{\{ki\}}\right] = \left[\hat{\vec{z}}_s, b_{\{ki\}}\right] = \left[\hat{p}_s, a_{\{ki\}}\right] = \left[\hat{p}_s, b_{\{ki\}}\right] = 0 \tag{4.12c}$$

B - The field representation

We are now in a position to present the field representation that will allow us to perform a WKB quantization of the model. In terms of the basical operators defined in the last subsection the field $\varphi(\vec{x}, t)$ will be written as (17)

$$\varphi(\vec{x}, t) = e^{i\theta}\left[B\, F_0(\vec{x} - \hat{\vec{z}}) + \frac{i\ell}{\sqrt{2}} \sum_{s=1}^{d} F_s(\vec{x} - \hat{\vec{z}}) \frac{\hat{p}_s}{\sqrt{A}} + \right.$$

$$\left. + \sum_{\{ki, K>1\}} (a_{\{ki\}} + i\, b_{\{ki\}})\, F_{\{ki\}}(\vec{x} - \hat{\vec{z}})\right] , \tag{4.13}$$

where the operator B is given by

$$B = \left[A - \frac{\ell^2}{2} \frac{\hat{\vec{p}}^2}{A} - \sum_{\{ki; K>1\}} (a^2_{\{ki\}} + b^2_{\{ki\}} - 1/2)\right]^{1/2} . \tag{4.14}$$

Since the basical operators are hermitian ones, the field $\varphi^{+}(\vec{x}, t)$ must be

$$\varphi^{+}(\vec{x}, t) = \left[B\, F_0(\vec{x} - \hat{\vec{z}}) - \frac{i\ell}{\sqrt{2}} \sum_{s=1}^{d} \frac{\hat{p}_s}{\sqrt{A}} F_s(\vec{x} - \hat{\vec{z}}) + \right.$$

$$\left. + \sum_{\{ki;\, K>1\}} (a_{\{ki\}} - i\, b_{\{ki\}})\, F_{\{ki\}}(\vec{x} - \hat{\vec{z}})\right] e^{-i\theta} \tag{4.15}$$

At this point it is appropriate to stablish some connection between the operator approach developed here and others semi - classical methods of quantization.

As we shall see shortly, the operator A is just the charge. It is the leading operator about which the Hamiltonian will be expanded. We will consider only those states where the eigenvalue of A is very large (A more specific definition of what we mean by very large will be given latter on). This fact allows us to see the scheme outlined before in a different way:

Suppose that we remove the non leading operators from the field representation (4.13), so defining what we call the leading part of the field $\varphi_L(\vec{x},t)$:

$$\varphi_L(\vec{x},t) = e^{i\theta}\sqrt{A}\left(\frac{1}{\ell\sqrt{\pi}}\right)^{d/2} \exp\left[-\frac{(\vec{x}-\vec{z})^2}{2\ell^2}\right] \qquad (4.16)$$

An expansion of any function of the fields about A is equivalent to an expansion around the leading field $\varphi_L(\vec{x},t)$. But $\varphi_L(\vec{x},t)$ — even though being an operator - has a remarkable ressemblance with the classical soliton (2.6) (a soliton whose center of mass position is $\vec{\hat{z}}$). In this sense we may say that the approximation which shall be done here is equivalent to an expansion of quantum fluctuation around the soliton itself.

C - Charge Quantization

Since $\{F_0; F_S ; F_{1Ki}\}$ constitute an orthonormal basis, the charge operator Q defined in (4.5) will be

$$B^2 + \frac{\ell^2}{2}\frac{\hat{\vec{P}}^2}{A} + \sum_{\{Ki;K>1\}}\left\{a^2_{1Ki} + b^2_{1Ki} + i\left[a_{1Ki}, b_{1Ki}\right]\right\} . \qquad (4.17)$$

Now using the commutation relations (4.9a) and the expression (4.14) for the operator B we conclude that

$$Q = A \tag{4.18}$$

as mentioned above. So the charge operator is the momentum conjugate to θ

$$[Q, \theta] = i \quad \text{or} \quad Q = i \frac{\partial}{\partial \theta} \tag{4.19}$$

Then the eigenfunctions of Q must be of the form

$$\psi_q(\theta, c_i) = e^{-iq\theta} R(c_i) \tag{4.20}$$

where q is the charge eigenvalue and c_i are others basical fields variables different from θ and Q.

Well, but θ is a cyclic variable, i.e: if we change θ into $\theta + 2n\pi$ (n being an integer), the field operators $\varphi(\vec{x}, t)$ and $\varphi^\dagger(\vec{x}, t)$ does not change (see (4.13) and (4.15)). So, in order that the wave-functional (4.20) be single valued in the space of field variables, it is necessary that the eigenvalues q be quantized in the following way

$$q_N = N \tag{4.21}$$

where N is an integer. And since the charge operator Q is defined in (4.4) as the sum of non negative operators, N must be non negative.

D - The WKB Approximation

Plugging the fields $\varphi(\vec{x}, t)$ and $\varphi^\dagger(\vec{x}, t)$ into the Hamiltonian (4.4) and performing the spatial integration we obtain this Hamiltonian as a function of the operators Q(or A), $\hat{P}$, $a_{(ki)}$ and $b_{(ki)}$

Note that, since H does not depends on θ nor $\hat{\vec{z}}$, it commutes with Q and $\hat{\vec{p}}$

Now we will explain what the WKB approximation is : Look first over the operators

$$R_{\{ki\}} = a^2_{\{ki\}} + b^2_{\{ki\}} - 1/2 \qquad (4.22)$$

From the commutations relations (4.9a) we deduce that the eigenvalues of $R_{\{ki\}}$ are non negative integers: $n_{\{ki\}}$

Let us consider the class of states $|N, \vec{p}, n_{\{ki\}}\rangle$ (where $\vec{p}$ is an eigenvalue of $\hat{\vec{p}}$) whose charge is very large when compared with $\ell^2 \vec{p}^2$ and $\sum_{\{ki;k>1\}} n_{\{ki\}}$ i.e: states where

$$N >> \ell^2 \vec{p}^2 \qquad (4.23.a)$$

and

$$N >> \sum_{\{ki;k>1\}} n_{\{ki\}} \qquad (4.23.b)$$

Condition (4.23.b) also implies that

$$N >> \langle a^2_{\{ki\}} \rangle \ , \ \langle b^2_{\{ki\}} \rangle \qquad (4.24)$$

Then, in dealing with states that obey (4.23) (or linear combinations among them), we take Q to be the leading operators. The operators $\hat{\vec{p}}$, $a_{\{ki\}}$ and $b_{\{ki\}}$ are considered as small fluctuation about it. In this way we expand our Hamiltonian around Q <u>retaining only terms up to second order</u> in $\hat{\vec{p}}$, $a_{\{ki\}}$ and $b_{\{ki\}}$. This is the WKB approximation.

E - The expanded Hamiltonian

Due to the fact that Q commutes with $\hat{\vec{p}}$, $a_{\{k_i\}}$ and $b_{\{k_i\}}$ the above mentioned expansion is unambiguous and easy to do. It is performed in Appendix A, and the resulting Hamiltonian is

$$H_0(Q) + H_1\left(\frac{\hat{\vec{p}}}{\sqrt{Q}}\right) + \sum_{\{k_i;\, k>1\}} H_{\{k_i\}}\left(a_{\{k_i\}}, b_{\{k_i\}}\right), \tag{4.25}$$

where

$$H_0(Q) = \frac{1}{2m\ell^2}\left[d+1 - \ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln Q\right], \tag{4.26}$$

the kinetic Hamiltonian $H_1\left(\hat{\vec{p}}/\sqrt{Q}\right)$ is given by

$$H_1\left(\frac{\hat{\vec{p}}}{\sqrt{Q}}\right) = \frac{1}{2m}\,\frac{\hat{\vec{p}}}{Q} \tag{4.27}$$

and each one of the partial Hamiltonian $H_{\{k_i\}}$ is

$$H_{\{k_i\}} = \frac{1}{m\ell^2}\left[k\, b^2_{\{k_i\}} + (k-1)\, a^2_{\{k_i\}}\right] \tag{4.28}$$

We recall that $k = \sum_{i=1}^{d} k_i$

The quantization of $H_0(Q)$ is trivial. Since we have seen above that the eigenvalues of the charge operator are q_N=N, the eigenvalues of $H_0(Q)$ must be

$$E_N = \frac{N}{2m\ell^2}\left[d+1 - \ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln N\right] \tag{4.29}$$

The diagonalization of each one of the partial Hamiltonians $H_{\{k_i\}}$ is very simple. Due to the fact the coefficients of $a^2_{\{k_i\}}$ and $b^2_{\{k_i\}}$, in (4.28), are positive numbers, we

recognize $H_{\{ki\}}$ as the Hamiltonian of an harmonic oscilator. Then it has a discrete set of eigenvalues:

$$E_{n_{\{ki\}}} = \frac{\sqrt{K(K-1)}}{m\ell^2}\, n_{\{ki\}} + \frac{1}{2}\,\frac{\sqrt{K(K-1)}}{m\ell^2} \tag{4.30}$$

where $n_{\{ki\}}$ is an integer.

Being $\vec{p}$ an eigenvalue of $\hat{\vec{p}}$, it is obvious that the eigenvalues of the kinetic Hamiltonian $H_1(\hat{\vec{p}}/\sqrt{Q})$ are

$$E_N(\vec{p}) = \frac{1}{2mN}\,\vec{p}^{\,2} \tag{4.31}$$

Then the spectrum of the total Hamiltonian (4.25) is given by

$$E(N, \vec{p}^{\,2}, n_{\{ki\}}) = \frac{N}{2m\ell^2}\left[d+1-\ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln N\right] + \sum_{\{ki;\,K>1\}} \frac{\sqrt{K(K-1)}}{m\ell^2}\, n_{\{ki\}} + \sum_{\{ki;\,K>1\}} \frac{1}{2}\,\frac{\sqrt{K(K-1)}}{m\ell^2} + \frac{1}{2mN}\,\vec{p}^{\,2} \tag{4.32}$$

The second sum in the above spectrum, the one independent from the state we are looking over, is the vacuum energy and can be dropped from the formula.

In order to compare the spectrum (4.32) with that obtained in section III by using the DHN formula, it is convenient to define, for a given $K = \sum_{i=1}^{d} ki$, the number

$$n_K = \sum_{\{k_i;\ \sum_{i=1}^{d} k_i = K\}} n_{\{k_i\}} \qquad (4.33)$$

So, in terms of n_K and after subtracting the vacuum energy, the spectrum (4.32) becomes

$$E(N, \vec{P}^2, n_{\{k_i\}}) = \frac{N}{2m\ell^2}\left[d+1-\ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln N\right] +$$

$$+ \sum_{K>1} \frac{\sqrt{K(K-1)}}{m\ell^2}\, n_K + \frac{1}{2mN}\,\vec{P}^2 \qquad (4.34)$$

When we take the value $\vec{P}^2 = 0$ in the above spectrum it agrees with that given by Eq. (3.9).

F - The momentum operator

Now we will show more clearly the connection between the operator $\hat{P}_s$ and the s^{th} component of the momentum operator $\hat{P}_s$ that is defined by

$$\hat{P}_s = -\int d^d\vec{x}\ \varphi^+(\vec{x}t)\, i\frac{\partial}{\partial x_s}\varphi(\vec{x},t) \qquad (4.35)$$

In the Appendix II it is shown that

$$[\hat{Z}_s, \hat{P}_{s'}] = i\,\delta_{ss'} + O(1/\sqrt{Q}) \tag{4.36}$$

and

$$\hat{P}_s^2 = \hat{p}_s^2 + (\text{terms of } 3^{th} \text{ and } 4^{th} \text{ order in } \hat{p}_s,\ a_{\{ki\}} \text{ and } b_{\{ki\}}) \tag{4.37}$$

From (4.36) we conclude that the semiclassical canonical commutation relations involving the c.m position operator $\hat{\vec{Z}}$ and the total momentum $\hat{\vec{P}}$ are preserved in this scheme of quantization.

In (4.37) the terms of 3^{th} and 4^{th} order in the non leading operator can be neglected, since this procedure is in agreement with the approximation done for the Hamiltonian (see subsections D and E). Then the kinetic Hamiltonian H_1 can be written as

$$H_1 = \frac{1}{2mQ}\,\hat{\vec{P}}^2 \tag{4.38}$$

whereas its spectrum is

$$E_N(\vec{P}) = \frac{1}{2mN}\,\vec{P}^2 \tag{4.39}$$

where $\vec{P}$ is an eigenvalue of the total momentum operator $\hat{\vec{P}}$.

G - Semiclassical Locality

In order to complete our quantization scheme we should say that the commutation relation among $\varphi(\vec{x},t)$ and its canonical

momentum $\pi(\vec{x},t)$ is given by

$$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta(\vec{x}-\vec{y}) + O(1/\sqrt{Q}) \tag{4.40}$$

This equation is deduced in Appendix III.

Then, when dealing with the class of states satisfying conditions (4.23.a) and (4.23.b), the term $O(1/\sqrt{Q})$ can be dropped form Eq. (4.40) and we get a local theory. We recall that the mentioned class of states is just the one for which the spectrum has been obtained.

V. CONCLUSION

We succeeded in quantizing semiclassically the nonrelativistic logarithmic model in any number of spatial dimension. Two methods of quantization were used: (a) applying the DHN quantization formula we were able to get the static spectrum of the model; (b) In section IV it was done a canonical quantization. There the charge was treated as a leading operator and we expanded the Hamiltonian about it retaining only terms up to second order in the non leading operators. Doing so, besides the static spectrum we also obtained the kinetic part of the energy.

From this kinetic energy (see (4.34) or (4.39)) we conclude that a state of charge N is the nonrelativistic description of a particle, whose mass is

$$M_N = m\,N\ , \tag{5.1}$$

that can be interpreted as a bound state of N solitons of mass m.

The binding energy of this N solitons system is given by the static spectrum (see (3.9) or (4.34)):

$$E(N, n_{Kij}) = \frac{N}{2m\ell^2}\left[d+1 - \ln\left(\frac{a}{\sqrt{\pi}\,\ell}\right)^d - \ln N\right] + \sum_{K>1} \frac{\sqrt{K(K-1)}}{m\ell^2}\, n_K \tag{5.2}$$

Now let us briefly discuss how could the present theory be turned more realistic. Consider the theory defined by Lagrangian (2.1) when $d = 3$ and

$$\ell \longrightarrow \frac{1}{m} \tag{5.3}$$

In this case, the states for which

$$N \lesssim \left(\frac{\sqrt{\pi}\,\ell}{a}\right)^d \tag{5.4}$$

have a binding energy whose modulus is much smaller than the mass M_N. Then we are describing a sort of nonrelativistic "nuclear physics" of non rotating particles[18].

Unfortunatelly - since the model has no fundamental state - when N is sufficiently large the binding energy may be equal to (or greater than) the mass M_N, and the above picture fails. This difficulty can eventually be remediated if we couple our field $\varphi(\vec{x},t)$ to the eletromagnetic field

$$g\left[\varphi^{+}\varphi A_{o} - i\,\varphi^{+}\partial_{\vec{x}}\varphi\,\vec{A}\right] \tag{5.15}$$

Doing so, the model should gain a fundamental state and maybe one can recover the description of a non relativistic "nuclear physics" even for very large values of the charge.

As a final remark we want to say that this method of considering the charge as the leading operator may - in principle - be applied to the semiclassical quantization of any theory invariante under gauge transformations of first kind and exhibiting solitons.

ACKNOWLEDGEMENTS

We thank M.Abud and J.A.Swieca for discussions on this subject.

FOOTNOTES and REFERENCES

1. R.Dashen, B.Hasslacher and A.Neveu, Phys. Rev. D11, 3424 (1975).
2. R.Rajaraman, Phys. Reports. vol. 21C # 5 (1975).
3. J.Goldstone and R.Jackiw, Phys. Rev. D11, 1486 (1975).
4. J.L.Gervais, A.Jevicki and B.Sakita, Phys. Rev. D12, 1038 (1975).
5. N.H.Christ and T.D.Lee, Phys. Rev. D12, 1606 (1975).
6. R.Friedberg, T.D.Lee and A.Sirlin, Phys. Rev. D13, 2739 (1975).
7. S.Coleman, "Classical Lumps and Their Quantum Descendents", proceedings of the 1975 School fo Subnuclear Physics Ettore Majorana.
8. A.Klein and F.Krejs, Phys. Rev. D13, 3295 (1976) - and references therein.
9. G.C.Marques and I.Ventura, Phys. Rev. D13, 1807 (1976).
10. I.Bialinicki-Birula and J.Mycielski, Bull. Acad. Polon Sci. Cl. III 23, 461 (1975).
11. G.C.Marques and I.Ventura, Phys. Rev. D14, 1056 (1976).
12. G.C.Marques and I.Ventura, São Paulo preprint.
13. I.Ventura and G.C.Marques - Talk given at the "Topical Meeting on Nonperturbative Phenomena", CERN, (July, 1976) - São Paulo preprint.
14. R.Kubo - "Statistical Mechanics", North Holland, Amsterdan, 36, (1971).
15. I.Ventura and G.C.Marques, Phys. Letters 64B, 43 (1976).

16. C.Montonen, Nucl. Phys. B112, 349 (1976)

17. Even though this field representation may seem somewhat artificial, it was designed to satisfy the semiclassical canonical commutation relations among the fields (see Eq. (4.40)) and also to lead to a simple expression for the charge operator (see Eq. (4.18)).

18. It is very easy to verify that rotating solitons may be obtained in Logarithmic Theories with a more complex internal symmetry. For instance, if we start - in three dimensions - with a four component field in Lagrangian (2.1) we shall get solitons having angular momentum.

APPENDIX A

In this Appendix we do a Taylor series expansion of the Hamiltonian (4.4) about the charge operator Q up to second order in the nonleading operators. First of all this expansion is done about the operator B defined in Eq. (4.4). After this the operator B itself is expanded around Q.

Before proceeding a remark must be stated: It is known that due to the non commutativity of the pairs of basical operators (in powers of which the Hamiltonian is developed) the mentioned expansion is not well defined. We have the freedom of ordering the operators that constitute a given term in various different ways. However - at least when we are concerned with approximations up to second order in the nonleading operators - each one of these various ways of ordering ammonnts only on a particular definition of what the vacuum energy is. Here for practical reasons our expansion will be guided by the following rule: The development of the Hamiltonian in Taylor series about the charge is done treating each one of the nonleading operators as c-numbers. Once the expansion is concluded they turn to be operators and the piece correspondent to the vacuum energy can be identified and subtracted.

In order to carry on the above mentioned expansion it is convenient to introduce an auxiliary field $W(\vec{x},t)$ that depends only upon the nonleading variables.

$$W(\vec{x},t) = \frac{i\ell}{\sqrt{2}} \sum_{s=1}^{d} F_s(\vec{x} - \hat{\vec{z}}) \frac{\hat{P}_s}{\sqrt{A}} + \cdot$$

$$+ \sum_{\{ki;k>1\}} (a_{\{ki\}} + i\, b_{\{ki\}}) F_{\{ki\}}(\vec{x} - \hat{\vec{z}}) \qquad (A.1)$$

Note that $W(\vec{x},t)$ is of first order in the non leading operators and that it is ortogonal to $F_o(\vec{x}-\hat{\vec{z}})$

The fields $\varphi(\vec{x},t)$ and $\varphi^+(\vec{x},t)$ are given by

$$\varphi(\vec{x},t) = e^{i\theta}\left[B\,F_o(\vec{x}-\hat{\vec{z}}) + W(\vec{x},t)\right] \tag{A.2.a}$$

and

$$\varphi^+(\vec{x},t) = \left[B\,F_o(\vec{x}-\hat{\vec{z}}) + W^+(\vec{x},t)\right] e^{-i\theta} \tag{A.2b}$$

Let us look over the product $\varphi^+(\vec{x},t)\,\varphi(\vec{x},t)$ that is a basical ingredient to construct the Hamiltonian.

$$\varphi^+(\vec{x},t)\,\varphi(\vec{x},t) = B^2 F_o^2(\vec{x}-\hat{\vec{z}}) + W^+(\vec{x},t)\,W(\vec{x},t) +$$

$$+\left[B\,F_o(\vec{x}-\hat{\vec{z}})\,W(\vec{x},t) + W^+(\vec{x},t)\,F_o(\vec{x}-\hat{\vec{z}})\right] \tag{A.3}$$

We are interested in the following function of the fields

$$T(\vec{x},t) = \varphi^+\varphi\left[\ln(\varphi^+\varphi\,a^d) - 1\right] \tag{A.4}$$

Then we expand $T(\vec{x},t)$ about the dominant operator $B^2F^2(\vec{x}-\hat{\vec{z}})$ going up to second order in $W(\vec{x},t)$ (that is equivalent to second order in the nonleading operators). We recall that at this step of the calculations the nonleading operators are treated as c-numbers. So $T(\vec{x},t)$ can be written as

$$T(\vec{x},t) = t_0 + t_1 + t_2 +$$

$$+ \left[\begin{array}{l}\text{terms of order greater than}\\ \quad 2 \text{ on } W(\vec{x},t)\end{array}\right] \tag{A.5}$$