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ABSTRACT

The "Bohr-Sommerfeld quantization rule" is shown to be, under certain circunstances, equivalent to the quantization of the charge. Charge conservation happens to imply the stability of the semiclassical spectrum of a model.

In the last few years much attention has been paid to semiclassical methods of quantization in field theory(1,2,3). Among these methods the simplest one is the "Bohr-Sommerfeld Quantization Rule" (BSQR)(2,3). Being ϕ (x) a quantum field of a given theory, and π (x) its canonical momentum, the BSQR reads(2,3)

$$\int_0^{\tau} dt \int d\vec{x} \; \pi^{cla}(\vec{x},t) \; \dot{\phi}^{cla}(\vec{x},t) = 2\pi n \tag{1}$$

where n is an integer and $\phi^{cla}(\vec{x},t)$ and $\pi^{cla}(\vec{x},t)$ are periodic solutions (with period τ) of the classical equations of motion.

Another kind of quantization we can perform on a classical theory refers to the "charge quantization". Consider a classical field theory invariant under gauge transformation of first kind. This gauge invariance leads to the conservation of a charge whose density is $\rho^{cla}(\vec{x},t)$ The "charge quantization" (CQ) mentioned above is implemented by imposing the condition

$$\int \rho^{cla}(\vec{x},t) \; d\vec{x} = N \tag{2}$$

where N is an integer.

The aim of this note is to show with examples that the BSQR is, under certain circunstances, equivalent to the CQ defined by eq. (2). Although we have verified this equivalence on a semiclassical level we conjecture that the <u>exact</u> quantization of these theories also demands a quantization of the charge; or, in other words, we conjecture that in a <u>quantum</u> field theory invariant under gauge transformations of first kind the charge is always an integer multiple of an elementary quantity - as is the case for the electric charge.

Let us look first to the charged scalar field theories defined by Lagrangian densities of the form

$$\mathcal{L}(\vec{x},t)= \left|\frac{\partial\phi}{\partial t}\right|^2 - \vec{\nabla}\phi^* \vec{\nabla}\phi - m^2|\phi|^2 - P(|\phi|^2) \tag{3}$$

$P(|\phi|^2)$ being a polynomial in $|\phi|^2$. These theories are invariant under gauge transformation of first kind. For them the BSQR is written as

$$\int_0^{\tau} dt \int d\vec{x} \; \overset{\circ}{\phi}{}^{*\,cla}(\vec{x},t) \; \overset{\circ}{\phi}{}^{cla}(\vec{x},t) = 2\pi n \tag{4}$$

where $\overset{\circ}{\phi}{}^{cla}(\vec{x},t)$ are periodic solutions (with period τ) of the classical equation of motion

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi + \frac{1}{2} P'(|\phi|^2)\phi = 0 \tag{5}$$

A large class of these periodic solutions is given by

$$\phi^{cla}(\vec{x},t) = f_{\tau}(\vec{x})\, e^{-i\frac{2\pi}{\tau}t} \tag{6}$$

where $f_{\tau}(\vec{X})$ is a solution (that we assume to be square integrable) of the equation

$$- \nabla^2 f + \left[m^2 - \left(\frac{2\pi}{\tau} \right)^2 \right] f + \frac{1}{2} P'(|f|^2) f = 0 \tag{7}$$

Inserting (6) on (4) we see that the BSQR implies

$$\frac{2\pi}{\tau} \int |f_\tau(\vec{x})|^2 d\vec{x} = n \tag{8}$$

Expression (8) "quantizes" the values of τ leading to a quantization of the energy(2,3).

The classical charge density associated to this complex field is

$$\rho^{cla}(\vec{x},t) = \frac{i}{2} \left[\phi^{*cla} \dot{\phi} - \dot{\phi}^{*cla} \phi \right] =$$

$$= \frac{2\pi}{\tau} \, |f(x)|^2 \tag{9}$$

Now, from eqs. (9) and (2) we conclude that the CQ requires that

$$\frac{2\pi}{\tau} \int |f_\tau(\vec{x})|^2 d\vec{x} = N \tag{10}$$

Then, comparing (8) and (10) we see that - for solutions of type (6) - the BSQR is equivalent to CQ.

Of course this phenomenon may occur in any field theoretical model invariant under gauge transformation of first kind. Another example where the equivalence studied here becomes manifest is the massive Thirring Model (MTM). There the charge density associated to the classical field $\psi^{cla}(x,t)$ is given by

$$\rho^{cla}(x,t) = \psi^{+cla}(x,t)\,\psi^{cla}(x,t) \tag{11}$$

whereas the charge quantization reads

$$\int \psi^{+}\psi\, dx = N \tag{12}$$

Here we mention that the CQ condition for the MTM (expression (12)) happens to be equal to a "normalization condition" defined in ref. (4).

In the MTM the equivalence between the BSQR and the CQ can be easily verified when we deal with solutions of the form

$$\psi(x,t) = \chi(x)\, e^{-i\omega t} \tag{13}$$

By using the BSQR[3] or the CQ we deduce that this theory has a discrete spectrum. The energy of the state of charge $Q_n = n$ will be

$$E_n = \frac{2m}{g}\sin\left(\frac{g}{2}|n|\right) \tag{14}$$

where $|n| = 0, 1, 2, \ldots < \pi/g$; g is the coupling constant and m is the mass of the classical Lagrangian density of the MTM.

We want to mention that charge conservation will forbid the decay of a particle of energy E_n into two others of energies E_p and E_q (consider for simplicity that n, p and q are positive). Charge conservation implies that n = p+q. But energy conservation requires

$$E_{p+q} = E_p + E_q + \text{Kinetic Energy} \tag{15}$$

or

$$\sin\left[\frac{g}{2}(p+q)\right] \geq \sin\left(\frac{g}{2}p\right) + \sin\left(\frac{g}{2}q\right) \tag{16}$$

because the Kinetic Energy is non negative

Inequality (16) cannot be satisfied since

$$0 < (g/2)(p+q) \leq \pi/2,$$

Then the decay under consideration is forbidden.

As pointed out in ref. (3), this spectrum is similar to that of the sine-Gordon Theory (SGT) - in agreement with Coleman's[(5)] proof of the equivalence between the MTM and the SGT. We mention that, in the context of the SGT, Dashen, Hasslacher, Neveu[(6)] and Faddeev[(7)] had conjectured that the decay studied above is not allowed. Our conclusion about the decays prohibition - in the context of the MTM - can be seen as another point in favor of them.

In conclusion, we mention that the equivalence discussed in this paper can be verified for any first kind gauge invariant theory having classical solutions whose time dependence is of the type (6) or (13). If this kind of time dependence is a necessary condition for the equivalence is an open question.

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