IFUSP/P-61

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ABSTRACT

We perform the "first quantization" of the sine-Gordon Theory. We obtain the classical potential that describes the long distance interaction between solitons. The "first quantization" is achieved by inserting that potential in the Schrodinger Equation. The nonrelativistic region of the DHN spectrum is easily reproduced by using our procedure. We also compute scattering amplitudes for non relativistic soliton-antisoliton and soliton-soliton scattering. We improve Faddeev's quantization rule. That improved version leads, in the non relativistic region, to the scattering amplitudes obtained in our approach.

1. INTRODUCTION

Recently much attention has been drawn to semiclassical methods in Quantum Field Theory[1-6]. That is the case for the W.K.B. approach. In this context we would like to mention the pioneering work of Dashen, Hasslacher and Neveu[2] (DHN) which, by extending the W.K.B. method to Q.F.T., succeeded in getting many features of the quantized sine-Gordon[7] Theory (SGT). Other interesting characteristics of the quantized version of S.G.T. has been studyed by other authors[3-6].

Although the technique employed by DHN was an approximate one, there exists a region in the coupling constants space ($\lambda/m^2 << 1$) where we would expect the method to be reliable. We will refer to that region as the DHN region.

When applied to S.G.T. the approach described in ref. (2) might leads us directly to answer many questions concerning the spectrum of the theory and scattering of particles without going over the intermediate step of a " first quantization" . The " first quantization " or quantization a " la Schödinger " of the SGT is what we study in this paper. That is the kind of quantization which, as we recall, allows us to get sensible results concerning energy levels of many bound state systems - for instance, the positronium - by using the Schrödinger Equation.

First of all we show that asymptotically (see below what we mean by asymptotically) the classical interaction between a nonrelativistic soliton-antisoliton ($\bar{S}A$) pair in the S.G.T. is well described by a classical potential of the form

$$V_{SA}(x) = -g M_c e^{-m|x|} \qquad (1.1)$$

where $|X|$ is the distance between the soliton and the antisoliton; m is the mass of the elementary meson whereas M_c is the classical soliton mass; and g is a dimensionless constant of order of magnitude one.

When we said that the potential description works asymptotically we meant that expression (1.1) describes the classical interaction between SA pairs whenever

$$|X| >> 1/m \tag{1.2a}$$

or

$$|t| >> 1/m\,v_\infty \tag{1.2b}$$

v_∞ being the modulus of the interacting soliton velocity for early and late times ($t \to \mp \infty$). We say that we have the nonrelativistic domain of a given process when $v_\infty^i << 1$ for all particles involved (v_∞^i stands for the asymptotic velocity of the i-th particle).

It can be verified that the soliton-soliton (SS) potential $V_{SS}(X)$ is, in the Asymptotic Region, given by

$$V_{SS}(X) = - V_{SA}(X) \tag{1.3}$$

Such a property confirms what one would expect from an intuitive reasoning.

The classical "size" of the soliton is of order (1/m). Then, in the Asymptotic Region we can consider the soliton as a point like particle. We point out that a similar approximation is performed also in the treatment of the Hidrogen Atom by a Coulomb potential where we consider the proton as being point like.

We justify this by arguing that the proton radius is much smaller than the Bohr radius.

It is known(2) that in the DHN Region the meson of mass m has a twofold role: it is the fundamental meson of S.G.T. and the lower bound state of the SA system. We note that the potential given by (1-1) is a one-dimensional Yukawa potential associated with that meson. In this way we conclude that this particle becomes the one which mediates the interaction between solitons at large distances

Once we have established that the classical interaction between solitons is determined, in the Asymptotic Region, by potential (1.1) we proceed to the non relativistic quantization. That quantization amounts to inserting the potential into the Schrodinger Equation. Fortunately, for the potential (1.1) the Schrodinger Equation is soluble, and due to this it is straightforward to get the binding energies and scattering amplitudes of the SA and SS system.

Before proceeding we would like to discuss the condictions under which we shall expect the method employed here to be a reliable one. Concerning to the non relativistic approximation, we feel tempted to say that this approach is valid whenever

$$|E| << M \tag{1.4a}$$

$$|V(x)| << M \tag{1.4b}$$

Where E is the energy of the state under description. Condition (1.4b) seems to be too much strong. If it was always necessary we couldn't understand the succesfull non relativistic description of the Hidrogen Atom. That's why we will addopt a weaker and more pragmatic condition

$$|\langle V(x)\rangle| << M \tag{1.5}$$

Another aspect of our approach which we would like to comment refers to taking only the long range tail of the potential. As it is known, states corresponding to large wave lenghts are not sensible to the behavior of the potential at short distances.

Such wavelength can be obtained once we know the wave function of each state.

Then, by computing the wave function, the energy and $\langle V\rangle$ we will be able to check a posteriori the validity of the simplifications introduced in our scheme.

After elucidating these points we understand also in which region in the DHN spectrum we should look in order to compare with our results. That region will be the non relativistic limit of DHN region. As expected, we achieved, in this part of the spectrum, a perfect agreement.

With regard to the scattering region, our results differs from those obtained by Jackiw and Woo[3]. One reason for that discrepancy is that the approach used by them doesn't work in the neighborhood of the threshold, which is just the non relativistic region where our method is reliable.

Still concerning the scattering of solitons, we would like to mention that Faddeev's[5] rule for "quantizing" the S-matrix has a little flaw. The S-matrix obtained by that procedure does not discriminates between even and odd parity states. In spite of that such rule can be improved in order to take into account parity. The scattering lengths obtained by that improved version agree with the ones computed by us in the DHN region.

This paper is organized as follows: the classical SGT is presented in section II whereas some of the DHN results are presented in section III. In section IV we get the asymptotic potential. Section V is dedicated to the calculation of the bound state energies and scattering amplitudes. We finish this paper with a section reserved to conclusions and two appendices which complement some parts of the text.

2. CLASSICAL SINE-GORDON THEORY

We shall present in this section a summary of the classical S.G.T.[7]. This two dimensional model is described by the following Lagrangian density

$$\mathcal{L}(x,t) = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right] \tag{2.1}$$

By minimizing the action obtained from (2.1) we get the sine-Gordon Equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi + \frac{m^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}\phi\right) = 0 \tag{2.2}$$

From (2.1) and (2.2) one can easily see that when $\lambda = 0$ the S.G.T. is a field theory of a free scalar meson of mass m.

Before presenting the solutions of (2.2) relevant for our considerations in this paper, it is convenient to change the variables x and t into dimensionless ones. That can be achieved by defining

$$\begin{aligned} x' &= m x \\ t' &= m t \end{aligned} \tag{2.3}$$

and

$$\phi'(x', t') = \frac{\sqrt{\lambda}}{m}\phi(x,t) \tag{2.4}$$

Now we proceed exhibiting some solutions of (2.2). A whole set of solutions can be obtained by making use of the "Backlund Transformation" (6,7). The procedure works as follows: suppose ψ_0 is a solution of the sine-Gordon Equation written in terms of light cone variables $\sigma = \frac{x'+t'}{2}$ and $\rho = \frac{x'-t'}{2}$. Then, another solution ψ_1 can be generated by plugging ψ_0 into the "Backlund Transformation"

$$\frac{1}{2}\frac{\partial}{\partial\sigma}(\psi_1-\psi_0) = a \sin\left(\frac{\psi_1+\psi_0}{2}\right)$$

$$\frac{1}{2}\frac{\partial}{\partial\rho}(\psi_1+\psi_0) = \frac{1}{a}\sin\left(\frac{\psi_1-\psi_0}{2}\right) \tag{2.5}$$

The "vacuum" $\psi_0 = 0$ is an obvious solution of (2.2). From it the procedure sketched above leads to the so called soliton solution:

$$\phi_s^{\prime u}(x't') = 4\tan^{-1}\exp\left[\frac{x'-ut'}{\sqrt{1-u^2}}\right] \tag{2.6}$$

where u stands for the soliton velocity. In figure (1.a) we sketch the function that represents the soliton. Other solutions on which we will be interested are the ones corresponding to the two soliton scattering. They can be generated by making $\psi_0 = \phi_s^u$. We shall get

$$\phi_{ss}^{u}(x't') = 4\tan^{-1}\left[\frac{\sinh\left(ut'/\sqrt{1-u^2}\right)}{u\cosh\left(x'/\sqrt{1-u^2}\right)}\right] \tag{2.7}$$

We note that (2.7) describes the SS scattering in the frame of the center of mass of the pair; and u is the modulus of the velocity of each particle when $|t| \to \infty$. The solution corresponding to SA scattering is

$$\phi_{SA}^{\prime u}(x' t') = 4 \tan^{-1}\left[\frac{u \sinh\left(x'/\sqrt{1-u^2}\right)}{\cosh\left(u t'/\sqrt{1-u^2}\right)}\right] \qquad (2.8)$$

We shall mention also the "Breather" solutions. The simplest of them is the doublet solution, which can be obtained from (2-8) by making the substitution $u \longrightarrow i v$ The solutions so obtained corresponds to a SA bound state.

An interesting feature exhibited by these classical solutions is that they describe extended objects. That property can be verified by looking at the energy density of some solutions. Following the usual classical treatment, we can associate to each solution ϕ_α an energy density given by

$$\mathcal{H}_\alpha = \left(\frac{\partial \phi_\alpha}{\partial t}\right)^2 - \mathcal{L}(\phi_\alpha) \qquad (2.9)$$

The soliton, for instance, is a bloc of energy which moves with velocity u without deformation (see fig (1.b)). The total soliton energy in its rest frame will be

$$M_c = \frac{8 m^3}{\lambda} \qquad (2.10)$$

which is interpreted as the classical mass of the soliton.

For a free soliton we can determine the position and the velocity of its center of mass. No uncertainty at all arises from simultaneous measurements of physical quantities (as expected from a classical description). The "first quantization", which is performed in section 5, should implement the uncertainty principle.

3. THE DHN TREATMENT

Quantum corrections to a classical theory can be obtained, within Feynman's Path Integral framework, by expanding the action functional around classical solutions. In particular, when we take into account fluctuations up to quadratic terms, that procedure is equivalent to the usual W.K.B. approach [1,2]

Dashen et al. [2] succeeded in applying the W.K.B. method to the S.G.T. We would like to exhibit some of their results.

The quantum correction to the soliton mass (represented from now on as M) is the following

$$M = M_c - \frac{m}{\pi} + O(\lambda) = \frac{8m^3}{\lambda} - \frac{m}{\pi} + O(\lambda)$$

$$= \frac{8m}{\gamma} + O(\gamma) \qquad (3.1)$$

where M_c is the classical soliton mass, and γ is given by

$$\gamma = \frac{\lambda/m^2}{1 - \lambda/8\pi m^2} \qquad (3.2)$$

The spectrum of bound states of the SA system, that was obtained from a kind of "Bohr-Sommerfeld quantization rule" [2], is

$$E_N = \frac{16m}{\gamma} \sin\left(\frac{n\gamma}{16}\right) \qquad (3.3)$$

where $n = 1, 2, \ldots < \frac{8N}{\gamma}$. The region defined by

$$\frac{\lambda}{m^2} \simeq \frac{m}{M} \simeq \frac{\gamma}{8} << 1 \tag{3.4}$$

we shall refer as the DHN Region. In there we should expect DHN results to be reliable. From now on we shall assume that the parameters of the theory satisfy condition (3.4). From (3.3) and (3.4) we can see that the lowest bound state energy is[2,6]

$$E_1 \simeq m \tag{3.5}$$

i.e. the meson of mass m is simultaneously the "elementary particle" of the theory as well as the lower bound state of the SA system.

It will be convenient for our pourposes to make an inverse ordering of the energy levels, i.e. to start ordering from the one at the top of the spectrum (3.3). That can be achieved in a very simple way. If N_{max} is the total number of bound states, there exists an ε satisfying

$$0 \leq \varepsilon < 1 \tag{3.6}$$

such that

$$(N_{max} + \varepsilon) \frac{m}{M} = \pi \tag{3.7}$$

Now we define p in the following way

$$p = N_{max} - n \tag{3.8}$$

Then if we obtain n in terms of p and ε (with the help of (3.7) and (3.8)), after substituting it in (3.3), we get the following dependence of the total energy on p

$$E^{p}_{Total} = 2M \cos\left[\frac{m}{M}(\varepsilon + p)\right] \tag{3.9}$$

Now, by inspection of (3.9) ,we see that by varying p we have an ordering from the top of the spectrum to the bottom, or, in other words, p=0 corresponds to the highest binding energy, p=1 is the one just below that and so one. In the DHN Region we can, for small enough p $(p \ll \frac{M}{m})$, expand the function in (3.9) obtaining

$$E^{p}_{Total} \simeq 2M - \frac{m^2}{4M}(p + \varepsilon)^2 \tag{3.10}$$

Expression (3.10) is a very convenient one in order to compare with some of our results. The range of values of p, for which (3.10) is a good approximation for the spectrum, corresponds to the non relativistic domain.

4. THE POTENTIAL

In this section we shall study some aspects of the Soliton-Antisoliton classical interaction. We will compute the potential $V_{SA}(x)$ which describes the asymptotic dynamics of the SA system for non relativistic processes. The $V_{SS}(X)$ potential, corresponding to the interaction between two solitons, can be computed by an analogous procedure but the calculations will not be presented here. We would like to report our finding which is expressed by (1.3).

We shall search for classical solutions of the sine-Gordon equation describing non relativistic SA scattering. In order to fix conventions we shall assume that the soliton is moving in the positive direction of the coordinate axis, and that the origin of the coordinate frame coincides with the position of the center of mass of the SA pair (see fig.2).

The problem of determining the asymptotic potential can be solved if we know the velocity v of the center of mass of the soliton in the asymptotic region. The argument goes as follows: we can always write

$$v = v_{\infty} + \Delta v \tag{4.1}$$

obviously in the asymptotic region Δv is only a small correction to v_{∞} . On the other hand energy conservation implies

$$M_c v_{\infty}^2 = M_c (v_{\infty} + \Delta v)^2 + V_{SA} \tag{4.2}$$

From what we said above we conclude that, for $|X| \gg 1/m$, the potential is

$$V_{SA}(X) \simeq -2 M_c v_\infty \Delta v \tag{4.3}$$

Expression (4.3) indicates how the knowledge of the asymptotic velocity leads to the asymptotic potential. With regard to that we would like to make some comments. Although the positon and velocity of the center of mass of the free soliton can be determined accurately, the same is not true when it is interacting. For example, when the separation distance $|X|$ between two solitons is of the order of magnitude of the soliton size, i.e. $|X| \sim 1/m$, the two particles form a single bloc of energy in such a way that it becomes impossible to say where is the center of mass of each one of them. Each particle loses its identity.

On the other hand, if $|X|$ is large enough (when $t \gg 1/m v_\infty$), we shall observe two distinct blocs of energy - each of them representing a quasi-free particle (see fig. 2). Under these circunstances we can determine, within a very good degree of accuracy, the position of the center of mass of each particle, allowing us to determine also its velocity. It is precisely for those values of $|X|$ that we calculate the potential.

A brief analysis of some features which characterizes the free soliton will shed some light on how to proceed in order to get information concerning the positions and velocities of the quasi-free solitons. Figure (1.a) represents a solution $\phi(x,t)$ corresponding to a free soliton moving with velocity v. We shall be very much interested in three points of the soliton which somehow characterizes its position. These points labeled as Y_0 , Y_1 and Y_2 are defined as solutions of the equations

$$\frac{\partial^2}{\partial t^2} \phi(y,t) \Big|_{y=Y_0} = 0 \qquad (4.4)$$

and

$$\frac{\partial^3}{\partial t^3} \phi(y,t) \Big|_{y=Y_1} = 0 = \frac{\partial^3}{\partial t^3} \phi(y,t) \Big|_{y=Y_2} \qquad (4.5)$$

Figure (1.b) exhibits the behavior of the energy density associated with the soliton. By comparision of figures (1.a) and (1.b) we conclude that Y_0 (the inflection point) gives the position of the center of mass, while, loosely speaking, we can say that the soliton extends from Y_1 to Y_2 . We shall note that in the case of a free soliton all points of it moves with the same speed v.

In figure 2 we represent an interacting SA pair in the asymptotic region. From a close observation of that figure we can see that when $|X| \gg 1/m$ the center of mass Y_{cm} of the quasi-free soliton is somewhere between Y_1 and Y_2 and, more specifically, close to the inflection point Y_0 (8). From that we should have

$$\sigma \simeq \sigma_0 \qquad (4.6\text{-a})$$

and

$$\sigma_1 \lesssim \sigma \lesssim \sigma_2 \qquad (4.6\text{-b})$$

where σ is the velocity of the center of mass of the soliton and σ_0, σ_1 and σ_2 are the velocities of the points Y_0, Y_1 and Y_2 respectively. These asymptotic velocities σ_0, σ_1 and σ_2 are computed in appendix I. The results obtained are

$$\sigma_0 = \sigma_\infty + \frac{1}{\sigma_\infty} e^{-2m|Y_0|} + O\left(e^{-4m|Y_0|}\right) \qquad (4.7\text{-a})$$

$$\sigma_1 = \sigma_\infty + \frac{(3-2\sqrt{2})}{\sigma_\infty} e^{-2m|Y_1|} + O\left(e^{-4m|Y_1|}\right) \qquad (4.7\text{-b})$$

and

$$\sigma_2 = \sigma_\infty + \frac{(3+2\sqrt{2})}{\sigma_\infty} e^{-2m|Y_2|} + O\left(e^{-4m|Y_2|}\right) \qquad (4.7\text{-c})$$

From expressions (4.6) and (4.7) we conclude that the velocity of the center of mass of the soliton, in the asymptotic region, can be written under the form

$$v \simeq v_\infty + \frac{g}{v_\infty} e^{-m|X|} \tag{4.8}$$

where

$$g \sim 1 \tag{4.9}$$

and we recall that $|X|$ in (4.8) is the distance between the centers of mass of the soliton and the antisoliton, i.e. $|X| = 2\,Y_{cm}$.

Comparing (4.8) and (4.1) we infer that

$$\Delta v \simeq \frac{g}{v_\infty} e^{-m|X|} \tag{4.10}$$

After substituting (4.10) into (4.3) we are led to the $V_{SA}(X)$ potential. It can be written under the form:

$$V_{SA}(x) = -\alpha g M_c e^{-m|X|} \tag{4.11}$$

This potential is responsible for the SA interaction at long distances.

We shall add here that the relevant characteristics of the non relativistic bound state spectrum is by no means dependent upon a specific value of g.

5. PHASE SHIFT AND ENERGY LEVELS

The quantization of the soliton-antisoliton system will be performed in this section. As previously explained, that procedure is implemented, in our approach, by substituting the potential (4.11) in the Schrödinger Equation

$$\frac{1}{M}\frac{d^2}{dx^2}\psi_E(x) + 2g\, M_c\, e^{-m|x|}\,\psi_E(x) = -E\,\psi_E(x) \tag{5.1}$$

where $\psi_E(x)$ is the wave function describing a stationary state of energy E.

Our primary goal will be to compute the S-matrix elements associated with the potential (4.11). From S(E) we shall compute the scattering amplitudes, whereas, by looking at the positions of its poles, the bound state energies will be determined.

Since the potential (4.11) is symmetric under the transformation $X \longrightarrow -X$, there exist solutions of equation (5.1) with well defined parity. This means that for a given energy E we shall have an "even " matrix element $\overset{\text{even}}{S(E)}$ and an "odd" one $\overset{\text{odd}}{S(E)}$, which are obtained from the asymptotic behavior ($X \longrightarrow \pm\infty$) of the wave function

$$\psi^{even}(x) \underset{x\to\infty}{\simeq} e^{-ikx} + \overset{even}{S(E)}\, e^{ikx} \tag{5.2-a}$$

$$\psi^{even}(x) \underset{x\to-\infty}{\simeq} e^{ikx} + \overset{even}{S(E)}\, e^{-ikx}$$

and

$$\psi^{odd}(x) \underset{x \to \infty}{\simeq} e^{-ikx} - S^{odd}(E)\, e^{ikx}$$

$$\psi^{odd}(x) \underset{x \to -\infty}{\simeq} -e^{ikx} + S^{odd}(E)\, e^{-ikx} \qquad (5.2\text{-b})$$

where k is the modulus of the momentum of each particle in the center of mass reference frame.

The SA forward scattering amplitude is

$$M(E)_{forward} = \frac{S^{even}(E) + S^{odd}(E)}{2} - 1 \qquad (5.3)$$

We note that this is an invariant amplitude that can be analytically continued to the SS channel[4].

The SA backward scattering amplitude is given by

$$M(E)_{backward} = \frac{S^{even}(E) - S^{odd}(E)}{2} \qquad (5.4)$$

Being $S_\nu(E)$ a generic matrix element we define the phase shift $\delta_\nu(E)$ as

$$\delta_\nu(E) = \frac{1}{2i} \ln S_\nu(E) \qquad (5.5)$$

while the scattering lengths a_ν are defined by the limit

$$a_\nu = \lim_{k \to 0_+} \frac{1}{k} \left[\delta_\nu(k) - \delta_\nu(0) \right] \qquad (5.6)$$

For the discussion which will follow we found convenient to define a function β (E) as

$$\beta(E) = \frac{\alpha\sqrt{-ME}}{m} \tag{5.7}$$

and a constant A as

$$A = \sqrt{8g}\,\frac{\sqrt{M_c M}}{m} \tag{5.8}$$

The meaning of all constants that appears in (5.7) and (5.8) can be understood by looking to (5.1).

In appendix II we show that the matrix elements $S_{\nu}(E)$ are given by

$$\overset{even}{S}(E) = -\frac{J'_{-\beta}(A)\,\Gamma(1-\beta)}{J'_{\beta}(A)\,\Gamma(1+\beta)}\left(\frac{A}{2}\right)^{2\beta} \tag{5.9-a}$$

and

$$\overset{odd}{S}(E) = \frac{J_{-\beta}(A)\,\Gamma(1-\beta)}{J_{\beta}(A)\,\Gamma(1+\beta)}\left(\frac{A}{2}\right)^{2\beta} \tag{5.9-b}$$

where J_β is the Bessel function of order β and J'_β stands for its derivative.

In order to obtain the bound state energies we will study the behavior of S(E) for $E < 0$. In that region β (E) can be represented as[9] .

$$\beta(E) \underset{E<0}{=} \alpha(E) = \frac{\alpha\sqrt{M|E|}}{m} \tag{5.10}$$

From (5.9) and (5.10) we conclude that the binding energies will be given by the positions of the zeros of $J'_\alpha(A)$ and $J_\alpha(A)$. Since in the DHN Region $A \gg 1$ (because there $M/m \gg 1$) we can make the following approximation

$$J'_\alpha(A) \simeq \sqrt{\frac{2}{\pi A}} \sin\theta(A,\alpha) \tag{5.11-a}$$

and

$$J_\alpha(A) \simeq -\sqrt{\frac{2}{\pi A}} \cos\theta(A,\alpha) \tag{5.11-b}$$

where $\theta(A,\alpha)$ is defined as

$$\theta(A,\alpha) = A - \frac{\alpha\pi}{2} - \frac{\pi}{4} \tag{5.12}$$

From (5.9) and (5.11) we can see that the positions of the poles of $S^{\text{even}}_{(E)}$ $\left[S^{\text{odd}}_{(E)}\right]$ are approximatelly given by the positions of the zeros of $\sin\theta$ ($\cos\theta$).

We can always write

$$A = \frac{\pi}{2}\left(N + \eta + \frac{1}{2}\right) \tag{5.13}$$

where N is, by choice, an integer number and

$$0 \leq \eta < 1 \tag{5.14}$$

By using (5.13) and (5.14) it is easy to see that, in the DHN Region, the binding energy spectrum will be given by

$$E_p = \frac{m^2}{4M}(p+\eta)^2 \tag{5.15}$$

where p is an integer such that if N is even (odd) the levels corresponding to p = 0,2,4... will have even (odd) wave functions and the levels corresponding to p = 1,3,5... will have odd (even) wave functions.

By comparing (5.15) with (3.10) and identifying η with ε (see also (5.14) and (3.6)) we verify that , in the nonrelativistic limit of DHN Region, our spectrum coincides with that of DHN .

It can be explicitly verified that our results obey the criteria for the validity of our approximations [see (1.4-a) and (1.5)].

In the scattering region ($E > 0$) β (E) will be a pure imaginary number, i.e.[9]

$$\beta(E) = -i\,n \tag{5.16}$$

where

$$n = \frac{\alpha\sqrt{ME}}{m} = \frac{\alpha\,k}{m} \tag{5.17}$$

The S-matrix elements are

$$S^{even}(E) = -\frac{J'_{in}(A)\,\Gamma(1+in)}{J'_{-in}(A)\,\Gamma(1-in)}\left(\frac{A}{2}\right)^{-2in}. \tag{5.18-a}$$

and

$$S^{odd}(E) = \frac{J_{i\eta}(A)\,\Gamma(1+i\eta)}{J_{-i\eta}(A)\,\Gamma(1-i\eta)}\left(\frac{A}{2}\right)^{-2i\eta} \tag{5.18-b}$$

By using (5.5) and (5.6) it is straightforward to compute phase shifts and scattering lengths. In the SA channel we get

$$a_{SA}^{even} = -\frac{2}{m}\left[\ln\left(\frac{M}{m}\right) + \frac{\pi}{4}\cotg\left(A - \frac{\pi}{4}\right) + \ln\sqrt{2g\frac{M_c}{M}} + \gamma\right] \tag{5.19-a}$$

and

$$a_{SA}^{odd} = -\frac{2}{m}\left[\ln\left(\frac{M}{m}\right) - \frac{\pi}{4}\operatorname{tg}\left(A - \frac{\pi}{4}\right) + \ln\sqrt{2g\frac{M_c}{M}} + \gamma\right] \tag{5.19-b}$$

where γ is the Euler constant.

In an analogous way we can compute the SS scattering lengths. If the solitons are fermions, as suggested in the literature(2,5,6,12), Pauli's exclusion principle will imply that we need to take into account only states described by odd wave functions. Then the SS scattering length will be

$$a_{SS}^{odd} = \frac{2}{m}\left[\ln\left(\frac{M}{m}\right) + \ln\sqrt{2g\frac{M_c}{M}} + \gamma\right] \tag{5.20}$$

In the introduction we have mentioned that Faddeev's(5) rule for " quantizing " the S-matrix(10) has a little flaw. Now we will clarify this point. It is known that parity is tipically a quantum concept, and, as a consequence of this, the classical S-matrix(10) does not discriminates between even and odd parity states. The flaw of Faddeev's rule lies on the fact that it extends this "blindness for parity " to the quantum S-matrix. Fortunately it is not difficult to remediate this problem. Recalling that the parity of the wave function of the n - th[11] SA bound state is given by $(-1)^{n+1}$, we can split Faddeev's quantum S-matrix into two parts: an S^{even} that contains the poles of the even parity bound states and an S^{odd} that contains the other set of poles.

Faddeev's quantum S-matrix can be written in the following form[10]

$$S^F = \prod_{n=1}^{N} Q_n \tag{5.21}$$

where Q_n is the factor which contains the pole of the n-th bound state. Note that in the scattering regions each Q_n is per se explicitily unitary. The splitting above mentioned consists in defining

$$S^{even} = \prod_{\substack{n=1\\ odd}}^{N} Q_n \tag{5.22-a}$$

and

$$S^{odd} = \prod_{\substack{n=2\\ even}}^{N} Q_n \tag{5.22-b}$$

One observes that the improved quantization rule leads to unitary S-matrix elements whereas the spectrum remains the same as that of ref.(5). It also leads to scattering amplitudes which agree with the ones obtained by us in the non relativistic limit of the DHN Region [Note that in the DHN Region $\ln\left(g\frac{M_c}{M}\right)$ and γ can be negleted when compared with $\ln(M/m)$]

6. CONCLUSION

By using a quite different approach, with regard to that employed by DHN, we were succeeded in giving a non relativistic quantum treatment to soliton interactions within the sine-Gordon Theory. Our method is essentially the well known non relativistic quantum mechanical approach. In order to do this, we had to compute first the potential which is responsible for the interactions between solitons. After that we studied the non relativistic motion by substituting the potential into the Schrödinger Equation. That way we were able to compute the soliton-antisoliton bound states, reproducing part of the DHN spectrum, and the SA and SS scattering amplitudes in the non relativistic region.

Coleman[12] has shown that the sine-Gordon Theory is equivalent to the massive Thirring Model, strongly suggesting that the soliton is the fermion of this model. On the other hand it is well known that the massive Thirring Model is equivalent to the two dimensional massive Vector Gluon model in the limit

$$\mu \to \infty \quad , \quad e \to \infty \; ; \tag{6.1}$$

$$\text{with } \frac{e}{\mu} = g \text{ fixed}$$

where μ is the mass of the Vector Gluon and e is the Fermion-Vector Gluon Coupling constant. From that equivalence we should naively expect that the fermion-fermion interaction potential should be given by[13]

$$V(x) = \lim_{\mu \to \infty} \left[g\mu \, e^{-\mu|x|} \right] = 2g \, \delta(x) \tag{6.2}$$

where $g\mu \exp(-\mu|x|)$ in (6.2) is the Yukawa potential associated with the Vector Gluon.

Of course the naive argument just above presented can not be true because the correct SS potential, given by (4.11), is rather different from (6.2). Then we conclude that here the dynamics exhibits a very interesting feature, namely the fact that the lightest fermion-antifermion bound state happens to be the responsible for the interaction between fermions at long distances.

APPENDIX I

We compute here the asymptotic velocities v_i. In the non relativistic domain ($v_\infty^2 << 1$) the SA solution can be approximated by the expression.

$$\phi_{SA}(y,t) = \frac{4m}{\sqrt{\lambda}} \tan^{-1}\left[\frac{\sinh(v_\infty m t)}{v_\infty \cosh(m x)}\right] \tag{A.1}$$

The positions of the inflection points $y_0(t)$ and $-y_0(t)$ are defined implicitly as solutions of (4.4). By solving (4.4) we conclude that

$$\cosh(m y_0) = f(t)\left[1 + \frac{2}{f^2(t)}\right]^{1/2} \tag{A.2}$$

where

$$f(t) = \frac{1}{v_\infty} \sinh(v_\infty m t) \tag{A.3}$$

On the other hand, the third derivative of ϕ_{SA} vanishes at the origin and at the points which we call $y_1\,(-y_1)$ and $y_2\,(-y_2)$. Then $y_1(t)$ and $y_2(t)$ are the solutions of the equations (4.5). After plugging into (4.5) $\phi_{SA}(y,t)$ given by (A.1) we will get .

$$\cosh(m y_1) = f(t)\left[3 - 2\sqrt{2}\sqrt{1 + \frac{2}{f^2} + \frac{9}{8f^4}}\right]^{1/2} \tag{A.4-a}$$

and

$$\cosh(m y_2) = f(t)\left[3 + 2\sqrt{2}\sqrt{1 + \frac{2}{f^2} + \frac{9}{8f^4}}\right]^{1/2} \qquad \text{(A.4-b)}$$

where f(t) was defined in (A.3). When $t \gg 1/m v_\infty$ we shall have (remember that $v_\infty^2 \ll 1$)

$$\cosh(m y_i) = \frac{a_i}{2 v_\infty} e^{v_\infty m t}\left[1 - e^{-2 v_\infty m t} + O\left(e^{-4 v_\infty m t}\right)\right] \qquad \text{(A.5)}$$

where the coefficients a_i are given by

$$a_0 = 1 \qquad \text{(A.6-a)}$$

$$a_1 = (3 - 2\sqrt{2})^{1/2} \qquad \text{(A.6-b)}$$

$$a_2 = (3 + 2\sqrt{2})^{1/2} \qquad \text{(A.6-c)}$$

From (A.5) we conclude that the asymptotic velocities v_i of each point y_i will be given by equations (4.7).

APPENDIX II

In this appendix we shall solve Schrödinger Equation (5.1). Being β and A given by expressions (5.7) and (5.8) respectively, that equation can be written as

$$\frac{1}{m^2}\frac{d^2}{dx^2}\psi + \frac{A^2}{4}e^{-m|x|}\psi = \frac{\beta^2}{4}\psi \tag{A.7}$$

we shall perform now a change of variable

$$\xi = A\, e^{-\frac{m|x|}{2}} \tag{A.8}$$

In terms of ξ equation (A.7) becomes

$$\xi^2\frac{d^2}{d\xi^2}\psi + \xi\frac{d}{d\xi}\psi + (\xi^2 - \beta^2)\psi = 0 \tag{A.9}$$

Equation (A.9) can be easily recognized as the Bessel Equation. The even solutions (even under the change $x \longrightarrow -x$) are given by

$$\psi_\beta^{even} = C\left[J'_{-\beta}(A)\, J_\beta(\xi) - J_\beta'(A)\, J_{-\beta}(\xi)\right] \tag{A.10-a}$$

where J_ρ is the Bessel function of order ρ , J_ρ' is its derivative and C is a constante. The odd solutions can be written, for $x > 0$, as

$$\psi_\beta^{odd} = C'\left[J_{-\beta}(A)\, J_\beta(\xi) - J_\beta(A)\, J_{-\beta}(\xi)\right] \tag{A.10-b}$$

Since in the limit $x \to \infty$, ξ tends to zero, we can make use, in this limit, of the following approximation for Bessel functions[14]

$$J_\rho(\xi) \underset{\xi \to 0}{\sim} \frac{1}{\Gamma(1+\rho)} \left(\frac{1}{2}\xi\right)^\rho \qquad \text{(A.11)}$$

From (A.10) and (A.11) it follows that when $x \to \infty$ we have

$$\psi_\beta^{even} \underset{x\to\infty}{\simeq} C\left[\frac{J'_\beta(A)}{\Gamma(1-\beta)}\left(\frac{A}{2}\right)^{-\beta} e^{\frac{\beta m x}{2}} - \frac{J'_{-\beta}(A)}{\Gamma(1+\beta)}\left(\frac{A}{2}\right)^{\beta} e^{-\beta\frac{m x}{2}}\right] \qquad \text{(A.12-a)}$$

and

$$\psi_\beta^{odd} \underset{x\to\infty}{\simeq} C'\left[\frac{J_\beta(A)}{\Gamma(1-\beta)}\left(\frac{A}{2}\right)^{-\beta} e^{\frac{\beta m x}{2}} - \frac{J_{-\beta}(A)}{\Gamma(1+\beta)}\left(\frac{A}{2}\right)^{\beta} e^{-\frac{\beta m x}{2}}\right] \qquad \text{(A.12-b)}$$

Remembering that in the scattering region $\beta(E)$ is represented by (5.16), and taking in account our definitions (5.2), we can deduce (5.9) simply adjusting suitably C and C'.

FOOTNOTES AND REFERENCES

1) R.Dashen, B.Hasslacher and A. Neveu, Phys. Rev. D10, 4114(1974); ibid D10, 4130(1974); ibid D10, 4138(1974).
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4) Sidney Coleman, Appendix to ref. (3).

5) L.D.Faddeev, IAS preprint (April 1975).

6) R.Rajaraman, IAS preprient (May 1975).

7) For a review on the classical sine-Gordon Theory see A.C.Scott, F.Y.F.Chu and D.W.Mc Laughlin, Proc.IEEE 61,1443(1973).

8) In the case of the interacting soliton the definitions of y_0, y_1 and y_2 are also given by (4.4) and (4.5).

9) The choice of the square roots signals in (5.10) and (5.16) agree with the convention that the scattering amplitude is obtained when we approach the right hand cut from above.

10) Faddeev's classical S-matrix is the one obtained in the tree approximation. It assumes the form

$$S(E) = \exp\left\{ \frac{N}{\pi} \int_0^{\pi} d\theta \left[\frac{\xi(E) + e^{i\theta}}{\xi(E) e^{i\theta} + 1} \right] \right\}$$

where $\xi(E)$ is defined in ref.(5). Faddeev's quantization is implemented in the following way

$$\frac{N}{\pi}\int_0^{\pi} d\theta \, \ln\left(\frac{\xi + e^{i\theta}}{\xi e^{i\theta}+1}\right) \longrightarrow \sum_n^N \ln\left(\frac{\xi + e^{i\theta_n}}{\xi e^{i\theta_n}+1}\right)$$

where $\theta_n = \frac{\pi}{N} n \quad ; \quad n = 1, 2, \dots N$

11) Here we are ordering the spectrum starting from the lower bound state which corresponds to n=1.

12) Sidney Coleman, Phys.Rev. D11, 2088(1975).

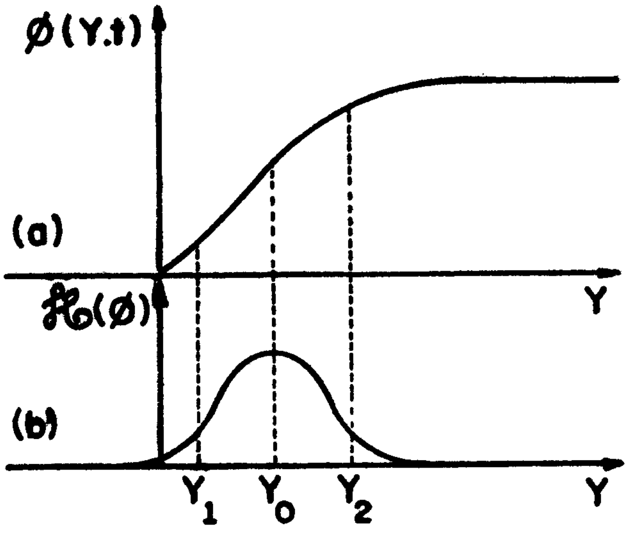
13) In the final discussions of ref.(12) Coleman had assumed that the fermion-fermion potential is a δ-function.

14) Milton Abramowitz and Irene A. Segun, Handbook of Mathematical Functions, Dover(1965).

FIGURE CAPTION

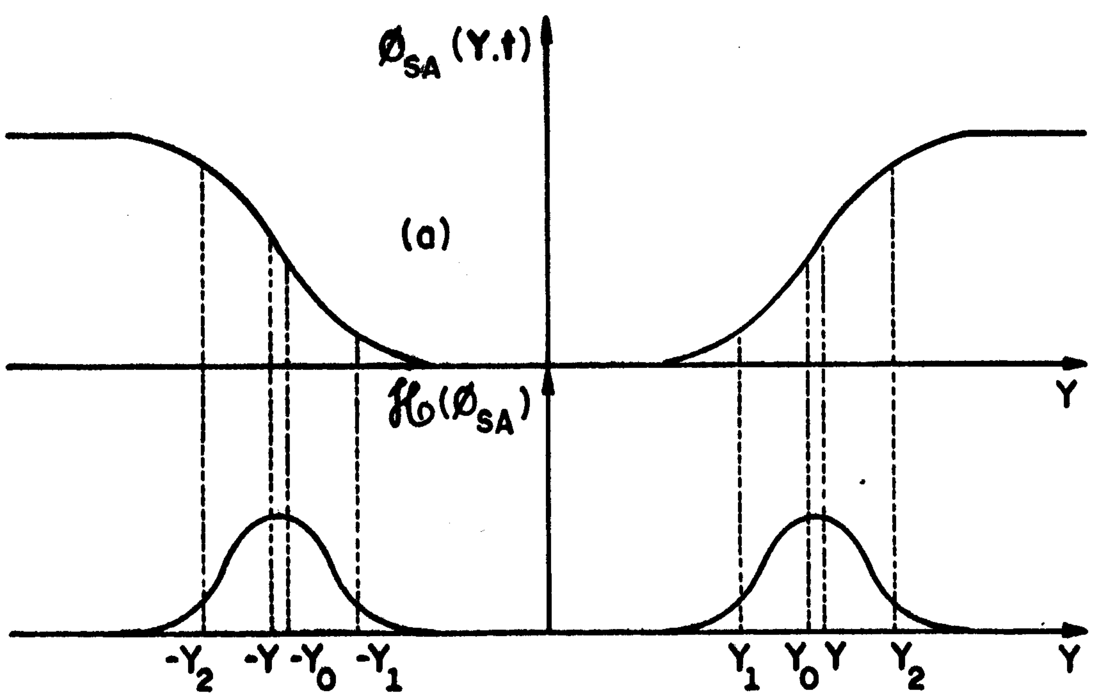
Fig.1. (a) Free soliton; (b) Energy density associated to it. y_o is the inflection point (that here is the center of mass); y_1 and y_2 are the points where the third derivative is zero.

Fig.2. (a) Soliton-antisoliton pair interacting in the asymptotic region; (b) Energy density associated to them. $\pm y_o$ are the inflection points; $\pm y_1$ and $\pm y_2$ are the points where the third derivative is zero; and $\pm y_{cm}$ are tipical candidates to be the centers of mass of each particle.



$\phi(Y.t)$
(a)
$\mathcal{H}_0(\phi)$
Y
(b)
Y_1
Y_0
Y_2
Y

FIG. 1



$\phi_{SA}(Y.t)$
(a)
$\mathcal{H}_0(\phi_{SA})$
Y
$-Y_2$
-Y
$-Y_0$
$-Y_1$
Y_1
Y_0
Y
Y_2
Y

FIG. 2