17396 96

IFUSP/P-53

BJÖRKEN LIMIT AND RENORMALIZATION GROUP EQUATIONS IN A CLASS OF GHOST-FREE GAUGE THEORIES

JOSIF FRENKEL

Instituto de Física
Universidade de São Paulo

June 1975

ABSTRACT

We consider the quantization of massless Yang-Mills theory in a class of generalized "axial" gauges, which do not lead to the appearance of ghosts. Then, we derive in the Björken limit an exact expression for the renormalized propagator. We argue that the consistency of this limit with the renormalization group equations implies that the β function has only one ultraviolet stable zero, which is located at the origin.

1. INTRODUCTION

In recent years, the renormalization group equations have played an increasingly important role in the study of the asymptotic behaviour of renormalizable field theories. This approach has acquired new importance due to the fact that non-abelian gauge theories are asymptotically free (1). In particular it was realized that an asymptotically free theory will exhibit Bjorken scaling. On the other hand, scaling implies, in fact, the same asymptotic behaviour of the current amplitudes as does the Bjorken limit (2). Since the coefficients β, γ, etc. of the renormalization group equations have only been computed to very low orders in perturbation theory around the origin, it has been suggested (3) that the consistency of these two approaches could place strong constraints on β and γ for domains of values of the coupling constant g, away from the origin.

In order to explain the reason for choosing a class of ghost-free non-abelian theories, consider the exact two point functions :

$$D^{ab}_{\mu\nu}(x) = \frac{1}{i}\int d^4x \; e^{iqx} \langle 0| T \, A^a_\mu(x) A^b_\nu(0)|0\rangle \qquad (1)$$

where A^a_μ is a vector boson field, μ, ν are Lorentz indices, and a, b are isotopic indices. As Bjorken has shown (2), the amplitude (1), in the limit $|q_0|\to\infty$, is determined by the following equal time commutation relations (ETCR) :

$$\lim_{\substack{|q_0|\to\infty \\ \vec{q}\text{ fixed}}} D^{ab}_{\mu\nu}(x) = \int d^3x \; e^{i\vec{q}\cdot\vec{x}} \Big\{ \frac{1}{q_0}\langle 0|[A^a_\mu(\vec{x},0), A^b_\nu(0)]|0\rangle + - \frac{1}{q_0^2}\langle 0|[\dot{A}^a_\mu(\vec{x},0), A^b_\nu(0)]|0\rangle \Big\} \qquad (2)$$

where we use the metric $q^2 = \vec{q}^{\,2} - q_0^2$. Since we want to know the exact form of the amplitude in this limit, we will need, to all orders in g, the above commutators. Now, the important observation is that, in a non-abelian gauge theory, $\dot{A}^a_\mu$ is not canonical to A^a_μ, the difference involving, among other things, the field A^a_0. Since A^a_0 is not a dynamical field, it must be eliminated by solving the equations of motion in some gauge.

Unfortunately, as shown in reference (3), in the Coulomb gauge it is not possible, in practice, to solve these equations explicitly to all orders in g. So, in order to derive the consistency conditions, one has to make certain assumptions concerning, for example, the ultraviolet behaviour of the wave function renormalization constant Z_3, etc. [(3)]. Due to this fact, we shall consider a set of gauges, which afford an explicit solution of the constraint equations. Such a gauge is $A_3^a = 0$, which was first considered by Arnowitt and Fickler [(4)]. Note that this gauge is not rotationally invariant.

In section II, we will consider the quantization of the Yang-Mills fields in a class of gauges, which exhibits 3-space rotational invariance and is characterized by :

$$n_i A_i^a = 0 \tag{3}$$

$\vec{n}$ being an arbitrary 3-vector. (In what follows, unless otherwise stated, sum over repeated indices is to be understood). We will show, by generalizing the arguments given in reference (4), that in these gauges it is possible to solve exactly the constraint equations.

In section III, we exhibit the exact expression for the renormalized amplitude $D_{ij,r}^{ab}$ in the limit $|q_0| \to \infty$. In order to do this, use is being made of the Ward-Taylor identities [(5)] in the absence of ghosts, which hold for our class of gauge conditions (3). We next derive a sum rule for the divergent part of the wave function renormalization constant Z_3 , where we exhibit explicitly the contribution of the one-particle intermediate state.

In section IV, we discuss the renormalization group equations, which, due to the Ward identities, acquire a particularly simple form in our class of gauges. Using these equations we derive a new expression for Z_3. We next compare the expressions for the renormalized wave function Z_3 derived in these two ways. The consistency conditions which result then, imply that the β and γ functions can only have one ultraviolet stable zero, which is located at the origin.

Since in this work we make use of the Bjorken limit and canonical ETCR, let us finally make the following remark. In the appendix, we compare in the lowest order of perturbation theory the ETCR defined via the Bjorken limit with the canonical ETCR. We find that these do not coincide, in general, a fact that is already well known (6). These anomalies are due to the singularities of the perturbation theory, and lead to the conclusion that scaling is violated and that the current algebra sum rules are divergent. On the other hand, experimental data suggest that the Bjorken limit is, indeed, correct and that the current algebra sum rules are finite. Since the non-abelian theory we are discussing implies scaling and scaling implies the same asymptotic behaviour as the Bjorken limit, we feel justified in using this limit together with the ETCR for the full theory (7).

II. QUANTIZATION IN THE GAUGE $n_i A_i^a = 0$

Consider a Yang-Mills theory (8) described by the first order Lagrangian :

$$L = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c) F_{\mu\nu}^a \tag{4}$$

Here, the fields A_μ^a and $F_{\mu\nu}^a$ are treated as independent variables, f_{abc} being the antisymmetric structure constants of the gauge group G. As is well known, this Lagrangian is invariant under the following transformations, characterized by the (infinitesimal) parameters ω^a:

$$F_{\mu\nu}^a \to F_{\mu\nu}^a + f_{abc}\, \omega^b F_{\mu\nu}^c \tag{5}$$

$$A_\mu^a \to A_\mu^a - \frac{1}{g}\partial_\mu \omega^a + f_{abc}\, \omega^b A_\mu^c$$

This invariance implies that the following identity is valid for arbitrary functions A_μ^a and $F_{\mu\nu}^a$:

$$D_\mu^{ab} D_\nu^{bc} F_{\mu\nu}^c = \tag{6a}$$

$$= \frac{1}{2} f_{abc} F_{\mu\nu}^b \left[(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f_{cde} A_\mu^d A_\nu^e) - F_{\mu\nu}^c \right]$$

where we have defined : $D_\mu^{ab} \equiv \delta_{ab}\partial_\mu + g f_{abc} A_\mu^c$ (6b)

Due to the fact that the above Lagrangian is invariant, in order to quantize the theory it is necessary to choose a gauge. Consider the following set of gauges :

$$n_\mu A_\mu^a = 0 \tag{7}$$

where n is a 4-vector, which will be discussed later. For the purpose of canonical quatization, we consider this relation as a constraint, excluding one of the fields A. This exclusion must be made, in general, before finding the field equations. However, in our class of gauges, we can show that this exclusion can be made even after finding the field equations. To see this, let us introduce the gauge condition (7) with the help of the Lagrange multiplier $\mathcal{B}^a$, which must be treated as a new dynamical field. Consider then the new Lagrangian :

$$L' = L + \mathcal{B}^a n_\mu A_\mu^a \tag{8}$$

We obtain the following Euler-Lagrange equations :

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c \tag{9a}$$

$$D_\mu^{ab} F_{\mu\nu}^b + n_\nu \mathcal{B}^a = 0 \tag{9b}$$

$$n_\mu A_\mu^a = 0 \tag{9c}$$

Applying the operator D on equation (9b), using (9a) and (9c), we obtain, with the help of (6a), the equation :

$$(\partial . n)\, \mathcal{B}^a = 0 \tag{10}$$

This relation, which decouples $\mathcal{B}^a$ from the vector-boson fields, shows that $\mathcal{B}^a$ is independent of $X = x.n/|n|$, for arbitrary n, and indicates that it might be a constant. In fact, using arguments similar to those discussed in reference (4), we can show that the consistency of equations (9) requires $\mathcal{B}^a$ to be zero. So they become identical to the equations of motion, which result from the Lagrangian (4), to which we add the gauge condition (7). (For the case $A_3^a = 0$, see also reference (9).)

From equation (4), we see that the canonical momenta are given by :

$$E_i^a \equiv \frac{\delta L}{\delta \dot{A}_i^a} = F_{i0}^a \tag{11}$$

With this definition we obtain from the set of equations (9) the equations of motion :

$$\partial_0 A_i^a = \partial_i A_0^a - E_i^a + g f_{abc} A_i^b A_0^c \tag{12a}$$

$$\partial_0 E_i^a = \partial_j F_{ij}^a + g f_{abc} A_j^b F_{ji}^c + g f_{abc} A_0^b E_i^c \tag{12b}$$

and the constraint equations :

$$\partial_i E_i^a + g f_{abc} A_i^b E_i^c = 0 \tag{13a}$$

$$F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g f_{abc} A_i^b A_j^c \tag{13b}$$

Consider now the equation of motion (12a). By multiplying it by n_i, using the condition (9c), we find :

$$(n.\partial\, \delta_{ac} + n_0 g f_{abc} A_0^b)\, A_0^c = n_i E_i^a \tag{14a}$$

The field A_0^c is not a dynamical variable and must be eliminated by solving equation (14a). (We will show later how to compute the right-hand side of (14a).) For the reasons discussed earlier in section I, we want an explicit expression for A_0, to all orders in g. For this reason, we will choose n_μ to be a space-like 4-vector, which, in the Lorentz frames where the quantization will be carried out, is given by $(\vec{n}, 0)$. In this case, the gauge condition becomes the one expressed in equation (3). Then, (14a) reduces to :

$$n_i \partial_i A_0^a = n_i E_i^a \tag{14b}$$

At this point, let us introduce a notation which will be used frequently. Any 3-vector can be decomposed in transversal and longitudinal components with respect to $\vec{n}$:

$$V_i \equiv V_i^T + V_i^L = V_i^T + V_j \frac{n_j}{|\vec{n}|} \frac{n_i}{|\vec{n}|} = V_i^T + V_L \frac{n_i}{|\vec{n}|} \tag{15a}$$

with $$V_j^T n_j = 0 \tag{15b}$$

(Note that the gauge condition (3) states that the fields A_i^a are purely transversal : $A_L^a = 0$ so that $A_i^a \equiv A_i^{Ta}$. This is not a Lorentz invariant gauge condition, althought it is invariant by a rotation in the 3-dimensional space.)

Then the solution to equation (14b) will be :

$$A_0^a(\vec{x}_T, x_L) = \int_{-\infty}^{x_L} dx'_L \; E_L^a(\vec{x}_T, x'_L) \tag{16}$$

To find E_L consider equation (13b) which can be written as :

$$\partial_L E_L^a = -(\partial_i^T E_i^{Ta} + g f_{abc} A_i^b E_i^{Tc}) \equiv D_i^{Tba} E_i^{Tb} \tag{17a}$$

whose solution is given by :

$$E_L^a(\vec{x}_T, x_L) = -\int_{-\infty}^{x_L} dx'_L \; D_i^{Tba} E_i^{Tb}(\vec{x}_T, x'_L) \tag{17b}$$

We will now impose the following canonical ETCR which are consistent with the gauge condition (3) :

$$[A_i^{Ta}(\vec{x},t), A_j^{Tb}(\vec{x}',t)] = [E_i^{Ta}(\vec{x},t), E_j^{Tb}(\vec{x}',t)] = 0 \tag{18a}$$

$$[A_i^{Ta}(\vec{x},t), E_j^{Tb}(\vec{x}',t)] = i\,\delta_{ab}\delta_{ij}^T\,\delta^3(\vec{x}-\vec{x}') \tag{18b}$$

with $\delta_{ij}^T \equiv \delta_{ij} - \frac{n_i n_j}{|\vec{n}|^2}$

We must finally examine the Heisenberg equations of motion for the independent canonical variables A_j^{Ta} and E_j^{Ta}:

$$i\partial_0 A_j^{Ta} = [A_j^{Ta}, H] \tag{19a}$$

$$i\partial_0 E_j^{Ta} = [E_j^{Ta}, H] \tag{19b}$$

In our theory the Hamiltonian is positive definite and takes the form :

$$H = E_i^{Ta}\dot{A}_i^{Ta} - L = \tag{20}$$

$$= \int d^3x \left[\tfrac{1}{2} E_i^{Ta}E_i^{Ta} + \tfrac{1}{2} E^{La}E^{La} + \tfrac{1}{4} F_{ij}^a F_{ij}^a\right]$$

where E^{La} and F_{ij}^a are to be expressed in terms of the canonical variables A_i^{Ta} and E_i^{Ta} by equations (13b) and (17b). Using (20), together with the canonical ETCR (18), after a straightforward calculation of the right-hand side of (19), we obtain that the set of equations (19a) and (19b) are identical to the equation of motion (12a) and the transversal part of the equation of motion (12b), respectively.

III. SUM RULE FOR Z_3^{div}

Due to the ETCR (18a), the first commutator appearing in the amplitude D_{ij}^{ab} considered in the Bjorken limit, equation (2), vanishes. In order to calculate the second commutator, we observe that, using (12a) together with (15), we have :

$$[\dot{A}_i^a(\vec{x},0), A_j^b(0)] = [-E_i^{Ta}(\vec{x}) - \frac{n_i}{|\vec{x}|}E^{La}(\vec{x}) + D_i^{ca}A_0^c(\vec{x}), A_j^b(0)] \quad (21)$$

With the help of relations (16) and (17b), using the canonical ETCR (18), we obtain :

$$i\langle 0|[\dot{A}_i^a(\vec{x},0), A_j^b(0)]|0\rangle = \delta_{ij}^T\ \delta^3(\vec{x}) + \quad (22)$$

$$+\Theta(x_L)\,x_L\, D_i^{Tad}\, D_j^{Tdb}\ \delta^2(\vec{x}_T)$$

where Θ is the usual step-function. This relation represents an exact result, valid for all values of the coupling constant g. Substituting (22) in (2), we find that, in the Bjorken limit, the amplitude D_{ij}^{ab} can be written as :

$$\lim_{|q_0|\to\infty} D_{ij}^{ab} = \frac{-1}{q_0^2}\int d^3x\ e^{i\vec{q}\cdot\vec{x}}\langle 0|[\dot{A}_i^a(\vec{x},0), A_j^b(0)]|0\rangle \equiv \frac{i}{q_0^2}D^{ab,B} \quad (23)$$

where

$$D_{ij}^{ab,B} = \delta_{ab}\Big\{\delta_{ij} - \frac{n_iq_j + n_jq_i}{(\vec{q}\cdot\vec{n})} + \frac{q_iq_j|\vec{n}|^2}{(\vec{q}\cdot\vec{n})^2} + \quad (24)$$

$$- Cg^2\int d^3x\ e^{i\vec{q}\cdot\vec{x}}\,\Theta(x_L)\,x_L\,\delta^2(\vec{x}_T)\langle 0|A_i^c(\vec{x},0)A_j^c(0)|0\rangle\Big\}$$

Note that the entire q_0 dependence is in the $1/q^2$ part of the propagator, which we have explicitly factored out. The positive constant C depends on the gauge group and is given by :

$$C = \frac{1}{N}\sum_{bc} f_{abc}\, f_{abc} \quad (24a)$$

where N is the number of generators of the group.

Until now, all the parameters and fields considered were unrenormalized quantities. In order to obtain the expression for the renormalized propagator, we recall that the Ward identities[5] in the absence of ghosts imply that the Green functions can be rendered finite by the scale transformation :

$$A_\nu^a = Z_3^{1/2} A_{\nu r}^a \qquad\qquad g = Z_3^{-1/2} g_r \tag{25}$$

Furthermore, it is necessary to rescale the gauge parameters n_i in such a way that the gauge condition (3) is invariant. This requires :

$$n_i = Z_3^{-1/2} n_{ir} \tag{26}$$

We remark that all renormalized quantities will depend on an arbitrary parameter μ , with dimension of mass, who sets the scale and which is necessary in order to perform the substractions which render the theory finite.

With (25) and (26), we obtain for the renormalized propagator in the Bjorken limit, the expression :

$$\lim_{|q_0|\to\infty} D_{ij,r}^{ab} = \frac{-1}{q_0^2}\int d^3x\, e^{i\vec{q}\cdot\vec{x}} \langle 0|[\dot{A}_{ir}^a(\vec{x},0), A_{jr}^b(0)]|0\rangle \equiv \frac{i}{q_0^2} D_{ij,r}^{ab,B} \tag{27}$$

where

$$D_{ij,r}^{ab,B} = \delta_{ab}\, Z_3^{-1}\left\{\delta_{ij} - \frac{n_{ir}q_j + n_{jr}q_i}{(\vec{q}\cdot\vec{n}_r)} + \frac{q_i q_j |\vec{n}_r|^2}{(\vec{q}\cdot\vec{n}_r)^2} + D_{ij}\right\} \tag{28a}$$

with

$$D_{ij} = -C g_r^2 \int d^3x\, e^{i\vec{q}\cdot\vec{x}}\, \theta(x_L)\, x_L\, \delta^2(\vec{x}_T) \langle 0| A_{ir}^c(\vec{x},0) A_{jr}^c(0)|0\rangle \tag{28b}$$

We remark that, since the renormalized function $D_{ij,r}^{ab,B}$ is finite, the above equation establishes a connection between the divergent part of the wave function renormalization constant Z_3 and the (exact) expression D_{ij} . In order to study this relationship, we observe that, due to the gauge condition (3), D_{ij} must have the following form :

$$D_{ij} = \left[\delta_{ij} - \frac{n_{ir}q_j + n_{jr}q_i}{(\vec{q}\cdot\vec{n}_r)} + \frac{q_i q_j |n_r|^2}{(\vec{q}\cdot\vec{n}_r)^2}\right] A + \frac{q_i^T q_j^T}{q_L^2} B \tag{29}$$

where A and B are dimensionless form factors, which are determined from equation (28b). So we can write (28a) as follows :

$$\begin{aligned} D_{ij,r}^{ab,B} = {} & \delta_{ab}\, Z_3^{-1}(1+A)\times \\ & \times\left\{\delta_{ij} - \frac{n_{ir}q_j + n_{jr}q_i}{(\vec{q}\cdot\vec{n}_r)} + \frac{q_i q_j |\vec{n}_r|^2}{(\vec{q}\cdot\vec{n}_r)^2} + \frac{q_i^T q_j^T}{q_L^2}\frac{B}{1+A}\right\} \end{aligned} \tag{30}$$

As shown in the appendix, in consequence of the Ward identities, the functions A and B are related in such a way that $B/1+A$ is finite. Furthermore, we have :

$$Z_3^{-1}(1+A) = \frac{1}{1+T_B} \tag{31}$$

where T_B is a dimensionless finite form factor. We fix Z_3 by the normalization condition :

$$T_B(g_r^2, |\vec{q}| = \mu, |q_L| = \mu) = 0 \tag{32}$$

which means that, at this point, the exact propagator (27) becomes equal to the free one. (Note that, given $\vec{q}$ and q_L, $\vec{q}^T$ is fixed.) We point out that the normalization condition (32) is a convenient one, but is not essential for our discussion.

We are actually interested in the (logarithmically) divergent part of Z_3. Since it is momentum-independent, we can evaluate it by considering equation (31) at the normalization point given by (32). Using equations (29) and (28b), we obtain :

$$Z_3^{div} = -\frac{1}{2} C g_r^2 \int_0^\infty dx_L\, x_L\, e^{iq_L x_L} \langle 0 | A^c_{\ell r}(\vec{x}_T = \vec{0}, x_L)\, A^c_{\ell r}(0) | 0 \rangle^{div}_{|q_L| = \mu} \tag{33}$$

where $A^c_{\ell r} \equiv A^{cT}_{\ell r}$ with $\ell = 1,2,3$ are independent renormalized fields. In order to get a sum rule for Z_3^{div}, we will insert a complete set of intermediate states. Then, we find :

$$Z_3^{div} = \frac{1}{2} C g_r^2 \sum_k \frac{1}{(\mu + k_L)^2} |\langle 0 | A^c_{\ell r}(0) | k \rangle|^2 \tag{34}$$

Remark that in our theory there are no Faddeev-Popov ghosts (9), so that only vector bosons will contribute to the sum above. It will be convenient to separate out explicitly the contribution of the one-particle state, which is normalized, as usual, in the same way as the free field. Finally, we obtain the following sum rule for the divergent part of the wave function renormalization constant Z_3:

$$Z_3^{div}\left(\frac{\Lambda^2}{\mu^2}, g_r^2\right) = \frac{1}{2} C g_r^2 \left\{ \frac{N}{6\pi^2} \log \frac{\Lambda^2}{\mu^2} + \sum_k{}' \frac{1}{(\mu + k_L)^2} |\langle 0 | A^c_{\ell r}(0) | k \rangle|^2 \right\} \tag{35}$$

Here Λ is a 3-vector ultraviolet cutoff and $\sum'$ represents the divergent contribution of the multi-particle intermediate states.

IV. DISCUSSION

We will now derive, using the renormalization group equations, a new expression for Z_3. These equations are most simply derived by noting that the unrenormalized Green functions, when expressed as functions of the cutoff, the unrenormalized coupling constant and the gauge parameters, are independent of μ [1]. We obtain two equations :

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g_r} + \alpha_i\frac{\partial}{\partial n_{ir}} + 2\gamma\right)\mathcal{D}^{ab,B}_{ij,r} = 0 \tag{36}$$

and

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g_r} + \alpha_i\frac{\partial}{\partial n_{ir}} - 2\gamma\right)Z_3 = 0 \tag{37}$$

where

$$\beta = \mu\frac{\partial g_r}{\partial\mu} \quad ; \quad \alpha_i = \mu\frac{\partial n_{ir}}{\partial\mu} \quad ; \quad \gamma = \mu\frac{\partial}{\partial\mu}\log Z_3^{1/2} \tag{38}$$

The solution of equation (36) is given by [10] :

$$\mathcal{D}^{ab,B}_{ij,r}\left(\lambda^2\frac{|\vec{q}|^2}{\mu^2}, \lambda^2\frac{q_L^2}{\mu^2}, g_r, n_{ir}\right) = \tag{39}$$

$$= \mathcal{D}^{ab,B}_{ij,r}\left(\frac{|\vec{q}|^2}{\mu^2}, \frac{q_L^2}{\mu^2}, \bar{g}_r, \bar{n}_i\right)\exp 2\int_0^{\tau}\gamma[\bar{g}(t), \bar{n}_i(t)]\,dt$$

where $\tau = \frac{1}{2}\log\lambda^2$ and $\bar{g}[g_r, n_{ir}, \tau]$ and $\bar{n}_i[g, n_{ir}, \tau]$ satisfy respectively the equations :

$$\frac{d\bar{g}}{d\tau} = \beta[\bar{g}, \bar{n}_i] \qquad\qquad \frac{d\bar{n}_i}{d\tau} = \alpha_i[\bar{g}, \bar{n}_i] \tag{40}$$

with the boundary conditions $\bar{g}[\tau=0] = g_r$; $\bar{n}_i[\tau=0] = n_{ir}$.
Letting $\lambda^2 = \Lambda^2/\mu^2$ we have :

$$Z_3(\lambda^2, g_r, n_{ir}) = Z_3(1, \bar{g}, \bar{n}_i)\exp -2\int_0^{\tau}\gamma[\bar{g}(t), \bar{n}_i(t)]\,dt \tag{41}$$

We can simplify this expression using relations (25) and (26), which are a consequence of the Ward identities. In this case we get :

$$\beta = g_r\gamma \qquad\qquad \alpha_i = n_{ir}\gamma \tag{42}$$

so that (40) can be written as :

$$\frac{d}{d\tau} \log \bar{g} = \frac{d}{d\tau} \log \bar{n}_i = \gamma\,[\bar{g}, \bar{n}_i] \tag{43}$$

Then we obtain for Z_3 the expression :

$$Z_3(\lambda^2, g_r, n_{ir}) = Z_3(1, \bar{g}, \bar{n}_i) \left[\frac{\bar{g}(0)}{\bar{g}(\tau)} \right]^2 \tag{44}$$

Now, let us compare expressions (35) and (44). As we have seen, the contribution of the one-particle state to Z_3^{div}, as given by equation (35), is a positive logarithmically divergent quantity. Furthermore, we observe that each intermediate state in $\sum'$ will give a non-negative contribution to Z_3^{div}. (In fact, we have verified that, to order g_r^2, the contribution to $\sum'$ resulting from two-particle intermediate states is positive definite.) So, we can conclude that the wave function renormalization constant Z_3 is a positive divergent quantity, for all values of the coupling constant.

On the other hand, we know that the β function has a negative slope at the origin, where it has a zero [1]. Therefore, if $\bar{g}(0)$ is in the domain of attraction of this zero, then $\lim_{\tau \to \infty} \bar{g}(\tau) = 0$, so that Z_3 as given by (44) diverges, in accordance with the previous discussion.

Let us now assume that there exists a second zero at g'. (If it does not, then, clearly, another ultraviolet stable zero cannot exist.) In this case, it will be reached with a positive slope, so that g' is an ultraviolet unstable point. Now we wish to argue that the β function (and therefore also the γ function) cannot have a zero elsewhere. Indeed, suppose it has a third zero at g'', which will have the desired negative slope. Consider now the region where $\bar{g}(0)$ is in the domain of attraction of this zero. Then, since g'' is a stagnation point, as $\tau \to \infty$, we will have :

$$\lim_{\tau \to \infty} \bar{g}(\tau) = g'' \tag{45a}$$

and using (43) :

$$\lim_{\tau \to \infty} \bar{n}_i(\tau) = \frac{g''}{g_r} n_i \equiv n_i'' \tag{45b}$$

so that :

$$\lim_{\zeta \to \infty} Z_3(1, \bar{g}, \bar{n}_i) \to Z_3(1, g'', n_i'') < \infty \tag{45c}$$

From equation (44) it follows that Z_3 will be finite as $\lambda^2 = \Lambda^2/\mu^2 \to \infty$. But this is in contradiction with the conclusion reached above, which states that Z_3 is a divergent function in the whole domain of variation of the coupling g_r.

Therefore, we conclude that the consistency of these two approaches implies, for our class of gauge conditions, that the β and γ functions can have only one ultraviolet stable point which is located at the origin.

We have not yet studied the problem represented by the introduction of fermions in the above theory of massless self-coupling Yang-Mills fields. This is an interesting case which deserves further consideration.

It is a pleasure to thank M.L.Frenkel for many useful discussions.

APPENDIX

Let us determine the most general form for the renormalized propagator $D^{ab}_{\mu\nu,r}$, which is consistent with the Ward identities, in a general class of gauges described by equation (7). It can be shown [5], that these gauge conditions can be generated by the gauge fixing Lagrangian :

$$L_G = -\frac{1}{2\beta}\left(n_\mu A^a_\mu\right)^2 \tag{A1}$$

in the limit $\beta \to 0$. (For notational simplicity, we drop the subscript r, although we are now considering only renormalized quantities.) Using (4) and (A1), we obtain that the renormalized two-point vertex is given by :

$$\Gamma^{ab}_{\mu\nu} = i\,\delta_{ab}\left(\delta_{\mu\nu}q^2 - q_\mu q_\nu - \frac{n_\mu n_\nu}{\beta} + \tilde{\Gamma}_{\mu\nu}\right) \tag{A2}$$

where, as a consequence of the Ward identities [5], the non-trivial part $\tilde{\Gamma}_{\mu\nu}$ must be transversal to the external momentum q. The most general form for $\tilde{\Gamma}_{\mu\nu}$ consistent with this requirement is :

$$\tilde{\Gamma}_{\mu\nu} = \left(\delta_{\mu\nu}q^2 - q_\mu q_\nu\right)\Gamma + \left[q_\mu q_\nu + \frac{q^2}{q.n}\left(\frac{q^2}{q.n}n_\mu n_\nu - n_\nu q_\mu - n_\mu q_\nu\right)\right]\gamma \tag{A3}$$

where Γ and γ are dimensionless finite form factors. Note that the counterterms generated by the rescalings (25) and (26) have the form : $(\delta_{\mu\nu}q^2 - q_\mu q_\nu)Z_3$, so that we can always fix Z_3 by a convenient normalization of the Γ function.

Since the renormalized propagator is essentially the inverse of $\Gamma^{ab}_{\mu\nu}$, using (A2) and (A3), we obtain that $D^{ab}_{\mu\nu,r}$, at the limit $\beta \to 0$, is given by :

$$D^{ab}_{\mu\nu,r} = -\frac{i}{q^2}\delta_{ab}\frac{1}{1+\Gamma}\Big\{\delta_{\mu\nu} - \frac{1}{q.n}\left(n_\mu q_\nu + n_\nu q_\mu\right) + \frac{n^2}{(q.n)^2}q_\mu q_\nu + \tag{A4}$$

$$+\left[\frac{n^2}{(q.n)^2}q_\mu q_\nu - \frac{1}{q.n}\left(n_\mu q_\nu + n_\nu q_\mu\right) + \frac{n_\mu n_\nu}{n^2}\right]F\Big\}$$

with
$$F = \frac{\gamma}{(q.n)^2/q^2n^2 + \gamma/1+\Gamma - \cancel{\ }} \tag{A5}$$

In particular, for the class of gauge conditions given by equation (3), i.e., $n_0 = 0$, we obtain, using the definitions (15), that :

$$D^{ab}_{ij,r} = \frac{-i}{q^2}\delta_{ab}\frac{1}{1+T}\left\{\delta_{ij} - \frac{n_i q_j + n_j q_i}{\vec{q}\cdot\vec{n}} + \frac{q_i q_j \vec{n}^2}{(\vec{q}\cdot\vec{n})^2} + \frac{q_i^T q_j^T}{q_L^2}F\right\} \qquad (A6)$$

Let us call T_B and F_B, respectively, the values in the Bjorken limit of the finite factors T and F. Identifying this equation with (27) and (30), we obtain :

$$Z_3^{-1}(1+A) = \frac{1}{1+T_B} \qquad \frac{B}{1+A} = F_B \qquad (A7)$$

We will now compare, in lowest order in perturbation theory, Z_3^{div} as given by equation (33) with Z_3^{div} obtained with the use of Feynman diagrams. We remark that the exact expression (33) was obtained in a non-perturbative way, using the Bjorken limit and ETCR (18). To order g^2, we obtain from this equation that :

$$Z_3^B = 1 + \frac{4G}{24\pi^2}g^2 \log\frac{\Lambda}{\mu} \qquad (A8)$$

where $G = \sum_{bc} f_{abc} f_{abc}$. Note that Z_3^B is positive definite and therefore gives rise to the correct negative slope of the β function around the origin.

On the other hand, we can also compute Z_3 from the vector-boson self energy graphs, shown in Figure 1.

Figure 1

The vertices are given by the Lagrangian (4) together with equation (9a), while the free propagator can be obtained from (A5). It is rather tedious to compute all contributions resulting from the graphs above. However, since we are interested only in Z_3^{div}, simplifications occur, due to the Ward identities. From (A3) we see that the divergent part must have the form :

$$M_{\mu\nu}^{ab} = \delta_{ab} \left(\delta_{\mu\nu} q^2 - q_\mu q_\nu \right) \mathcal{D} \qquad \text{(A9)}$$

as the form factor Υ must already be finite. We can therefore isolate the logarithmically divergent factor $\mathcal{D}$ by the operation :

$$O = \frac{1}{2} q_\alpha q_\beta \left(\partial^2 / \partial_\alpha \partial_\beta \right)_{q=0} \qquad \text{(A10)}$$

After a straightforward calculation, using techniques discussed in reference (11), we obtain :

$$Z_3 = 1 + \frac{11G}{24\pi^2} g^2 \log \frac{\Lambda}{\mu} \qquad \text{(A11)}$$

which is also positive definite, but differs from Z_3^B given by equation (A8). If one wished to reproduce (A11) via ETCR, it would be necessary to modify correspondingly the ETCR (18b), while the ETCR (18a) are left unchanged. However, we will not do so since, as discussed in section I, we believe that these anomalies are due to the peculiar features of perturbation theory. As we have not used perturbative arguments in this work, our conclusions are not affected.

REFERENCES

(1) - D.J.Gross and F.Wilczek, Phys. Rev. D8 (1973) 3633 ;
Phys. Rev. D9 (1974) 980.

(2) - J.D.Bjorken, Phys. Rev. 148 (1969) 1467 ;
Phys. Rev. 179 (1969) 1547.

(3) - A.Ali and J.Bernstein, Phys. Rev. D11 (1975) 336.
J.Bernstein, "Adler's Theorem in finite massless QED and possible extensions to non-abelian gauge theories", CERN preprint REF.TH.2010.

(4) - R.Arnowitt and S.Fickler, Phys. Rev. 127 (1962) 1821.

(5) - B.W.Lee, Phys. Rev. D9 (1974) 933.
G.Costa and M.Tonin, La Revista del Nuovo Cimento, vol 5, 1 (1975) 29.

(6) - R.Jackiw and G.preparata, Phys. Rev. Letters 22 (1969) 975.
S.Adler and W.Tung, Phys. Rev. Letters 22 (1969) 978.

(7) - D.J.Gross, "Weak and electromagnetic interactions", Princeton Series of Physics - Lectures on Current Algebra and its Applications, pp328, Princeton University Press, (1972).

(8) - C.N.Yang and R.L.Mills, Phys. Rev. 96 (1954) 191.

(9) - L.D.Faddeev and V.N.Popov, Phys. Letters 25B (1967) 29.
E.S.Fradkin and I.V.Tyutin, Phys. Rev. D2 (1970) 2841.
S.Coleman, Lectures at 1973 International Summer School of Physics Ettore Majorana, Erice, Harward Preprint, (1973).

(10) - S.Coleman, "Dilatations" in A.Zichichi (ed), Properties of Fundamental Interactions, Part A, pp359, Academic Press (1973).

(11) - W.Kainz, W.Kummer and M.Schweda, Nucl. Phys. B79 (1974) pp484.