IFUSP/P-104

SOLID NEUTRON MATTER: THE ENERGY DENSITY IN THE RELATIVISTIC HARMONIC APPROXIMATION

B.I.F.-USP

by

M.Cattani and N.C.Fernandes

Instituto de Física, Universidade de São Paulo
C.P.nº 20516 - São Paulo, S.P., Brazil

ABSTRACT

A relativistic expression for the energy density as a function of particle density for solid neutron matter is obtained using Dirac's equation with a truncated harmonic potential. Ultra baric and superluminous effects are not found in our approach.

1. Introduction

In the past few years special attention has been devoted to the problem of the possible solidification of the core of a neutron star (Canuto, 1975). Several attempts to calculate the equation of state of the dense neutron matter have been published. In our opinion the most plausible ones are those of Banerjee et al (1970), Canuto and Chitre (1974), Pandharipande (1973) and Guyer and Takemori (1975).

Banerjee et al. (1970) proposed a solid - body model for the dense matter at zero temperature. The neutrons are assumed to form a body-centered cubic lattice, with a lattice parameter Δ. This approach is based on the assumption that when the nuclear forces become sufficiently repulsive, a possible minimum energy state can be achieved by keeping the neutrons as far a part as possible by localizing them at lattice sites. The calculations are done in the harmonic approximation using the classical Debye model with Reid's soft core potential as the interaction between neutrons. All calculations were performed in a non-relativistic approximation.

Canuto and Chitre (1974), used a non - relativistic quantum-mechanical treatment to calculate the energy of the neutron lattice. They assumed that the neutrons oscillate harmonically around their equilibrium positions at the lattice sites. The characteristic frequency ω of the harmonic oscillator and the spread of the wave function of the particle are assumed to depend on the lattice distance Δ. The frequency ω is obtained by the Hartree method taking the two body nucleon - nucleon potential as

Reid's phenomenological soft-core potential. Correlations between pairs of oscillators are taken into account. In addition, a FCC structure for the neutron lattice was used to obtain the lowest possible value for the energy of the particles. They considered densities up to 8×10^{15} g/cm^3 and have shown that only for densities larger than 1.5×10^{15} g/cm^3 the lattice is stable.

On the other hand, Pandharipande (1973) and Takemori and Guyer (1975) analysed the possibility of solidification using several versions of Reid's interaction. They conclude that for none of these no liquid - solid phase transition is found for densities up to 5×10^{15} g/cm^3 .

In our opinion, the problem of solidification remains unsolved and we can only speculate about this point.

If solidification is possible relativistic effects will be significant for densities higher than 2×10^{15} g/cm^3 , since the region of confinement of the neutron (see Canuto and Chitre) will be smaller than 0.8 fermi , comparable to the neutron Compton wavelength, which is about 0.2 fermi .

In the present work our purpose is, assuming that the core of the neutron star is a solid, to obtain an equation of state for densities higher than 10^{15} g/cm^3 taking into account relativistic effects. We evaluate the relativistic eigenvalues for the bound states of the neutron using Dirac's equation.

To carry out the calculations we assume that: a) the neutrons are arranged in a lattice, b) they vibrate around an equilibrium position under a potential $V(r)$ that is harmonic for $r \leq a$ and is constant and equal to $V_o = (1/2)\, m\omega^2 a^2$, for

$r \gg a$; c) since the vibration frequencies are extremely high we neglect correlations among neighboring neutrons.

This model can be justified by the following arguments: the very strong gravitational field in the core of the star tends to confine each neutron inside a cell delimited by surrounding neutrons.

Several forms for the nucleon - nucleon interaction potential have been employed during the last twenty years. The best theoretically understood model for the nucleon forces is based on the exchange of mesons between two nucleons.

Walecka (1974) proposed a theory for a highly condensed matter assuming that the nucleons interact via scalar and vector mesons. He solved the field equations, for uniform baryon density, replacing the scalar field by their expectation values: $\phi \to \phi_o$ and $A_\mu \to i\,\delta_{\mu 4} A_o = i\,\delta_{\mu 4} \varphi$.

This approximation was first employed by Zeldovich (1962) for a pure vector interaction.

Using a different approach for the strongly interacting system of baryons, Bowers et al. (1973) considered only the pseudo-scalar mesons.

However, as one can see explicitly in Walecka's work, for high densities, the pseudo-scalar mesons play a negligible role and the vector mesons dominate.

We assume, according to Walecka, that the vector mesons play a dominant role and that $\vec{A} = 0$ and $A_4 = i\varphi$. Furthermore, φ is taken as soft, that is, with a finite maximum at the origin (Otsuki et al. 1964). We must note that this last approximation will not be rigorously consistent with our scheme if more mesons than vector mesons are needed to describe the soft-core at the origin. However, up to now, this is an open question (Canuto 1975).

So, in our model, the neutron inside the cell feels an average potential $< \varphi > = V(r)$ which is soft and that, for simplicity, is assumed to have an harmonic shape. The maximum value for $V(r)$ is V_o .

Assumption (c) is equivalent to an Einstein model of a solid and it seems to be justified in the relativistic case because very high vibracional frequencies appear due to the confinement of the neutron.

A similar procedure was adopted by Cazzola et al. (1966) but, instead of the harmonic cell, they have used a square well. Our description, involving a harmonic potential seems to be more realistic than the previous treatment since the nucleon-nucleon interaction, as required by the experimental facts, is soft.

In section (2) we shown how to obtain the eigenvalues for the bound states of the neutron corresponding to the model stated above.

In section (3) we use the results of section (2) to obtain an energy per particle-density relation that is compared with the non - relativistic calculations.

2. Solutions of Dirac's Equation

Let us first solve Dirac's equation for the harmonic potential.

Considering the radial Dirac's equation (Plesset 1932; Messiah 1966) and assuming that $\vec{A} = 0$ and $V = \frac{1}{2} Kr^2$ (see section 1), we have:

$$\frac{dR_a}{dr} = \frac{\chi}{r} R_a + \left(\frac{mc^2 - E}{\hbar c} + \frac{V}{\hbar c} \right) R_b \tag{1}$$

$$\frac{dR_b}{dr} = - \frac{\chi}{r} R_b + \left(\frac{mc^2 + E}{\hbar c} - \frac{V}{\hbar c} \right) R_a \tag{2}$$

where R_a and R_b are the small and large components, respectively, and $\chi = -(\ell + 1)$ if $j = \ell + 1/2$ and $\chi = +\ell$ if $j = \ell - 1/2$.

Defining $k \equiv m\omega^2$, $\xi \equiv (\frac{m\omega}{\hbar})^{1/2} r$ and $E \equiv$ $\equiv \eta \hbar\omega + mc^2$, equations (1) and (2) become:

$$\frac{dR_a}{d\xi} = \frac{\chi}{\xi} R_a + (\varepsilon_- + A \xi^2) R_b \tag{3}$$

$$\frac{dR_b}{d\xi} = - \frac{\chi}{\xi} R_b + (\varepsilon_+ - A \xi^2) R_a \tag{4}$$

where $\varepsilon_- = \eta\varepsilon$, $\varepsilon_+ = \eta\varepsilon + \frac{2}{\varepsilon}$, $A = \frac{\varepsilon}{2}$ and $\varepsilon = (\frac{\hbar\omega}{mc^2})^{1/2}$.
One easily verifies that for $\varepsilon \to 0$ (Nikolsky 1930) the non-relativistic limit is obtained, namely, $R_a \to 0$ and R_b obeys the radial equation for the non-relativistic harmonic oscillator:

$$\left[\frac{d^2}{d\xi^2} - \frac{\ell(\ell+1)}{\xi^2} - \xi^2 + 2\eta \right] R_b = 0$$

Equations (3) and (4) can be solved by expanding R_a and R_b into power series. For $\chi = +\ell$ $(j = \ell - 1/2)$ we have:

$$R_a = \xi^{\ell} \frac{\varepsilon}{2} \sum_{m=0}^{\infty} A_m \xi^{2m} \tag{5}$$

and

$$R_b = \xi^{\ell+1} \sum_{m=0}^{\infty} B_m \xi^{2m} \tag{6}$$

where $A_o = 1$,

$$A_m = \frac{1}{2m} \left[\frac{\varepsilon_+ \varepsilon_- A_{m-1}}{2\ell + 2m - 1} + A\, A_{m-2} \left(\frac{\varepsilon_+}{2\ell + 2m - 3} - \frac{\varepsilon_-}{2\ell + 2m - 1} \right) - \right.$$
$$\left. - A^2 \frac{A_{m-3}}{2\ell + 2m - 3} \right]$$

for $m \geq 1$, with $A_{-m} = 0$ and

$$B_m = \frac{\varepsilon}{2(2\ell + 2m + 1)} \left(\varepsilon_+ A_m - A\, A_{m-1} \right)$$

For $\chi = -(\ell + 1)$ $(j = \ell + 1/2)$ we have:

$$R_a = \xi^{\ell+2} \sum_{m=0}^{\infty} A_m \xi^{2m} \tag{7}$$

and

$$R_b = \xi^{\ell+1} \sum_{m=0}^{\infty} B_m \xi^{2m} \tag{8}$$

where $A_o = \dfrac{\varepsilon_-}{2\ell + 3}$, $A_1 = \dfrac{1}{2\ell + 5} \left[\dfrac{\varepsilon_+ \varepsilon_-^2}{2(2\ell + 3)} + A \right]$,

$$B_1 = \frac{1}{2} \frac{\varepsilon_+ \varepsilon_-}{(2\ell + 3)} , \quad B_2 = \frac{1}{4} \left(\varepsilon_+ A_1 - A\, A_o \right) ,$$

$$A_2 = \frac{1}{2\ell + 7} \left(\varepsilon_- B_2 + A\, B_1 \right) ,$$

$$A_m = \frac{1}{2\ell + 2m + 3}\left[\frac{\varepsilon_+\varepsilon_-}{2m} A_{m-1} + A\, A_{m-2}\left(\frac{\varepsilon_+}{2m-2} - \frac{\varepsilon_-}{2m}\right) - \frac{A^2\, A_{m-3}}{2m-2}\right] \quad \text{for} \quad m \geq 3$$

and $$B_m = \frac{1}{2m}\left(\varepsilon_+ A_{m-1} - A\, A_{m-2}\right) \quad \text{for} \quad m \geq 1 .$$

It is in general not possible to write R_a and R_b in closed form. This can be done only for $\varepsilon = 0$.

Figures 1, 2, 3 and 4 are a plot of $R_a(\xi)/\xi$ and $R_b(\xi)/\xi$ as function of ξ and ε .

(INSERT FIGURES 1, 2, 3, 4)

Figures 1 and 2 show the case $\chi = +\ell$ with $\ell = 1$ and $\eta = 5/2$; figures 3 and 4 show the case $\chi = -(\ell+1)$ with $\ell = 0$ and $\eta = 3/2$.

For $\varepsilon = 0$ (non-relativistic limit) we obtain from equation (6) that $R_b = \xi \exp(-\xi^2/2)$ for $\chi = -(\ell+1) = -1$ and from equation (8) that $R_b = \xi^2 \exp(-\xi^2/2)$ for $\chi = +\ell = +1$, which are the radial wavefunctions for the harmonic oscillator for the states 1s and 1p , respectively. From equations (5) and (7) we see that $R_a = 0$, in both cases, if $\varepsilon = 0$.

For $\varepsilon > 0$ and for very large values of ξ we see from equations (3) and (4) that $R_a \cong \cos(A\,\xi^3/3 + \theta)$ and $R_b \cong - \sin(A\,\xi^3/3 + \theta)$ which means that the neutron cannot be bound by the harmonic potential. This effects is known as Klein's paradox (Klein 1929, Plesset 1932, Tomonaga 1968). We shall show that with a truncated harmonic potential, bound states do exist.

The radial eigenfunctions for the constant potential

$V(r) = V_o$ can be found from equations (1) and (2). It is known (Cazzola et al. 1966) that the eigenfunctions that correspond to bound states are:

$$R_a = \sqrt{e - v_o - 1}\ \xi\ h_{\ell'}^{(1)}\ (i\,q\,\xi/\varepsilon)\ \tilde{\omega}\ \varepsilon(e - v_o)$$

and

$$R_b = \sqrt{e - v_o + 1}\ \xi\ h_{\ell}^{(1)}\ (i\,q\,\xi/\varepsilon) \tag{9}$$

where $e = E/mc^2$, $v_o = V_o/mc^2$, $\ell' = \ell + 1$ for $\chi = -(\ell + 1)$, $\ell' = \ell - 1$ for $\chi = \ell$, $\varepsilon(e - v_o)$ is the signal function of $e - v_o$, $\tilde{\omega}$ the parity of the state, $h_{\ell}^{(1)}$ the spherical Hankel functions of first order and $q = \sqrt{1 - (e - v_o)^2}$ is real.

The complete eigenfunctions ψ_a^{int} and ψ_b^{int} for $r \leq a$ are:

$$\psi_a^{int} = i\, C_{int}\ \frac{R_a}{\xi}\ Y_{\ell' j}^m$$

$$\psi_b^{int} = C_{int}\ \frac{R_b}{\xi}\ Y_{\ell j}^m \tag{10}$$

For $r \geq a$ we have:

$$\psi_a^{ext} = i\, C_{ext} \sqrt{e - v_o - 1}\ h_{\ell'}^{(1)}\ (i\,q\,\xi/\varepsilon)\ \tilde{\omega}\ \varepsilon(e - v_o)\ Y_{\ell' j}^m$$

$$\psi_b^{ext} = C_{ext} \sqrt{e - v_o + 1}\ h_{\ell}^{(1)}\ (i\,q\,\xi/\varepsilon)\ Y_{\ell j}^m \tag{11}$$

where R_a and R_b for $\chi = +\ell$ are given by equations (5) and (6), respectively, and for $\chi = -(\ell + 1)$ by (7) and (8), respectively, C_{int} and C_{ext} are normalization constants and $Y_{\ell j}^m$ the function os the total angular momentum (j, m), formed by the composition of a spin 1/2 with the spherical harmonics of order ℓ.

For $r = a$, corresponding to $\xi = \varepsilon a/\lambda\!\!\!^{-}$, where $\lambda\!\!\!^{-}$

is the Compton wavelength of the neutron, we must have $\psi_a^{int} = \psi_a^{ext}$ and $\psi_b^{int} = \psi_b^{ext}$. This gives:

$$\frac{R_a(\varepsilon a/\lambda\!\!\!^{-})}{\sqrt{e - v_o - 1}\; h_{\ell'}^{(1)}(i q a/\lambda\!\!\!^{-})} = \frac{R_b(\varepsilon a/\lambda\!\!\!^{-})}{\sqrt{e - v_o + 1}\; h_{\ell}^{(1)}(i q a/\lambda\!\!\!^{-})}\, \varepsilon (e - v_o)\, \tilde{\omega} \qquad (12)$$

From this relation we can determine the energy levels e in a graphical way.

3. Density of Energy and Comments

In this section we obtain the density of energy $\underline{u}$ as a function of the density ρ of the neutron matter at zero temperature. According to the model suggested in section (1), we have $u = n\, e_o$, where n is the neutron density and e_o is the ground state energy of one neutron for a given lattice distance Δ .

To calculate e_o we proceed as follows: we fix $\underline{a}$ in equation (12) and look for a potential V_o that gives the minimum value for the ground state energy $\underline{e}$ of the neutron. This minimum value is taken to be e_o .

In all cases analysed here the wavefunctions are concentrated essentially in the harmonic well. So, the lattice distance Δ is given approximately by $\Delta \cong 2a$.

Since $V_o = (1/2)\, m\omega^2 a^2$ the characteristic frequency ω depends on Δ and consequently on the matter density as occurs in the model of Canuto and Chitre (1974).

We considered only two cases: (1) $j = 1/2$, $\ell = 1$ $(\ell' = 0)$ and (2) $j = 1/2$, $\ell = 0$ $(\ell' = 1)$. These cases are the two lowest energy states having the lowest centrifugal barrier.

Following Canuto and Chitre, we considered the FCC structure as the favorite for the solidified phase. Another structure, like BCC , gives higher values for the energy.

In Figure 5 our results for the energy density u (erg/cm^3) , as a function of the density ρ (g/cm^3) , are compared with those of Banerjee et al. and of Canuto and Chitre. The energy density for $j = 1/2$ and $\ell = 1$ ($\ell' = 0$) will be indicated by $u_1(\rho)$ and for $j = 1/2$ and $\ell = 0$ ($\ell' = 1$) by $u_2(\rho)$.

(INSERT FIGURE 5)

We observe that only for densities less than 4×10^{15} g/cm^3 the energy density $u_{CC}(\rho)$ obtained by Canuto and Chitre is lower than $u_2(\rho)$. Our results $u_1(\rho)$ coincide with $u_{CC}(\rho)$ for $\rho \lesssim 10^{15}$ g/cm^3 and are lower than $u_{CC}(\rho)$ for $\rho > 10^{15}$ g/cm^3 . For densities higher than 40×10^{15} g/cm^3 , $u_2(\rho)$ becomes lower than $u_1(\rho)$.

For case (2) V_o varies from 0.35 up to 4 BeV when ρ goes from 10^{15} g/cm^3 up to 100×10^{15} g/cm^3 . For case (1) V_o varies from 1.8 up to 4.20 BeV when ρ goes from 10^{15} g/cm^3 up to 40×10^{15} g/cm^3 . These results are in agreement with the generally accepted values for the strength of the nucleon - nucleon interaction (Canuto 1975).

We have shown in section (2) that when $\varepsilon \to 0$ the non-relativistic approach can be used. On the other hand, as ε becomes of the order of, or greater than 1, the relativistic treatment must be used. Of course, ε increases as ρ increases. For our conditions, ε runs from 0.5 up to 2.2 , meaning that ω assumes values ranging from 3.7×10^{23} rd/s to 7.7×10^{24} rd/s.

For $\rho \gtrsim 5.0 \times 10^{15}$ g/cm^3, we can put, approximately, $u_1 = 10^{13.25} \rho^{1.50}$ and $u_2 = 10^{15.45} \rho^{1.36}$. So, for these densities, the pressure defined by $P = \left[\rho^2 \frac{\partial}{\partial \rho} (u/\rho) \right]_{T=0}$ is given by $P_1 = 0.50\, u_1$ and $P_2 = 0.36\, u_2$, respectively. This means that the sound velocity c_s divided by the light velocity c is, in case (1), given by $(c_s/c)^2 \cong 0.50$ and, in case (2), by $(c_s/c)^2 \cong 0.36$.

Probably, somewhat better results can be obtained by taking into account the many body effects due to the coupling of modes but this is not the aim of the present work. The purpose of our calculation is to estimate the importance of relativistic effects for densities higher than 10^{15} g/cm^3 .

An important result of our simple model is that $(c_s/c)^2$ is of the order of 1/3 , rather than 1 (Canuto 1975). Therefore, no superluminous or ultrabaric effects appear in this model.

REFERENCES

BANERJEE, B., CHITRE, S.M. and GARDE, V.K. 1970. Phys. Rev. Letters 25, 1125.

BOWERS, R.L., CAMPBELL, J.A. and ZIMMERMAN, R.L. 1973. Phys. Rev. D7, 2278.

CANUTO, V. and CHITRE, S.M. 1974. Phys. Rev. D9, 1587.

CANUTO, V. 1975. LXV Enrico Fermi International School of Physics, Varenna, Italy.

CAZZOLA, P., LUCARONI, L. and SCARINCI, C. 1966. Nuovo Cimento 43, 250.

KLEIN, O. 1929. Zeits. f. Physik 53, 157.

MESSIAH, A. 1966. Quantum Mechanics Vol.II. John Wuley & Sons, Inc., New York.

NIKOLSKY, K. 1930. Zeits. f. Physik 62, 677.

OTSUKI, S., TAMAGAKI, R. and WADA, M. 1964. Progr. Theor. Phys. 32, 220.

PANDHARIPANDE, V.R. 1973. Nucl. Phys. A217, 1.

PLESSET, M.S. 1932. Phys. Rev. 41, 278.

TAKEMORI, M.T. and GUYER, R.A. 1975. Phys. Rev. D11, 2697.

TOMONAGA, S. 1968. Quantum Mechanics Vol. II. North-Holland Publishing Company, Amsterdam.

WALECKA, J.D. 1974. Ann. of Phys. 83, 491.

ZELDOVICH, Ya.B. 1962. JETP 14, 1143.

FIGURE CAPTIONS

Fig. 1 - $R_a(\xi)/\xi$ as a function of ξ and ε, for $\chi = +\ell = 1$ and $\eta = 5/2$.

Fig. 2 - $R_b(\xi)/\xi$ as a function of ξ and ε, for $\chi = +\ell = 1$ and $\eta = 5/2$.

Fig. 3 - $R_a(\xi)/\xi$ as a function of ξ and ε, for $\chi = -(\ell+1) = -1$ and $\eta = 3/2$.

Fig. 4 - $R_b(\xi)/\xi$ as a function of ξ and ε, for $\chi = -(\ell+1) = -1$ and $\eta = 3/2$.

Fig. 5 - The energy density u as a function of the density ρ. The computed values are indicated by: Banerjee et al. (— · —); Canuto and Chitre (—— —— ——); our u_1 (———) and our u_2 (· · ·).

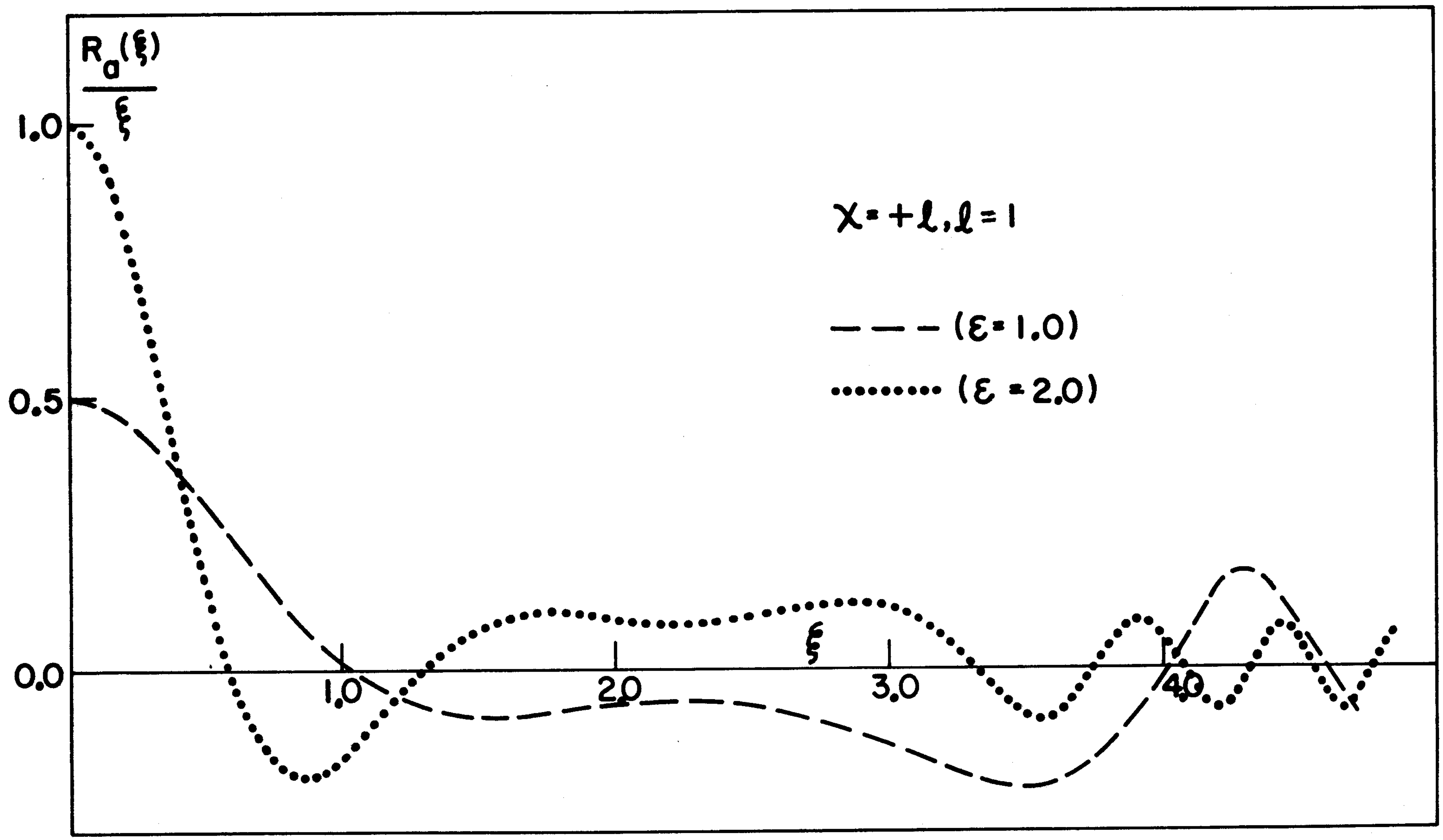


fig. 1

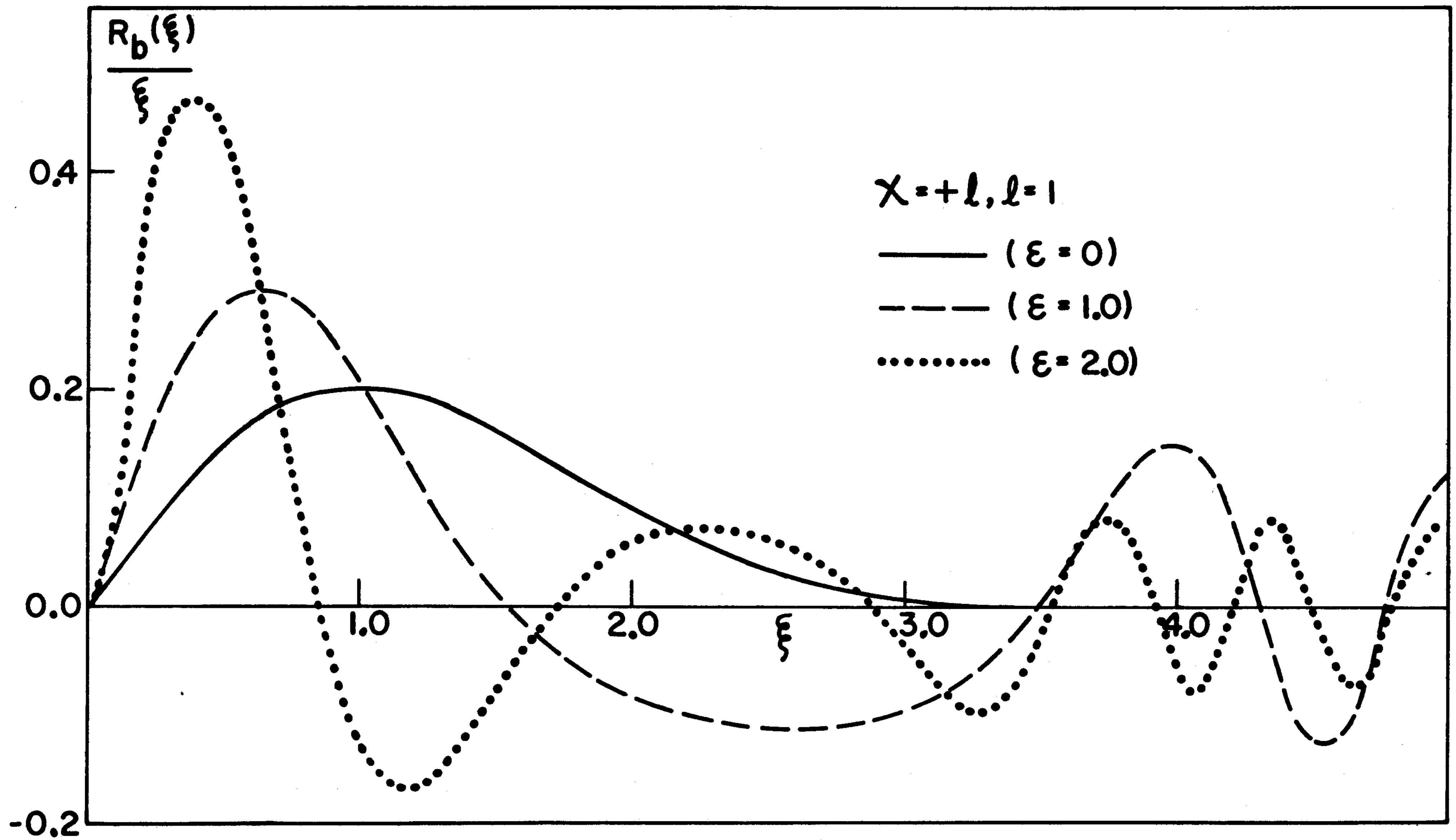


fig. 2

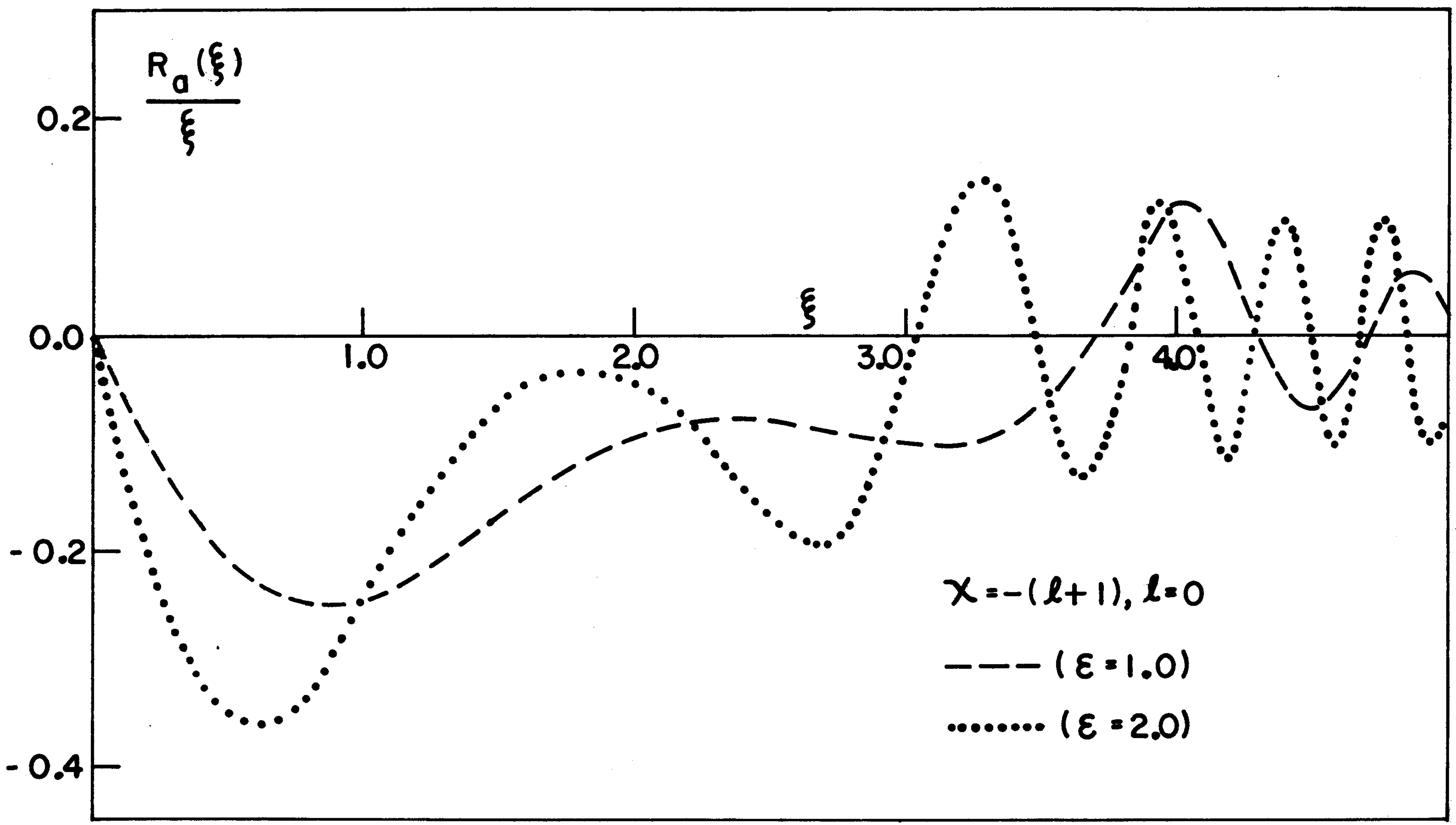
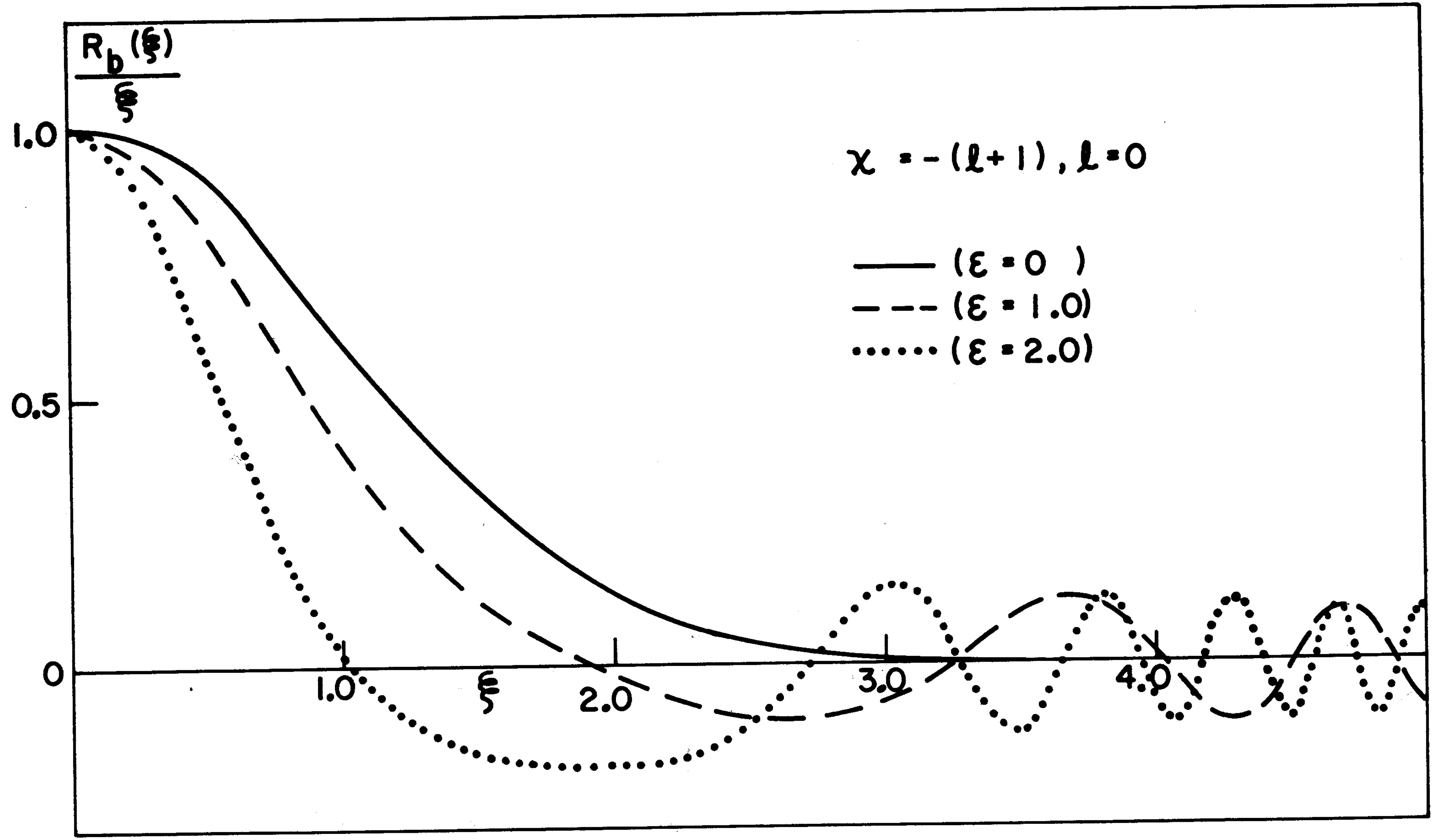


fig. 3



$R_b(\xi)/\xi$
1.0
0.5
0
1.0
ξ
2.0
3.0
4.0
$\chi = -(\ell+1), \ell=0$
$(\varepsilon = 0)$
$(\varepsilon = 1.0)$
$(\varepsilon = 2.0)$

fig. 4

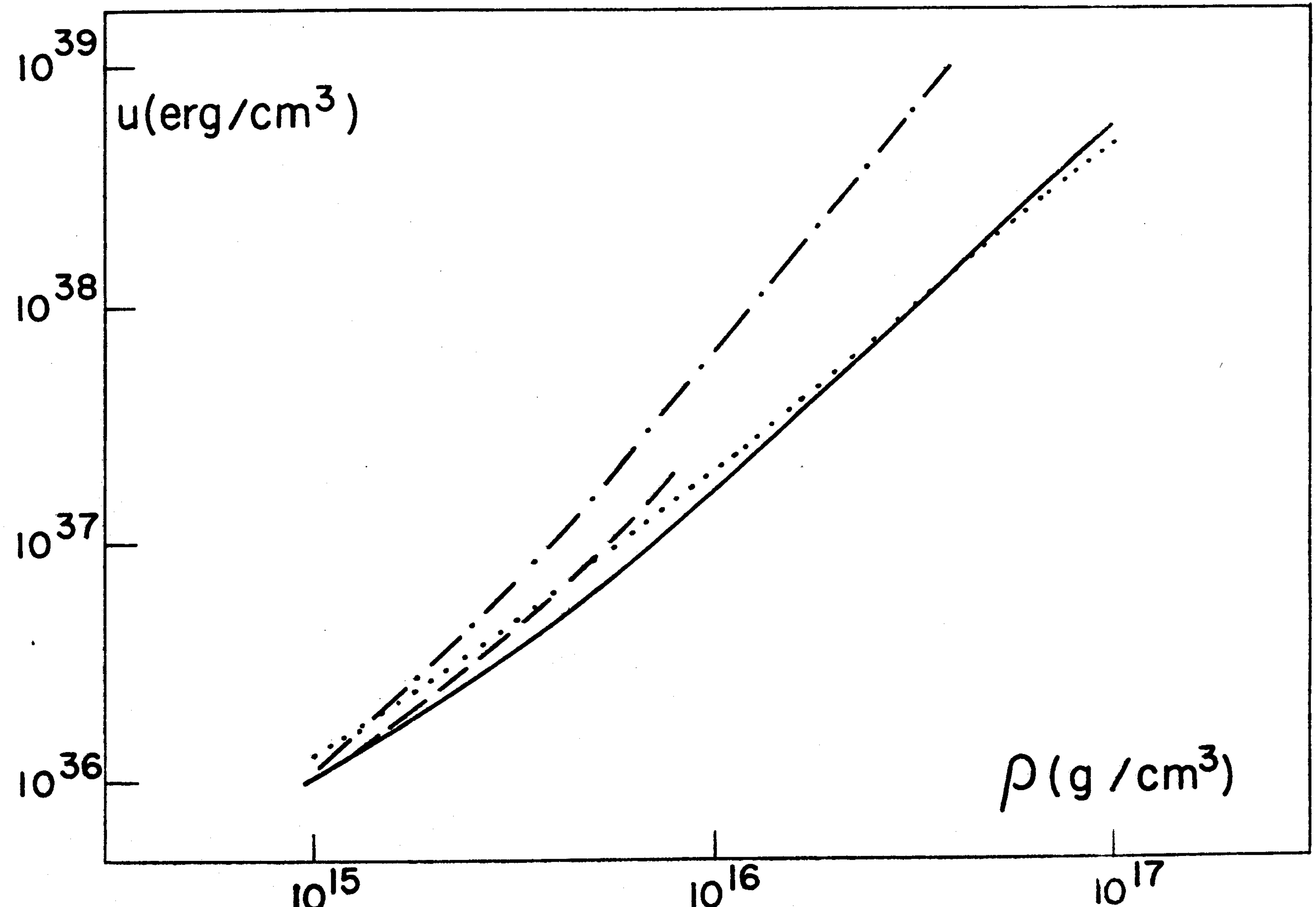


fig. 5