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by

Isidoro Kimel

Instituto de Física - Universidade de S.Paulo

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I. KIMEL

Instituto de Física, Universidade de São Paulo
Caixa Postal 20.516 - Cidade Universitária
São Paulo - SP - Brasil

ABSTRACT

The equivalence between the sine-Gordon and massive Tirring models, that Coleman showed to exist for the quantized theories, is illustrated in a classical context. Insistence on a pure massive Tirring model with no extra dynamical variables seems to require Skyrme's condition at least in the classical theory.

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I. INTRODUCTION

The pioneering work of Skyrme[1] calling attention to the possible applications of the sine-Gordon equation to Particle Physics is almost two decades old. Some of the classical solutions of this equation, the so called solitons, exhibit particle like behaviour and conserved quantities appear which one can hope to associate to fermionic charge. A lot of effort has been devoted in the last years to see what happens at the quantum level using semiclassical methods[2,3].

More recently Coleman[4] compared the quantum sine-Gordon with the massive Tirring model[5] in the charge zero sector and noticed that one can make a correspondence between field operators (and at the same time among masses and coupling constants) of both theories. The relations obtained lead to very interesting conclusions. It looks as if the sine-Gordon "elementary" boson were a bound state of Tirring's fermion-antifermion. Since, on the other hand, this same boson can also be thought as being a soliton-antisoliton bound state, one could identify the soliton with the elementary fermion of the Tirring model. This identification relies on the explicit form of the relations found by Coleman.

Some of Coleman's results, obtained in the quantized theory, do not seem to have an immediate interpretation in classical terms. On the other hand, it is known that in most cases classical results survive quantization and remain valid as lowest order approximations[6]. Thus, it would be interesting to have an understanding of Coleman's relations at the classical level also. To try to develop such an understanding is our main aim in the present work.

In the following section we obtain sine-Gordon as a special case of the σ-model[7]. Coleman's results are written down in section III. In section IV and V we show how to go from the Tirring model to sine-Gordon and viceversa. The last section contains concluding remarks.

II. SINE-GORDON AS THE NON-LINEAR σ-MODEL

An equivalence between sine-Gordon and σ-model was already noticed by Skyrme[1,8]. In this section we will elaborate on that equivalence emphasizing the Lagrange constraint aspect of the problem since similar techniques will be applied later on in showing the Tirring model-sine-Gordon connection.

Let us start for simplicity with the U(1) $\otimes$ U(1) σ-model with bosons only and explicity broken chiral symmetry. The Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\right] - \frac{\mu_o}{2}(\sigma^2+\pi^2)+C\sigma-B(\sigma^2+\pi^2-A^2)^2, \tag{1}$$

where σ is a scalar field, π a pseudoscalar and everything else are constants.

The chiral symmetry can be realized linearly with σ and π being independent fields. Or, one can have a nonlinear realization in which now σ is no more an independent field, being related to π by the constraint equation

$$\sigma^2 + \pi^2 = A^2 . \tag{2}$$

In this case the fields can be parametrized in terms of a new pseudoscalar ϕ as

$$\sigma = A \cos \beta\phi, \qquad \pi = A \cos \beta\phi, \tag{3}$$

where β is a constant.

Our Lagrangian is now

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{\beta^2} \cos \beta\phi \tag{4}$$

where we have set

$$A = \beta^{-1}, \quad C = \frac{m^2}{\beta^2} . \tag{5}$$

Eq. (4) is the Lagrangian of the sine-Gordon model, with mass m and coupling constant $(m^2\beta^2)$.

The point we would like to stress is that one can interpret the last term in Lagrangian (1) as a constraint imposed on a theory that otherwise would describe two noninteracting fields. According to this interpretation the parameter B would be a sort of Lagrange multiplier. Such Lagrange multiplier methods will be used in section IV in constructing a model that illustrates the sine-Gordon-Tirring model relationship.

III. THE COLEMAN CONNECTION

Coleman has gone a long way in showing that the quantized version of the sine-Gordon model is equivalent to the massive Tirring model in the charge zero sector.

The classical massive Tirring model[5] Lagrangian is

$$\mathcal{L}_T = i\,\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi - \frac{1}{2}\,g j^\mu j_\mu \,, \tag{6}$$

where the self interaction is given in terms of the current j^μ. This current and the pseudocurrent are defined by

$$j^\mu = \bar{\psi}\gamma^\mu\psi \quad , \qquad j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \ . \tag{7}$$

The Hamiltonian density of the Tirring model can be written as

$$\mathcal{H}_T = \frac{1}{2}\left[(j_5^0) + (j_5^1)\right] + M\bar{\psi}\psi \tag{8}$$

while for the sine-Gordon model this density is

$$\mathcal{H}_{SG} = \frac{1}{2}\left[(\partial^0\phi)^2 + (\partial^1\phi)^2\right] - \frac{m^2}{\beta^2}(\cos\beta\phi - 1) \ . \tag{9}$$

A comparison of Eqs. (8) and (9) could provide the motivation of Coleman's strategy for the identification of both models:

1) Show that in the quantized theories there exists the operator equivalence

$$-M\bar{\psi}\psi = \frac{m^2}{\beta^2}\cos\beta\phi \tag{10}$$

in the sence that the Green functions for both field operators can be made to be equal.

2) Once that is achieved, comparing the commutators

$$\left[j^\mu , \bar{\psi}\psi\right] \quad ; \qquad \left[\partial^\mu\phi, \cos\beta\phi\right] \ , \tag{11}$$

one can establish an equivalence between j^μ and $\partial^\mu\phi$.

More precisely, from a comparison of Green functions and commutators one arrives at the identifications

$$-Z\ s\ \bar{\psi}(1 \pm \gamma_5)\psi = m\ N_m\ e^{\pm i\beta\phi} \quad , \tag{12}$$

and

$$j^{\mu} = -\frac{\beta}{2\pi}\ \varepsilon^{\mu\nu}\partial_{\nu}\phi \quad , \tag{13}$$

where N_m refers to the normal ordering operation with respect to the mass m and Z is an (infinite) renormalization constant from which we extracted the finite number s.

As a necessary condition for Eqs. (12) and (13), the coupling constants have to be related by

$$(1 + g/\pi)^{-1} = \frac{\beta^2}{4\pi} \tag{14}$$

and to go from Eq. (12) to (10) (up to the renormalization constant Z) one also needs

$$M = s\ \frac{m}{\beta^2} \quad . \tag{15}$$

From Eqs. (14) and (15) we see that the mass m of the sine-Gordon boson is given in terms of the mass M of the Tirring fermion by

$$m = \frac{4\pi/s}{1+g/\pi}\ M \quad , \tag{16}$$

which suggest that the boson is a bound state of a fermion-antifermion pair.

On the other hand semiclassical studies yielded the conclusion that the sine-Gordon fundamental boson is a bound state of soliton-antisoliton modes[3]. The classical soliton mass is given precisely by Eq. (15) with s=8. Considerations of this kind, among others, lead Coleman to the conjecture that the sine-Gordon soliton is the fermion of the massive Tirring model.

The very interesting analysis of Coleman was done for the quantized theory. A natural question to ask would be: Is it possible to obtain Coleman's results in a classical context? The following sections are devoted to answering that question.

IV. FROM TIRRING MODEL TO SINE-GORDON

The appearance of infinitely many superselection sectors in the sine-Gordon model is what encourages one to look for something like a fermionic number. As a matter of fact a good candidate for the fermionic charge would be

$$Q = -\frac{\beta}{2\pi}\int_{-\infty}^{\infty} \varepsilon^{01}\partial^{1}\phi \, dx^{1} = \frac{\beta}{2\pi}\left[\phi(\infty) - \phi(-\infty)\right], \tag{17}$$

since as we go from one superselection sector to the next Q changes by one[2,3]. Thus if there is a relationship between the sine-Gordon model and fermions, $\varepsilon^{\mu\nu}\partial_{\nu}\phi$ would be the natural candidate for the fermionic current.

A comparison of the Hamiltonians (8) and (9) leads us to expect some connection between $\bar{\psi}\psi$ and ($\cos\beta\phi$+ constant). In that case, chiral symmetry would also dictate a similar relation between $\bar{\psi}\gamma_5\psi$ and $\sin\beta\phi$. And now we have to see how to incorporate all these features in a model.

After all the preparatory considerations let us start with a self coupled massive fermion ψ and a free massless boson ϕ' with no interaction between boson and fermion. The Lagrangian is

$$\mathcal{L}_0 = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - M\bar{\psi}\psi - \frac{1}{2}g\bar{\psi}\gamma^{\mu}\psi\bar{\psi}\gamma_{\mu}\psi + \frac{1}{2}y(\partial_{\mu}\phi')^2 \tag{18}$$

(the reason for introducing the constant y will be explained later).

At this point ψ and ϕ' are independent variables and, as usual, the variation of $\mathcal{L}_o$ with respect to them yields the equations of motion.

If, on the other hand, we knew that ψ and ϕ' are coupled somehow (or, equivalently, that there are constraints between them) we could not perform independent variations of the Lagrangian unless we incorporate to it the constraints according to the Lagrange multiplier method. Let us follow that procedure in our case adding to Eq. (18) the expected constraints

$$\mathcal{L}_c = \lambda_1(\bar{\psi}\gamma^\mu\gamma_5\psi + a\partial^\mu\phi')^2 + \lambda_2\{[\bar{\psi}\psi + \mu(\cos\beta'\phi' + b)]^2 + + (i\bar{\psi}\gamma_5\psi - \mu\sin\beta'\phi')^2\}, \tag{19}$$

where a, b, μ, λ_1 and λ_2 are constants, the last two having the role of Lagrange multipliers.

It might be worthwhile at this point to compare what we are doing here with what was done in Ref (4). Hoping to find equivalences (between the boson and fermion models) of the type of Eqs. (10) and (13) Coleman inquired whether the n point functions of the field operators under consideration could be made equal in both theories and under what conditions would that happen. In our case, looking for a model with that kind of equivalences, we introduce them in the Lagrangian (through the constraints (19)) to see what relations (if any) emmerge.

Our constrained Lagrangian is now

$$\mathcal{L}_I = \mathcal{L}_o + \mathcal{L}_c = i\bar{\psi}\gamma^\mu\partial_\mu\psi - (M-2b\mu\lambda_2)\bar{\psi}\psi$$

$$- \frac{1}{2}(g - 2\lambda_2 + 2\lambda_1)\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi + 2\lambda_1 a\bar{\psi}\gamma^\mu\gamma_5\psi\partial_\mu\phi' +$$

$$+ \lambda_2\mu\,\bar{\psi}(\cos\beta'\phi' - i\gamma_5\sin\beta'\phi')\psi +$$

$$+ \frac{1}{2}(y + 2\lambda_1 a^2)(\partial_\mu\phi')^2 + 2b\mu^2\lambda_2\cos\beta'\phi', \tag{20}$$

where a constant was dropped and use was made of

$$(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2 = \bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi, \tag{21}$$

which can be easily checked using the explicit representation $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$ and $\gamma_5 = -\sigma_3$.

We see that in the Lagrangian (20) the four-fermion coupling and the mass term disappear if

$$g = 2(\lambda_2 - \lambda_1), \tag{22}$$

and

$$M = 2\mu b\lambda_2. \tag{23}$$

In this case, we can introduce

$$\psi' = e^{-\frac{i}{2}(\beta'\phi'+\pi)\gamma_5}\psi \tag{24}$$

in terms of which the Lagrangian reads

$$\mathcal{L}_1 = i\bar{\psi}'\gamma^\mu\psi' + (2\lambda_1 a - \beta'/2)\bar{\psi}'\gamma^\mu\gamma_5\psi'\partial_\mu\phi' - 2\mu\lambda_2\bar{\psi}'\psi'$$

$$+ \frac{1}{2}(y + 2\lambda_1 a^2)(\partial_\mu\phi')^2 + 2b\mu^2\lambda_2\cos\beta'\phi'. \tag{25}$$

The derivative coupling term is easily gotten rid off with

$$\lambda_1 = \beta'/4a, \tag{26}$$

and what remains is

$$\mathcal{L}_1 = i\bar{\psi}'\gamma^{\mu}\partial_{\mu}\psi' - M_o\bar{\psi}'\psi' + \frac{1}{2}(\partial_{\mu}\phi)^2 + \frac{m^2}{\beta^2}\cos\beta\phi \,, \qquad (27)$$

with

$$M_o = 2\mu\lambda_2 \,, \qquad (28)$$

$$m^2/\beta^2 = 2b\mu^2\lambda_2 \qquad (29)$$

and

$$\phi = (y + 2\lambda_1 a^2)^{1_2} \phi', \quad \beta'\phi' = \beta\phi \,. \qquad (30)$$

Eq. (27) is, of course, the Lagrangian for a sine-Gordon field plus a massive free fermion.

From Eqs. (23) and (29) we get

$$m^2/\beta^2 = \mu M \,. \qquad (31)$$

M can be adjusted to be the classical soliton mass with

$$\mu = m/8 \qquad (32)$$

which gives for the second of the Lagrange multiplies the value

$$\lambda_2 = \left(\frac{32}{b}\right)\beta^{-2} \qquad (33)$$

obtained from Eqs. (23), (31) and (32).

If, as is suggested by Eq. (17), we take for the parameter a, in the constraint relation (18), the value

$$a = \beta'/2\pi \qquad (34)$$

the first Lagrange multiplier (Eq. (26)) turns out to be

$$\lambda_1 = \pi/2 \,. \qquad (35)$$

The four fermion coupling in the Lagrangian (18) is now almost completely determined. From Eqs. (22), (33) and (35) we have

$$g = \left(\frac{64}{b}\right)\beta^{-2} - \pi \ , \qquad (36)$$

which would reproduce Coleman's relation (14) if $b=16/\pi$. Why should such a special value for b arise at the classical level we have not been able to figure out.

With the manipulations we have just gone through, we have shown how to get from a massive Tirring plus massless free boson (plus constrainsts) model to a free fermion plus sine-Gordon model. The starting point could even be a pure Tirring model since ϕ' can be eliminated from Eq. (18) by putting the constant y equal to zero.

The journey from Tirring model to sine-Gordon can also be done in the oposite direction as will be shown in the next section.

V. FROM SINE-GORDON TO TIRRING MODEL

Now the starting point is a model with a free massive fermion plus a sine-Gordon field based on the Lagrangian

$$\mathcal{L}_2 = i\bar{\psi}'\gamma^\mu\partial_\mu \psi' - M_o\bar{\psi}'\psi' + \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{\beta^2} \ (\cos \beta\phi - 1) \ . \qquad (37)$$

In terms of the new fermion

$$\psi = e^{\frac{i}{2}(\beta\phi+\pi)\gamma_5} \psi' \qquad (38)$$

the Lagrangian, up to an additive constant, can be written as

$$\mathcal{L}_2 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{2\lambda_2 m^2}{\beta^2 M_o}\bar{\psi}\psi - (\lambda_2-\lambda_1)\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$$
$$+ \frac{1}{2}(1-\beta^2/8\lambda_1)(\partial_\mu\phi)^2 + \lambda_1(\bar{\psi}\gamma^\mu\gamma_5\psi + \frac{\beta}{4\lambda_1}\partial_\mu\phi)^2$$
$$+ \lambda_2\Big[\Big(\bar{\psi}\psi + \frac{M_o}{2\lambda_2}\cos\beta\phi + \frac{m^2}{\beta^2 M_o}\Big)^2 + \Big(i\bar{\psi}\gamma_5\psi - \frac{M_o}{2\lambda_2}\sin\beta\phi\Big)^2\Big] . \tag{39}$$

If once the theory is quantized, it turns out that all matrix elements of

$$(\bar{\psi}\gamma^\mu\gamma_5\psi + \frac{\beta}{4\lambda_1}\partial^\mu\phi),\ (\bar{\psi}\psi + \frac{M_o}{2\lambda_2}\cos\beta\phi + \frac{m^2}{\beta^2 M_o}),$$
$$(i\bar{\psi}\gamma_5\psi - \frac{M_o}{2\lambda_2}\sin\beta\phi), \tag{40}$$

vanish in the charge zero sector, then we effectively have in that sector a massive Tirring plus free boson model. The mass of the Tirring fermion comes out

$$M = \frac{2\lambda_2 m^2}{\beta^2 M_o} \tag{41}$$

With the procedure (plus wishful thinking) outlined in this section all the results of the last one (Eqs. (33), (35), (36), etc) are recovered. The boson ϕ can be eliminated from Eq. (39) as an independent dynamical field if we put

$$\beta^2 = 8\lambda_1 . \tag{42}$$

This, together with Eq. (35) would fix β at the value

$$\beta^2 = 4\pi \tag{43}$$

Eq. (43) is Skyrme's condition. He speculated[1] that in order to have an equivalence between sine-Gordon and Tirring models, Eq. (43) has to hold.

In the classical massive Tirring model one has

$$\partial_\mu j^\mu_5 = 2iM\bar{\psi}\gamma_5\psi \tag{44}$$

Thus, if the identifications (13), (15) and

$$s\ i\ \bar{\psi}\gamma_5\psi = m \sin \beta\phi \tag{45}$$

hold, it follows that

$$\Box\phi = -\frac{4\pi}{\beta}\ \frac{m^2}{\beta^2}\ \sin \beta\phi \tag{46}$$

which is the sine-Gordon equation if $\beta = 4\pi$ (Skyrme's condition again).

At the quantum level Eq. (44) is not so easy to prove. Eq. (45), on the other hand, follows directly from Eq. (12) obtained by Coleman.

VI. CONCLUSIONS

The work reported here grew from an atempt to understand, already at the classical level, the conditions for the sine-Gordon-Tirring model equivalence derived by Coleman using quantum perturbation techniques. We have seen how one can obtain in a simple manner the relations among coupling constants and masses given in Ref. (4). In particular, the nice relation (14) leading to the Eq. (16) which suggest the interpretation that the sine-Gordon boson is a bound state of fermion antifermion of the Tirring model. This, in turn, points to the identification of the massive Tirring fermion with the sine-Gordon soliton.

One difference with respect to Ref. (4) is that our simple treatment gives Eq. (14) up to an undetermined constant (the b in Eq. (36)).

Skyrme's condition $\beta = 4\pi$ appears to be related to the complete elimination of extra dynamical fields when the theory is described in the Tirring model language.

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