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ABSTRACT

We study the stability of soliton-like solutions of a classical relativistic field theory with logarithmic non-linearities. For a given charge (not greater than a certain value q_{max}) the model exhibits a stable state and an unstable one. The latter state instead of being nonphysical can be interpreted as a resonance. Problems related to the semi-classical quantization of this theory are also discussed. All calculations are done in any number of spatial dimensions.

1. Introduction

Due to their nonperturbative character, the semiclassical approximations to quantum field theories have gained considerable attention nowadays[1,2,3]. Among these methods of quantization we can cite the Bohr - Sommerfeld quantization rule (BSQR)[1,4,5] and the extension of WKB to field theory[2,3].

In order to implement any one of the mentioned methods, exact and well behaved (with finite energy) solutions of the classical equations of motion are needed. In this context, the question of the stability of these classical solutions happens to be a crucial one. The definition of stability commonly used[1] can be seen as an extension to classical field theories of Liapunov's criterion of stability introduced in classical mechanics[6].

Recently Bialynicki-Birula and Mycielski[7,8] studied two classical equations with logarithmic nonlinearities - the Schrödinger -like and the Klein-Gordon-like equation. In both cases they were able to obtain exact soliton-like solutions of the Gaussian type. We intend to study the stability of the solutions of their relativistic equation on the light of the stability criterion mentioned above.

An interesting feature of these theories is that their classical version and the stability equations can be solved in any dimension of the space-time, and in particular in the 1+3 dimensional real world.

The relativistic model is presented in section 2. There, its stability equation is also solved. Our conclusion is that for a given value of the charge (not greater than a certain value q_{max}) there exist a stable soliton - which we interpret as a bound state- and an unstable one - that we interpret as a resonance, since its mass is larger than that of the former particle.

These questions are discussed in details in section 3. There, we also conclude that when the charge modulus is greater than q_{max} there is no more bound state, nor resonance.

Section 4 is left to conclusions and discussions about the possibility of quantizing the model.

2. The stability of solitons in the relativistic theory

In ref. (7) Birula and Mycielski studied nonlinear wave equations with logarithmic nonlinearities. They were mainly concerned with a nonrelativistic version of the model, but a certain class of relativistic classical solitons were also obtained by them.

The relativistic theory is the one studied by us. It is defined by the following Lagrangian density[8]

$$\mathcal{L}(\vec{x},t) = \partial_\mu \phi^* \partial^\mu \phi - (\bar{\lambda}^2 + \bar{\ell}^2)\,\phi^*\phi + \bar{\ell}^2 \ln(\phi^*\phi a^{n-1})\phi^*\phi, \qquad (2.1)$$

where ϕ is a charged scalar field and λ , ℓ and a are dimensional parameters of the theory. n stands for the number of spatial dimensions that will remain unspecified.

The Euler-Lagrange equation corresponding to the model defined by (2.1) is

$$\left[\Box + \bar{\lambda}^2 - \bar{\ell}^2 \ln(\phi^*\phi a^{n-1})\right]\phi = 0 \, . \qquad (2.2)$$

In the soliton rest frame, the soliton-like solution of interest for us is[7,8]

$$\phi_\omega(\vec{x},t) = A\, exp\left(- \vec{x}^2/2\ell^2 - i\omega t\right), \qquad (2.3)$$

where A is such that

$$\ln\left(|A|^2 a^{n-1}\right) = n + (\ell/\lambda)^2 - (\omega\ell)^2. \qquad (2,4)$$

In what follows we consider A as a real number, without loss of generality.

The study of stability a la Liapanov[6] can be implemented if we add to $\phi_\omega(\vec{x}\cdot t)$ a small fluctuation

$$\phi(\vec{x},t) = \phi_\omega(\vec{x},t) + e^{-i\omega t}\eta(\vec{x}\cdot t). \qquad (2.5)$$

Inserting $\phi(\vec{x}\cdot t)$ into Eq. (2.2), and retaining only terms of first order in $\eta(\vec{x}\cdot t)$, we obtain the linearized stability equation[1,2,3]:

$$\left(\frac{\partial^2}{\partial t^2} - 2i\omega\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - (n+1)/\ell^2 + \vec{x}^2/\ell^4\right)\eta = (1/\ell^2)\,\eta^*. \qquad (2.6)$$

Now, we say that the soliton described by $\phi_\omega(\vec{x}\cdot t)$ is stable if $|\eta(\vec{x}\cdot t)|$ is bounded for all t, i.e., for a given $\vec{x}$ there exists a $\eta_{sup}(\vec{x})$, positive and finite, such that the inequality

$$\eta(\vec{x}\cdot t) < \eta_{sup}(\vec{x}) \qquad (2.7)$$

holds for any value of t.

A typical solution of Eq. (2.6) is of the form

$$\eta_{m_1\dots m_n}(\vec{x},t) = \left[\prod_{j=1}^{n} h_{m_j}(x_j/\ell)\right]\cdot$$

$$\cdot\left[a_{m_1\dots m_n}\exp(i\gamma_{m_1\dots m_n}t) + b_{m_1\dots m_n}\exp(-i\gamma^*_{m_1\dots m_n}t)\right], \tag{2.8}$$

where h_{m_j} is the m_j^{th} eigensolution of the unidimensional harmonic oscilator.

Here we point out that in order to have stability in the sense of (2.7), all $\gamma_{m_1\dots m_n}$ must be real.

Substituting (2.8) into Eq. (2.6) we obtain an homogeneous linear system for $a_{m_1\dots m_n}$ and $b_{m_1\dots m_n}$. Such homogeneity leads to the vanishing of the determinant of that system, and we get

$$\left(\gamma^{\pm}_{m_1\dots m_n}\right)^2 = (2/\ell^2)\left[\alpha_{m_1\dots m_n} \pm \left(\alpha^2_{m_1\dots m_n} - \beta_{m_1\dots m_n}\right)^{1/2}\right], \tag{2.9}$$

where

$$\alpha_{m_1\dots m_n} = (\omega\ell)^2 - 1/2 + \sum_j m_j \tag{2.10.a}$$

and

$$\beta_{m_1\dots m_n} = \left(\sum_j m_j\right)\left[\left(\sum_j m_j\right) - 1\right]. \tag{2.10.b}$$

Since $\beta_{m_1\dots m_n}$ is positive semidefinite we shall separate the set $\{m_j\}$ into two kinds of sets

a) $\{m_j\}_1$ is such that $\beta_{m_1\dots m_n} > 0$. In this case the $\gamma^{\pm}_{m_1\dots m_n}$ are non zero real, and it does not emerge any suspicion of instability.

b) $\{m_j\}_2$ is such that $\beta_{m_1\dots m_n} = 0$. This case is realizide in two situations (see (2.10.b)):

b.1) $m_k = 1$, and $m_j = 0$ for all j $\neq$ k. Here we have

$$\left(\gamma^-_{0,0\ldots 1\ldots 0}\right)^2 = 0 . \qquad (2.11.a)$$

Note that this zero of $\left(\gamma^-_{0,0\ldots 1\ldots 0}\right)^2$ has a degeneracy of degree n. This is a consequence of translational invariance in any one of the n spatial dimensions. In the language of refs. (2) and (3) we should say that these are zero frequency modes associated to invariance under translation.

The value of $\left(\gamma^+_{0,0\ldots 1\ldots 0}\right)^2$ will be

$$\left(\gamma^+_{0,0\ldots 1\ldots 0}\right)^2 = \left(2/\ell^2\right)\left[(\omega\ell)^2 + 1/2\right] . \qquad (2.11.b)$$

Here again we do not have any reason for having instability because the $\gamma^+_{0,0\ldots 1\ldots 0}$ are all nonzero real numbers.

b.2) $m_j = 0$ for all j. Now we have

$$\left(\gamma^-_{0,0\ldots 0}\right)^2 = 0 . \qquad (2.11.c)$$

This zero of $\left(\gamma^-_{0,0\ldots 0}\right)^2$ has a degeneracy of order 1 that is associated to the gauge invariance of first kind or to the the invariance under time translation-which for solutions of type (2.3) is equivalent to gauge invariance.

The trouble with instability becomes evident when we look to the value of $\left(\gamma^+_{0,0\ldots 0}\right)^2$

$$\left(\gamma^+_{0,0\ldots 0}\right)^2 = \left(2/\ell^2\right)\left[(\omega\ell)^2 - 1/2\right] . \qquad (2.11.d)$$

From (2.11.d) we note that there exists complex values of $\gamma^+_{0,0\ldots 0}$ whenever

$$\omega^2 < \omega_c^2 = 1/2\ell^2 , \qquad (2.12)$$

where by ω_c we mean a critical value of ω . Thus, all solitons, for which (2.12) holds, are unstable. Otherwise they are stable.

In the next section we shall study the consequences of the conditional stability (2.12).

An interesting question emerges when $\omega = \frac{1}{2} e^2$. In this case $\gamma^{+}_{0,0,\dots 0}$ is also zero and the solution associated to that value of ω is expected to have an additional symmetry. This problem will be clarified in the next section.

3. The spectrum: Bound states and resonances

Here we will obtain the classical spectrum of the model whose Lagrangian density is given by (2.1). Based upon the results of this section we shall see that the unstable classical solutions-instead of being discarded- can be interpreted as resonances.

The energy is computed by integrating the classical Hamiltonian density

$$\mathcal{H}(\vec{x},t) = 2\left(\frac{\partial \phi^*}{\partial t}\right)\left(\frac{\partial \phi}{\partial t}\right) - \mathcal{L}(\vec{x},t). \tag{3.1}$$

A classical soliton of the model considered here is characterized by its energy and another conserved quantity: the charge. This is due to the invariance under gauge transformation of first kind. The charge of a given state can be obtained from the classical charge density

$$\rho(\vec{x},t) = i\left[\left(\frac{\partial \phi^*}{\partial t}\right)\phi - \phi^*\left(\frac{\partial \phi}{\partial t}\right)\right]. \tag{3.2}$$

After plugging (2.3) into (3.1) and integrating we shall get the energy of the solution of frequency ω (ϕ_ω)

$$E(\omega) = (C/\ell)\left[(\ell\omega)^2 + 1/2\right] \exp\left[-(\ell\omega)^2\right], \quad (3.3)$$

where

$$C = 2\sqrt{\pi}\left(\ell\sqrt{\pi}/a\right)^{n-1} \exp\left[n + (\ell/\lambda)^2\right] \quad (3.4)$$

The charge associated to ϕ_ω is computed by integrating (3.2). Doing so, we get

$$q(\omega) = C\,(\ell\omega)\exp\left[-(\ell\omega)^2\right]. \quad (3.5)$$

In fig. (1.a) and (1.b) we plot respectively the energy and the charge as a function of ω^2 (recall that $q(-\omega) = -q(\omega)$). From these figures we see that the energy, as well as the modulus of the charge, attains its maximum value just for that value of ω that we have named critical: $\omega_c^2 = 1/2\,\ell^2$.

We can also see that for a given value of the charge q (not greater than q_{max}) there exist two soliton-like solutions of Eq. (2.2) having different energies - as shown in fig. (2). This fact becomes more interesting when we observe that the state of lower energy is stable, whereas the other is unstable. In this context we are tempted to interpret the heaviest particle (the unstable one) as a resonance, instead of discarding it.

In fig. (2) $E^0(q)$ is the energy of the bound state of charge q, whereas $E^*(q)$ is the energy of the resonance. An elementary analysis shows that $E^0(q)$ is a concave function, i.e.

$$E^0(q_1 + q_2) < E^0(q_1) + E^0(q_2). \quad (3.6)$$

Then the decay of a bound state of charge $q_1 + q_2$ into two others of charges q_1 and q_2 respectively is forbidden, because the final system should have a negative kinetic energy(9).

The preceding discussion justifies the stability of the bound states already at the classical level.

Inequality (3.6) does not holds if on its left hand side we put $E^*(q_1+q_2)$ - instead of $E^0(q_1+q_2)$. Then, we cannot forbid the decay of a resonance of charge q_1+q_2 into two bound states of charges q_1 and q_2 . From this fact we have an understanding of the instability of the resonance at the classical level.

In conclusion, we expect that in a given charge sector of the quantized version of this theory there exist a resonance and a bound state. The resonance (that is unstable) can decay, whereas the bound state cannot. The stable and unstable solitons that we have studied are supposed to be classical manifestations of those quantum objects.

Now, we will answer the question rised in the final part of section 2. The question is: what is the additional symmetry associated to the zero frequency mode gained when $\omega = \omega_c$? As mentioned above, the point ω_c is a point of maximum for the energy and for the modulus of the charge. This means that both these quantities are stationary under small variations of frequency , i.e.:

$$E\,(\omega_c + \delta\omega) = E_{max} - O\left[(\delta\omega)^2\right] \qquad (3.7.a)$$

and

$$q(\omega_c + \delta\omega) = q_{max} - O\left[(\delta\omega)^2\right]. \qquad (3.7.b)$$

Then, for $\omega = \omega_c$, all quantities of physical interest (the energy and the charge) will not change under small fluctuations of ω, and we can say that the additional symmetry gained is the invariance under infinitesimal "frequency translations".

4. On the possibility of quantizing the theory

In this section we report some difficulties one confronts and some hopes one sees when trying to quantize the model, whose Lagrangian is given by (2.1), by means of the WKB method.

At first we mention that a quantization a la Bohr-Sommerfeld can easily be done, since for theories invariant under gauge transformation of first kind this quantization is equivalent to charge quantization - as shown in ref. (5). Then, in order to perform the BSQR it is sufficient to impose that $|q| = 0, 1, \ldots \leq |q_{max}|$. Of course, this procedure leads to a discrete spectrum of bound states and resonances that can be seen in fig. (2), taking only integer values of $|q|$.

Things are not so simple in the case of WKB. First of all, one observes that the interaction Lagrangian is not analytical at $|\phi|^2 = 0$ (the vacuum). Then, the introduction of fluctuations about this value of the field cannot eventually be a legitimate procedure. As a consequence of this we have troubles in subtracting the vacuum energy from the soliton energy in the process of renormalization.

Our hope is that there exists a special type of counterterms - different from the usual ones [2,3] - that are appropriate for the present problem.

We point out that the process of renormalization can lead to the appearance of resonance widths, because the sum of the resonance stability angles[2,3] has a finite imaginary part.

Another delicate point deserving caution is that the potential is not bounded from below

$$V(|\phi|) \underset{|\phi| \to \infty}{\simeq} - \ln|\phi|\,|\phi|^2. \qquad (4.1)$$

The quantum mechanics (field theory with zero spatial dimension) of this potential does not addmits states confined near the origin of the potential, since the particle can tunnel through the barrier.

We expect that in field theory (1 or more spatial dimension) the tunnelling will be forbidden by topollogical reasons, as is the case for the soliton in the sine-Gordon Theory. Although the quantum mechanics of the potential cos χ has a continuous tunnelling - manifested by its band structure, with no bound states - a soliton sector on its field theoretical version cannot tunnel to another one.

Finally we mention that the stability of the nonrelativistic solitons of ref. (7) can be studied by a similar procedure. The conclusion is that the nonrelativistic solutions are all stable.

FOOTNOTES AND REFERENCES

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(6) See, for instance, H.Leipholz - Stability Theory, Academic Press (1970).

(7) I.Bialynicki- Birula and J.Mycielski - Bull.Acad.Polon.Sci. Cl. III, Vol. XXIII, nº 4 (1975).

(8) These theories were also studied by P.Schick - Wisconsin preprint C00 - 479 (1975).

(9) In ref. (5) we study the stability of the semiclassical spectrum of the massive Thirring Model. There, a similar phenomenum occurs.

FIGURE CAPTIONS

FIG. 1 : Energy and charge as functions of the frequency of the classical soliton

FIG. 2 : Energy of the soliton as function of its charge. The continuous curve refers to the bound states, whereas the dashed one refers to the resonances.

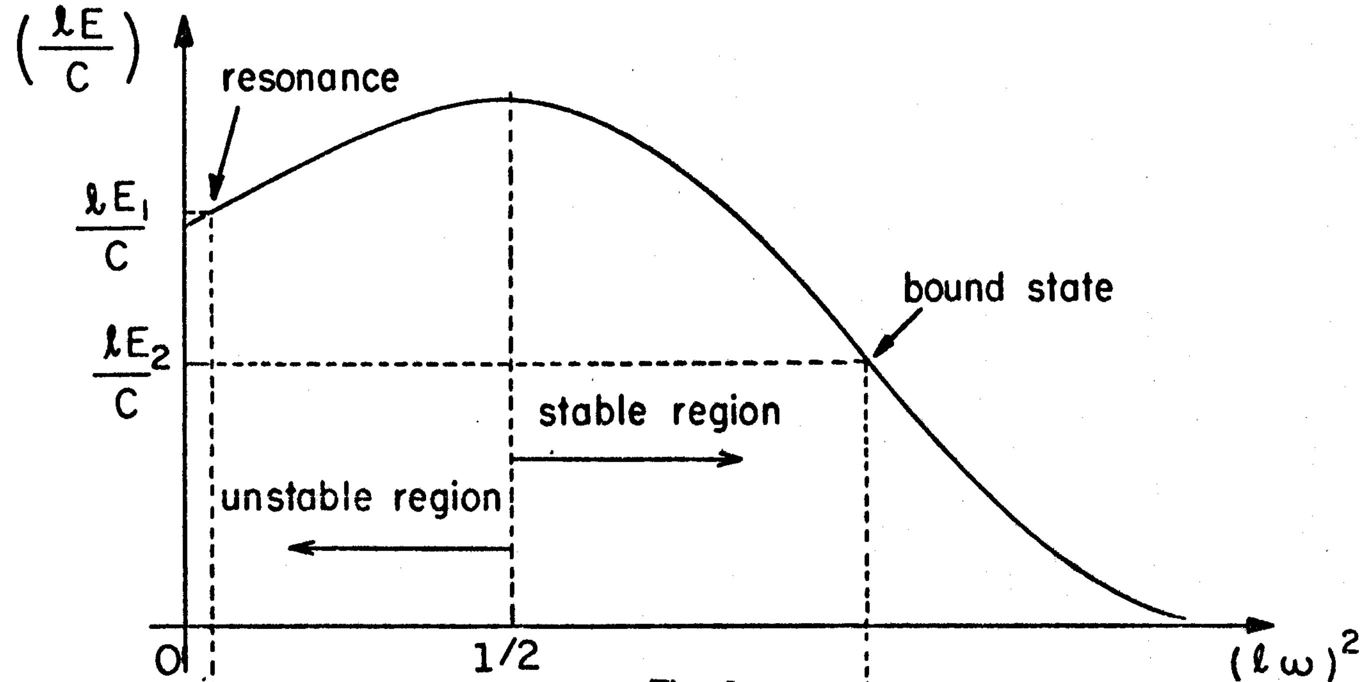

$\left(\frac{\ell E}{C}\right)$
resonance
$\frac{\ell E_1}{C}$
$\frac{\ell E_2}{C}$
bound state
stable region
unstable region
O
1/2
$(\ell\omega)^2$

Fig. 1a

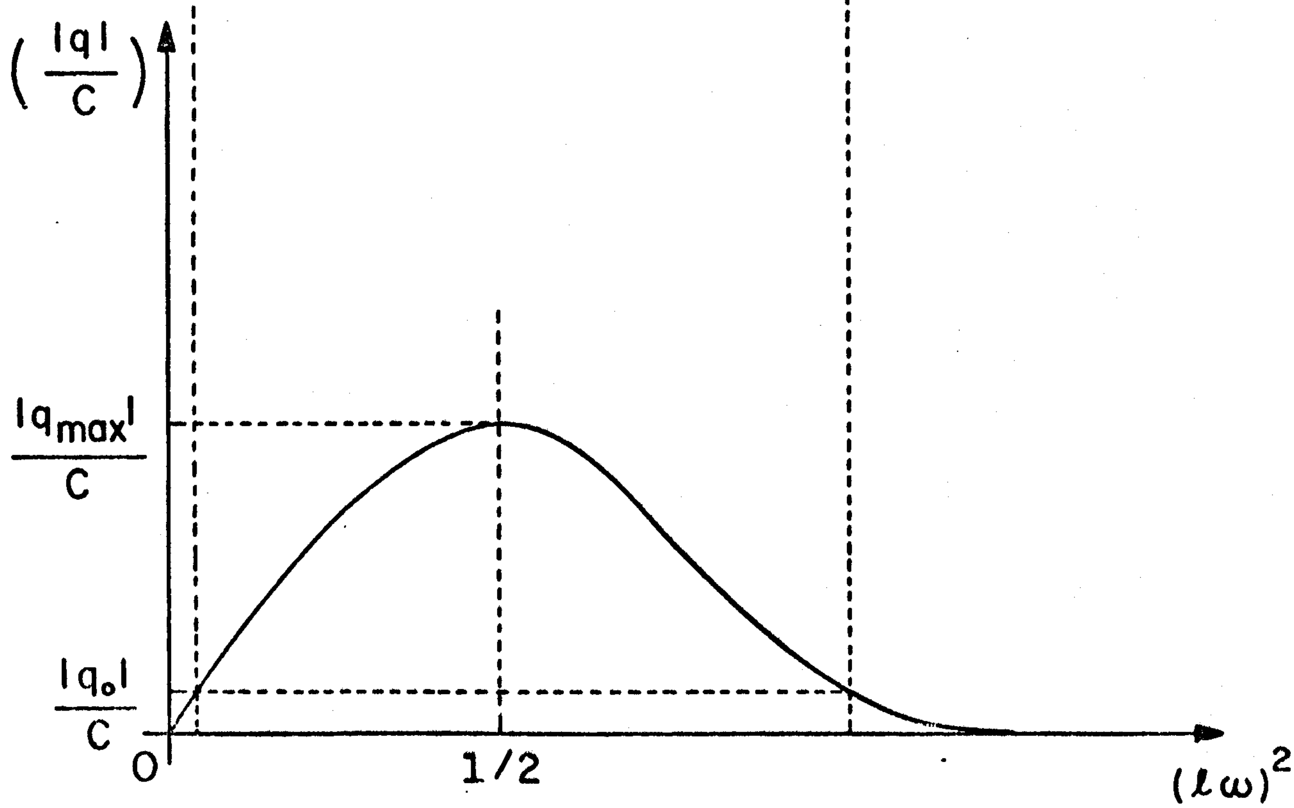
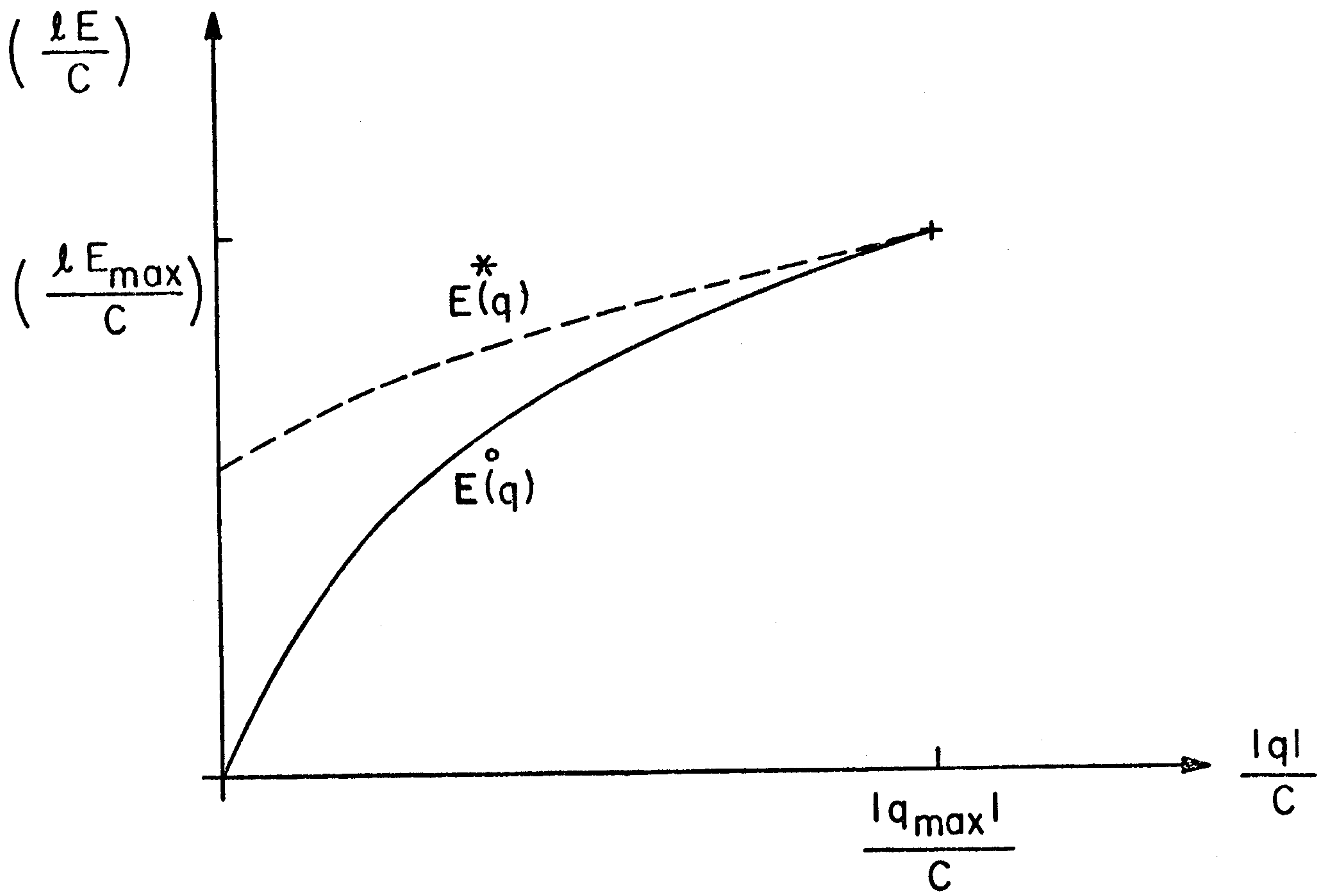

$\left(\frac{|q|}{C}\right)$
$\frac{|q_{max}|}{C}$
$\frac{|q_o|}{C}$
O
1/2
$(\ell\omega)^2$

Fig. 1b



$\left(\frac{\ell E}{C}\right)$
$\left(\frac{\ell E_{max}}{C}\right)$
$\overset{*}{E}(q)$
$\overset{\circ}{E}(q)$
$\frac{|q|}{C}$
$\frac{|q_{max}|}{C}$

Fig. 2