IFUSP/P-122

SOME ASPECTS OF THE CLASSICAL MOTION OF EXTENDED CHARGES

H.M.França, G.C.Marques and A.J.da Silva

Instituto de Física, Universidade de São Paulo,SP

Brasil

SOME ASPECTS OF THE CLASSICAL MOTION OF EXTENDED CHARGES

H.M.França, G.C.Marques and A.J. da Silva

Instituto de Física, Universidade de São Paulo, S.P. Brasil.

A B S T R A C T

Starting from a covariant equation, we have shown how a consistent equation of motion and an equation for the energy balance are achieved in the nonrelativistic limit. In that limit we can integrate such an equation for any charge distribution by finding its Green's function. Conclusions concerning to the existence of unphysical solutions (and explicit solutions for a large class of models) can be drawn from an analysis of the location of the poles of the Green's function. A new condition for the nonexistence of pathological solutions is discussed.

JULHO/1977

I. INTRODUCTION

The theory of the classical motion of point-like particles (taking into account the effect of the radiation reaction) is beset by troubles which can be circunvented at the price of a violation of causality. The time scale in which such a violation occurs suggests that a consistent description would be achieved only within the quantum theory(1). If we are aiming at a consistent theory at the classical level the alternative is to assume that the particle is an extended object.

The equation of motion for charged particles, taking into account the reaction due to the radiation was originally developed by Abraham, Lorentz and Dirac. For spherically symmetric charges of finite extension such an equation, as derived in Jackson's book(2), can be written as:

$$m_0 \frac{d^2 \mathbf{z}(t)}{dt^2} = \mathbf{F}_{ext}(t) - \frac{2}{3} \sum_{n=0}^{\infty} \gamma_n \frac{(-1)^n}{n!} \frac{d^{n+2}}{dt^{n+2}} \mathbf{z}(t) \tag{I.1}$$

where m_0 is the mechanical mass, $\mathbf{F}_{ext}(t)$ is the external force, $\mathbf{z}(t)$ is the center of charge position and the γ_n are given, in terms of the charge density $\rho(\mathbf{x},t)$, by

$$\gamma_n = \int d^3x \int d^3x' \, \rho(\mathbf{x},t) \, \rho(\mathbf{x}',t) \, |\mathbf{x} - \mathbf{x}'|^{n-1} \tag{I.2}$$

One defect which plagues the classical theory of charged particles is the manifestation of runaway solutions and preaccelerations. Based upon the study of simple charge distributions[1,3] it has been stated that the theory is free of such unphysical solutions as long as

$$m_0 > 0 \tag{I.3}$$

One striking feature of condition (I.3) is that it has nothing to do with electromagnetic interactions (m_0 is of non electromagnetic nature).

Another undesirable feature of (I.1) is the 4/3 factor in the contribution of the electromagnetic self energy to the mass of the particle[4]. It is known that an equation obtained as the non relativistic limit of a Lorentz covariant equation of motion should circunvent such a problem. A Lorentz covariant formulation of classical electrodynamics has been tackled by some authors[5-7]. Based upon these results we show in section II that in the nonrelativistic domain ($v \ll 1$) the equation of motion presents some new nonlinear terms besides a modification of the linear ones.

For spherically symmetric charge distributions the linearized equation differs from (I.1) only by the fact that the mechanical mass m_0 is replaced by $m_0 - \frac{m_e}{3}$ (here m_e is the electromagnetic mass).

Another way of studying the motion of extended particles, which has been used by several authors[6,8], is a method based upon an integral equation. We will show that this method and the one based upon equation (I.1) are equivalent.

We shall show in section III, that by using the integral equation most results which has been got for very specific charge distributions can actually be extended to any charge distributions. The equation of motion can be integrated, for any (time dependent) applied external force, by finding its Green's function.

It is shown here that in order to get explicit solutions for a very large class of charge distributions, all we need to do is to solve an eigenvalue equation. An analysis of this equation shows that the theory does not violates the principle of causality as long as:

$$m_0 - \frac{m_e}{3} > 0 \tag{I.4}$$

As has been pointed out first by Kaup[6], the change of condition (I.3) to (I.4) is a consequence of the requirement of formulating the theory in a Lorentz covariant way from the very beginning. In this context it is worth noting that the $m_e/3$ term in (I.3) is the one that fixes the 4/3 puzzle. More than that, however, we will show that this term actually leads us to a very interesting, unsuspected up to now, picture on the question of why unphysical solutions appear. We have no unphysical solutions as long as the charge does not exceeds a critical value which depends on the radius of the particle (assumed fixed, within this framework).

By taking the nonrelativistic limit of the fourth component of the relativistic equation of motion we obtain, in section IV, an equation for the energy balance. The formulae will be worked out in such a way that all terms in this equation can be easily interpreted. As a result we get explicit expressions for the radiated power and Schott's energy for extended particles.

Section V is reserved to the discussion of simple examples. They are aimed at a better understanding of some points discussed in the paper.

The conclusions that can be drawn are presented in section VI.

II. NONRELATIVISTIC LIMIT OF THE CLASSICAL EQUATION OF MOTION

If one denotes by dp_μ/ds the rate of change of the mechanical 4-momentum of a particle with respect to its proper time, the relativistically covariant equation of motion of an extended particle under the influence of electromagnetic fields is given by[6]

$$\frac{dp_\mu}{ds} = \int F_{\mu\nu}(x)\, J^\nu(x)\left[1 - (x-z)\cdot a\right] u^\rho\, d\sigma_\rho \qquad \text{(II.1)}$$

Where $F_{\mu\nu}(x)$ stands for the electromagnetic tensor and $J^\nu(x)$ the electromagnetic current. The integration hyperplane σ_ρ is taken orthogonal to the 4-velocity of the particle, x_μ stands for an arbitrary point in such a hyperplane and z_μ (the charge center) is the point where the world line (whose lenght is s) intersects it, $u_\nu = dz_\nu/ds$ is the 4-velocity and $a_\nu = du_\nu/ds$ is the 4-acceleration of the particle.

Equation (II.1) was obtained from the energy-momentum conservation and does not involve the hypothesis that the particle is rigid (a similar equation has been obtained by Nodvik[7] under the rigidity assumption).

The usefulness of (II.1) is limited due to our lack of knowledge of the nonelectromagnetic structure of the particle. A way of gaining some insight on the motion of extended charges is a phenomenological approach(*) in which one imposes simple properties on the particle. Assuming that the particle has a fixed shape in its rest frame one can construct the electromagnetic current.[7] Under this hypothesis, and assuming that the particle is non rotating, the mechanical momentum can be written as[6] $p_\nu = m_0 u_\nu$.

Upon separating the electromagnetic field tensor into a self and external contributions

$$F_{\mu\nu}(x) = F^{ext}_{\mu\nu}(x) + F^{self}_{\mu\nu}(x) \qquad \text{(II.2)},$$

expanding the external fields in (II.1) around the world line and neglecting dipole moments and higher order terms (the particle is assumed to be spherically symmetric in its rest frame) we get

$$m_0 \frac{du_\nu}{ds} = -q F^{ext}_{\nu\mu}(z)\, u^\mu - \int F^{self}_{\nu\mu}(x)\, J^\mu(x) \left[1-(x-z)\cdot a\right] u^\rho \, d\sigma_\rho \qquad \text{(II.3)}$$

Where q is the electric charge carried out by the particle namely

$$q = \int J_\nu(x)\, d\sigma^\nu \qquad \text{(II.4)}$$

(*) We are presently investigating a nonphenomenological approach in which extended particles are associated to classical solutions of nonlinear field theoretical equations (solitons).

Throughout this paper we shall be mainly concerned with the nonrelativistic limit of the classical motion. Under the assumptions made previously, the nonrelativistic limit of the $J_\mu(x)$ current is

$$J_\mu(x) = (1, \dot{\mathbf{z}}(t))\,\rho(|\mathbf{x}-\mathbf{z}(t)|) \equiv (1, \dot{\mathbf{z}}(t))\,\rho(\mathbf{x}, t) \qquad \text{(II.5)}$$

Where $\mathbf{z}(t)$ is the classical charge center position and $\dot{\mathbf{z}}(t)$ is its first time derivative.

The nonrelativistic equation is obtained by considering the $v \ll 1$ limit of (II.3). One difficulty which one faces with in this limit is that small velocities do not lead right away to a linear nonrelativistic equation. In fact, the derivation presented in Jackson's book suggests that this equation is a nonlinear one. We shall cast the equation (II.3) into a form which permits us to see that the neglect of these nonlinearities is, in fact, justified.

The spatial component of (II.3), in the $v \ll 1$ limit assumes the form:

$$m_0 \ddot{\mathbf{z}}(t) = \mathbb{F}_{ext} + \mathbb{R} \qquad \text{(II.6)}$$

Where the term in (II.6) which takes into account the effect of the self electromagnetic interaction is $\mathbb{R}$. After dropping the magnetic force term (which is negligible in the $v \ll 1$ limit) and denoting the 4-vector electromagnetic potential by $A_\mu(x) = (A_0, \mathbb{A})$, the explicit expression for $\mathbb{R}$ is

$$\mathbb{R} = -\int d^3x \left(\frac{\partial \mathbb{A}}{\partial t} + \nabla A_0\right)\rho(\mathbf{x}, t)\left[1 + (\mathbf{x} - \mathbf{z}(t)) \cdot \ddot{\mathbf{z}}(t)\right] \qquad \text{(II.7)}$$

We shall separate $\mathbb{R}$ into two terms:

$$\mathbb{R} = \mathbb{R}_1 + \mathbb{R}_2 \tag{II.8}$$

where $\mathbb{R}_1$ and $\mathbb{R}_2$ are defined as:

$$\mathbb{R}_1 = -\int d^3x\, \rho(\mathbb{x},t)\left[1 + (\mathbb{x} - \mathbb{z}(t)) \cdot \ddot{\mathbb{z}}(t)\right] \nabla A_0 \tag{II.9}$$

$$\mathbb{R}_2 = -\int d^3x\, \rho(\mathbb{x},t)\left[1 + (\mathbb{x} - \mathbb{z}(t)) \cdot \ddot{\mathbb{z}}(t)\right] \frac{\partial \mathbb{A}}{\partial t} \tag{II.10}$$

In order to present a clear discussion on the negligibility of the nonlinear contributions it is prefereable to work in the Coulomb gauge[2].

In this gauge we have:

$$A_0(\mathbb{x},t) = \int d^3x' \frac{\rho(\mathbb{x}',t)}{|\mathbb{x} - \mathbb{x}'|} \tag{II.11}$$

and

$$\mathbb{A}(\mathbb{x},t) = \int d^3x' \frac{\mathbb{J}_\perp(\mathbb{x}', t - |\mathbb{x} - \mathbb{x}'|)}{|\mathbb{x} - \mathbb{x}'|} \equiv$$

$$\frac{4\pi}{(2\pi)^{3/2}} \int_{-\infty}^{t} dt' \int \frac{d^3k}{k}\, \tilde{\rho}(\mathbb{k}) \sin[k(t-t')] \frac{\mathbb{k} \times [\dot{\mathbb{z}}(t') \times \mathbb{k}]}{k^2} \exp\left[i\,\mathbb{k} \cdot (\mathbb{x} - \mathbb{z}(t'))\right] \tag{II.12}$$

where $\tilde{\rho}(\mathbb{k})$ is defined by:

$$\rho(\mathbb{x},t) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, \tilde{\rho}(\mathbb{k}) \exp\left[i\,\mathbb{k} \cdot (\mathbb{x} - \mathbb{z}(t))\right] \tag{II.13}$$

It is straightforward to check that, if one plugs (II.11) into (II.9), we shall get, for spherically symmetric charge distributions

$$\mathbb{R}_1 = \frac{m_e}{3} \ddot{\mathbf{z}}(t) \tag{II.14}$$

where m_e is the electromagnetic mass:

$$m_e = \frac{1}{2}\int d^3x \int d^3x' \frac{\rho(\mathbf{x},t)\,\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} \tag{II.15}$$

whereas the $\mathbb{R}_2$ term can be written as:

$$\mathbb{R}_2 = 4\pi \int_{-\infty}^{t} dt' \int d^3k\, \tilde{\rho}(\mathbf{k}) \cos[k(t-t')] \frac{\mathbf{k}\times[\dot{\mathbf{z}}(t')\times\mathbf{k}]}{k^2} \times$$

$$\exp[i\mathbf{k}\cdot(\mathbf{z}(t)-\mathbf{z}(t'))]\left(1 - i\ddot{\mathbf{z}}(t)\cdot\nabla_k\right)\tilde{\rho}^*(\mathbf{k}) \tag{II.16}$$

If one change variables, calling $\tau = t-t'$, (II.16) assumes the form

$$\mathbb{R}_2 = -4\pi \int_0^{\infty} d\tau \int d^3k\, \tilde{\rho}(\mathbf{k}) \cos k\tau \frac{\mathbf{k}\times[\dot{\mathbf{z}}(t-\tau)\times\mathbf{k}]}{k^2} \times$$

$$\exp[i\mathbf{k}\cdot(\mathbf{z}(t)-\mathbf{z}(t-\tau))]\left(1 - i\ddot{\mathbf{z}}(t)\cdot\nabla_k\right)\tilde{\rho}^*(\mathbf{k}) \tag{II.17}$$

Expression (II.16) confirms our previous assertion that it is not clear, a priori, that the resulting nonrelativistic equation of motion is a linear one. In this context we would like to say that a fully covariant treatment introduces nonlinear contributions to the equation of motion. That all non-linear terms in (II.17) can be neglected, can be justified as follows[8].

Let us consider ther term $\mathbf{z}(t)-\mathbf{z}(t-\tau)$ which appears in (II.17). Such a term is essentially the distance covered by the particle during the time interval τ. The relevant region of integration in the τ variable is the one such that $\tau \sim 2a$ where a is the radius of the particle, since this is the time needed for the electromagnetic disturbance to cross the particle. Consequently we can say that

$$\mathbf{z}(t)-\mathbf{z}(t-\tau) \simeq \dot{\mathbf{z}}(t)\,2a + O(a^2) \qquad \text{(II.18)}$$

If one looks back at (II.17), then in the $v \ll 1$ limit and in view of (II.18), one can safely replaces

$$\dot{\mathbf{z}}(t-\tau)\exp\left[i\mathbf{k}\cdot(\mathbf{z}(t)-\mathbf{z}(t-\tau))\right] \longrightarrow \dot{\mathbf{z}}(t-\tau) \qquad \text{(II.19)}$$

Under the approximation (II.19) we shall get for (II.17)

$$\mathbb{R}_2 = -4\pi\int_0^\infty d\tau\int d^3k\,\tilde{\rho}(k)\cos k\tau\,\frac{\mathbf{k}\times[\dot{\mathbf{z}}(t-\tau)\times\mathbf{k}]}{k^2}\left(1-i\,\ddot{\mathbf{z}}(t)\cdot\nabla_k\right)\tilde{\rho}^*(\mathbf{k}) \qquad \text{(II.20)}$$

Integrating by parts with respect to τ we obtain

$$\mathbb{R}_2 = -4\pi\int_0^\infty d\tau\int d^3k\,\tilde{\rho}(k)\frac{\sin k\tau}{k}\,\frac{\mathbf{k}\times[\ddot{\mathbf{z}}(t-\tau)\times\mathbf{k}]}{k^2}\left(1-i\,\ddot{\mathbf{z}}(t)\cdot\nabla_k\right)\tilde{\rho}^*(\mathbf{k}) \qquad \text{(II.21)}$$

It is worth noting that the approximation (II.19) is not sufficient for getting rid of all nonlinear terms. However, it can be checked that the surviving non linearity (the term proportional to $\ddot{\mathbb{Z}}(t).\ddot{\mathbb{Z}}(t-\tau)$ in (II.21)) gives no contribution at all for spherical charges. Thus, after angular integration, we have

$$\mathbb{R}_2 = -\frac{32\pi^2}{3}\int_0^\infty d\tau\, \ddot{\mathbb{Z}}(t-\tau)\int_0^\infty k\,dk\,|\tilde{\rho}(k)|^2 \sin k\tau \tag{II.22}$$

From (II.22), (II.14) and (II.8), equation (II.6) becomes, in configuration space,

$$\left(m_0 - \frac{m_e}{3}\right)\ddot{\mathbb{Z}}(t) = \mathbb{F}_{ext}(t) - \frac{2}{3}\int d^3x \int d^3x' \,\frac{\rho(\mathbf{x},t)\,\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,\ddot{\mathbb{Z}}(t-|\mathbf{x}-\mathbf{x}'|) \tag{II.23}$$

We would like to mention at this point that we can cast (II.23) into two different forms. The first one consists in expanding $\ddot{\mathbb{Z}}(t-|\mathbf{x}-\mathbf{x}'|)$ by using the formal expression

$$\left[\ddot{\mathbb{Z}}(t)\right]_{ret} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}|\mathbf{x}-\mathbf{x}'|^n \frac{d^n}{dt^n}\ddot{\mathbb{Z}}(t) \tag{II.24}$$

Doing so, (II.23) can be written as:

$$m\frac{d^2\mathbb{Z}}{dt^2}(t) = \mathbb{F}_{ext}(t) - \frac{2}{3}q^2\frac{d^3\mathbb{Z}}{dt^3}(t) + \sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\gamma_n\frac{d^{n+2}}{dt^{n+2}}\mathbb{Z}(t) \tag{II.25}$$

where m is the physical mass ($m=m_0+m_e$) and the γ_n coefficients are defined in (I.2). Except by the fact that the term which appears multiplying the acceleration in (II.25) is the physical mass (in contradistinction with (II.1) which leads to an incorrect mass coefficient), the equation (II.25) is equivalent to (I.1).

An alternative way of writing (II.23) is to keep it under the form of an integral equation. That can be done if we recall the expression (II.22) and define

$$H(\tau) = \frac{32\pi^2}{3}\int_0^\infty k\,dk\,|\tilde{\rho}(k)|^2 \sin k\tau \equiv$$

$$\equiv \frac{8\pi}{3}\tau\int d^3x'\;\rho(|\mathbf{x}'|)\,\rho(|\mathbf{x}+\mathbf{x}'|) \quad , \qquad \text{(II.26)}$$

where $\tau = |\mathbf{x}|$.

In terms of $H(\tau)$ the equation of motion assumes the form

$$\left(m_0 - \frac{m_e}{3}\right)\ddot{\mathbf{z}}(t) + \int_0^\infty d\tau\, H(\tau)\,\ddot{\mathbf{z}}(t-\tau) = \mathbf{F}_{ext}(t) \quad . \qquad \text{(II.27)}$$

By using the fact that

$$\int_0^\infty d\tau\, H(\tau) = \frac{4}{3} m_e \quad , \qquad \text{(II.28)}$$

we can write (II.27) as

$$m\,\ddot{\mathbf{z}}(t) = \mathbf{F}_{ext}(t) + \mathbf{F}_{rad}(t) \quad , \qquad \text{(II.29)}$$

where

$$\mathbf{F}_{rad}(t) = \int_0^\infty d\tau\, H(\tau)\left[\ddot{\mathbf{z}}(t) - \ddot{\mathbf{z}}(t-\tau)\right] \qquad \text{(II.30)}$$

represents the radiation reaction force.

Equation (II.27) was first obtained by Markov, D.Bohn and M.Weinstein[8] (with the wrong mass) and by D.J.Kaup[6]. Many features of the classical motion for arbitrary charge distributions (spherically symmetric) can be abstracted more easily if one uses (II.27) instead of (II.25). That will be shown next.

III. THE GREEN'S FUNCTION APPROACH

The general solution of (II.27) can be written as

$$\ddot{\mathbb{Z}}(t) = \ddot{\mathbb{Z}}_0(t) + \int_{-\infty}^{\infty} dt'\, G(t-t')\,\mathbb{F}(t') \qquad \text{(III.1)}$$

where $\ddot{\mathbb{Z}}_0(t)$ is the solution of the homogeneous equation, and $G(t-t')$ is its Green's function

$$G(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\exp[i\omega(t-t')]}{m_0 - \frac{m_e}{3} + M(\omega)} \qquad \text{(III.2)}$$

$M(\omega)$ is given by

$$M(\omega) = \int_0^{\infty} d\tau\, H(\tau)\exp(-i\omega\tau) \qquad \text{(III.3)}$$

which, for $\mathrm{Im}\,\omega < 0$, admits the representation

$$M(\omega) = \frac{16\pi^2}{3}\int_{-\infty}^{\infty} dk\, \frac{k^2}{k^2-\omega^2}\,\tilde{\rho}^2(k) \qquad \text{(III.4)}$$

where $\tilde{\rho}(k)$ is defined in (II.13).

The explicit integration of (III.1) might be a very hard task for arbitrary charge distributions and/or external forces. However for a very large class of charge distributions, many features of the motion can be understood by just studying the analytic structure of the Fourier transform of the Green's function.

The first question that we adress ourselves is causality. This is a very relevant question since it is intimately associated to the problem of preacceleration and, in a less direct way, to the existence of runaway solutions.

The causality condition:

$$G(t-t')=0 \qquad \text{for} \qquad t<t' \tag{III.5}$$

is satisfied whenever the eigenvalue equation:

$$m_0-\frac{m_e}{3}+M(\omega)=0 \tag{III.6}$$

does not have solutions for $\operatorname{Im}\omega<0$. We shall analyse this case a little further.

Suppose that we have a solution of (III.6) in the lower half ω-plane ($\omega=\omega_R+i\omega_I,\ \omega_I<0$). By using the representation (III.4), and separating the real and imaginary parts of (III.6) we get:

$$m_0-\frac{m_e}{3}=-\frac{16\pi^2}{3}\int_{-\infty}^{\infty}dk\,k^2\,\tilde{\rho}^2(k)\,\frac{(k^2-\omega_R^2+\omega_I^2)}{(k^2-\omega_R^2+\omega_I^2)^2+4\omega_R^2\omega_I^2} \tag{III.7a}$$

$$\omega_R \omega_I \int_{-\infty}^{\infty} dk \frac{k^2 \tilde{\rho}(k)^2}{(k^2-\omega_R^2+\omega_I^2)^2 + 4\omega_R^2\omega_I^2} = 0 \qquad \text{(III.7b)}$$

The implication of (III.7b) is obvious. The poles (if any) of the Green's function in the lower half ω -plane lie on the imaginary axis. Thus one can write for (III.7a)

$$m_0 - \frac{m_e}{3} = -\frac{16\pi^2}{3}\int_{-\infty}^{\infty} dk \frac{k^2 \tilde{\rho}(k)^2}{k^2+\omega_I^2} \qquad \text{(III.8)}$$

This expression, allows us to state that the Green's function is causal whenever:

$$m_0 - \frac{m_e}{3} > 0 \qquad \text{(III.9a)}$$

or

$$m_0 - \frac{m_e}{3} < -\frac{16\pi^2}{3}\int_{-\infty}^{\infty} dk\, \tilde{\rho}(k)^2 = -M(0) = -\frac{4}{3} m_e \qquad \text{(III.9b)}$$

(condition (III.9b) implies that the physical mass $m = m_0 + m_e$ is negative and will be discarded).

By the same token we can say that if $m_0 - \frac{m_e}{3} < 0$ there is a violation of causality. Such a violation is exponentially damped for t $<$ t', and leads to a preacceleration.

Condition (III.9a) also prevents the appearence of runaway solutions.

In fact, by trying a runaway type solution

$$z(t) = e^{i\omega t}\ , \quad \mathrm{Im}\,\omega < 0 \qquad \text{(III.10)}$$

to the homogeneous part of (II.27) we shall get the same condition (III.9a).

For a fairly wide class of charge distributions - those for which $M(\omega)$ is defined in the whole ω-plane (this is the case of finite - extended particles, or sufficiently rapidly vanishing distributions at infinity) - the general structure of the Green's function can be pictured by just analysing the eigenvalue equation (III.6).

Without specifying $M(\omega)$ we cannot say anything more than that the poles are anywhere in the upper half ω-plane, or on the real axis, and that the reality of $m_0 - \frac{m_e}{3}$ implies that each pole with $\omega_R \neq 0$ has a counterpart on the opposite side of the imaginary axis (see (III.3) and (III.6)). So, the solutions will be of the form of damped exponentials, cossines x damped exponentials or just cossines. Cossine type solutions would imply a steady state of motion excited by the external force. They are possible only for extended particles, and its existence depends a great deal on the nature of the charge distribution. A detailed study of these solutions was carried out by Bohm-Weinstein[8].

Something should be said about initial conditions. Since we haven't runaways we can restrict ourselves to the case $\ddot{z}_0(t) = 0$ (One should observe that the character of the homogeneous part of (II.27), which involves a retardation requires that $\ddot{z}_0(t) = 0$ over a finite interval of time). This is equivalent to saying that the motion of the particle was uniform before the application of the external force.

Let us now analyse the implication of the $m_0 - \frac{m_e}{3} > 0$ condition.

Suppose that the size of the charge distribution is characterized by a length parameter a. Then, from dimensional analysis arguments one can say that condition $m_0 - \frac{m_e}{3} > 0$ can be written as

$$m_0 - \frac{\alpha}{3}\frac{q^2}{a} > 0 \qquad \text{(III.11)}$$

where α is a constant which depends only upon the form of the charge distribution and q is the total charge. Then the condition for not having unphysical solution can also be written under the form:

$$|q| < \sqrt{\frac{3a\,m_0}{\alpha}} \qquad \text{(III.12)}$$

which means that for a fixed value of m_0 , a and for a specific way of patching the constituents (determined by α), one cannot place more charges than it is "classically allowed". The bound, which we call the critical charge is

$$q_{crit} = \sqrt{\frac{3a\,m_0}{\alpha}} \qquad \text{(III.13)}$$

IV. ENERGY BALANCE

In the nonrelativistic limit, the fourth component of (II.3) becomes

$$\frac{d}{dt}\left(\frac{m_0}{2}\dot{z}^2(t)\right) = \dot{\mathbf{z}}(t)\cdot\mathbf{F}_{ext}(t) +$$

$$-\dot{\mathbf{z}}(t)\cdot\int d^3x\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla A_0\right)\rho(\mathbf{x},t)\left[1 + (\mathbf{x} - \mathbf{z}(t))\cdot\ddot{\mathbf{z}}(t)\right] \qquad \text{(IV.1)}$$

The last term in this expression can be easily recognized as $\dot{\mathbf{z}}(t) \cdot \mathbb{R}(t)$ where $\mathbb{R}(t)$ is defined in (II.8). In this way we can write

$$\frac{d}{dt}\left(\frac{1}{2} m_0 \dot{z}^2(t)\right) = \dot{\mathbf{z}}(t) \cdot \left[\mathbb{F}_{ext}(t) + \mathbb{R}(t)\right] \qquad \text{(IV.2)}$$

By using (II.14), (II.22) and (II.26) we can write (IV.2) under the form

$$\frac{d}{dt}\left(\frac{1}{2} m \dot{z}^2\right) = \dot{\mathbf{z}}(t) \cdot \mathbb{F}_{ext}(t) - \dot{\mathbf{z}}(t) \int_0^\infty d\tau \, H(\tau) \left[\ddot{\mathbf{z}}(t-\tau) - \ddot{\mathbf{z}}(t)\right] \qquad \text{(IV.3)}$$

(here we have used $\int_0^\infty d\tau \, H(\tau) = \frac{4}{3} m_e$)

The last term in (IV.3), the rate of work done by the radiation reaction, will be shown to be separable into the sum of two terms: one representing the radiated outgoing power, and the other the time derivative of an energy stored in the surrounding electromagnetic fields (an extension of that named Schott's energy[9-11]).

In order to achieve this goal let us introduce the quantities

$$W(t) = \frac{1}{8\pi} \int d^3x \left[(\nabla \times \mathbb{A})^2 + \left(\frac{\partial \mathbb{A}}{\partial t}\right)^2 \right] \qquad \text{(IV.4)}$$

and

$$I(t) = \int d^3x \; \mathbb{J}_\perp \cdot \frac{\partial \mathbb{A}}{\partial t} \qquad \text{(IV.5)}$$

representing, respectively, a part of the energy stored in the electromagnetic fields, and a part of the work done by unit time, by the self force.

By using the expression (II.12) for the vector potential $\mathbb{A}$ in Coulomb's gauge, and an approximation similar to (II.19) it is straightforward to check that W(t) in (IV.4) can be written as

$$W(t) = \frac{1}{2}\int_0^\infty d\tau \int_0^\infty d\sigma\, \dot{\mathbb{Z}}(t-\tau)\cdot\dot{\mathbb{Z}}(t-\sigma)\, \frac{\partial}{\partial\tau} H(\tau-\sigma) \qquad \text{(IV.6)}$$

In a similar way we get:

$$I(t) = \dot{\mathbb{Z}}(t)\cdot\int_0^\infty d\tau\, \dot{\mathbb{Z}}(t-\tau)\, \frac{\partial}{\partial\tau} H(\tau) \qquad \text{(IV.7)}$$

If one integrate (IV.7) by parts, remembering that $H(0) = H(\infty) = 0$, we shall get:

$$I(t) = \dot{\mathbb{Z}}(t)\cdot\int_0^\infty d\tau\, H(\tau)\, \ddot{\mathbb{Z}}(t-\tau) \qquad \text{(IV.8)}$$

The radiated power due to the motion of the particle (here represented by P_{rad}) is obtained by integrating the Poynting's vector $\mathbb{S} = \frac{1}{4\pi}\mathbb{E}\times\mathbb{B}$ over a closed surface at infinity

$$P_{rad} = \int \frac{da}{4\pi}\, (\mathbb{E}\times\mathbb{B})\cdot \mathbb{n} \qquad \text{(IV.9)}$$

By using the Gauss theorem to transform (IV-9) into a volume integral and after some manipulations with Maxwell equations, we get

$$P_{rad} = I(t) - \frac{dW}{dt}(t) \tag{IV.10}$$

Let us define $Q(t)$ as

$$Q(t) = -W(t) + \frac{2}{3} m_e \dot{z}^2(t) \tag{IV.11}$$

This can be written in terms of the velocity of the particle, If one uses (IV.6) we shall get

$$Q(t) = \frac{2}{3} m_e \dot{z}^2(t) - \frac{1}{2}\int_0^\infty d\tau \int_0^\infty d\sigma\, \dot{z}(t-\tau) . \dot{z}(t-\sigma) \frac{\partial}{\partial\tau} H(\tau-\sigma) \tag{IV.12}$$

By using (IV.8), (IV.10) and (IV.11) we can write (IV.3) as

$$\frac{d}{dt}\left[\frac{1}{2} m \dot{z}^2(t) - Q(t)\right] = \dot{z}(t) . F_{ext}(t) - P_{rad}(t) \tag{IV.13}$$

This equation is the promised result. The only term which deserves some comment is the Q term.

Turning back to expressions (IV.11) and (IV.4) we can see that an accelerated charge acquires an energy of electromagnetic nature, besides the rest mass and the kinetic term (Schott's energy). This form of energy is associated to the acceleration [9] and higher order time derivatives of the velocity.

In the point-like limit our expression for $Q(t)$ reduces to

$$Q(t) = \frac{2}{3} q^2 \dot{z}(t) . \ddot{z}(t) \tag{IV.14}$$

Whereas P_{rad} reduces to Larmor's expression for the radiated power

$$P_{rad} = \frac{2}{3} q^2 \ddot{\mathbb{Z}}^2(t) \qquad \text{(IV.14)}$$

V. SIMPLE EXAMPLES

In the first part of this section we shall present a simple example of how explicit solutions for (II.27) can be obtained once we know the solutions of the eigenvalue equation (III.2). The extension to more general type of forces in straightforward .

In the second part a position dependent force is studied. Explicit results are presented for the sphericall shell charge distribution and some aspects of the motion are discussed.

The question of energy balance is analysed within both subsections.

A) The step function force

Let us draw the general character of solution (III.1) for the simple external force

$$\mathbb{F}_{ext}(t) = \mathbb{F}_0 \left[\Theta(t) - \Theta(t-T) \right] \qquad \text{(V.1)}$$

where $\Theta(t)$ is the Heaviside step-function.

By substituting (V.1) into the solution (III.1) and using the representation (III.2) for the Green's function, we get after integration in t':

$$\ddot{\mathbb{Z}}(t) = \frac{\mathbb{F}_0}{2\pi i} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left[\frac{\exp(i\omega t) - \exp(i\omega(t-T))}{m_0 - \frac{m_e}{3} + M(\omega)} \right] \qquad \text{(V.2)}$$

Assuming that the charge distribution does not allow for real zeros of (III.6), then, from our previous discussion, and after separating out the $\omega = 0$ pole contribution to both integrals in (V.2), we shall obtain for $0 < t < T$.

$$\ddot{z}(t) = \frac{F_0}{m} + \sum_n A(\omega_n) \exp\left(i\omega_n^R t - \omega_n^I t\right) \tag{V.3}$$

while for $t > T$

$$\ddot{z}(t) = \sum_n A(\omega_n)\left\{\exp\left(i\omega_n^R t - \omega_n^I t\right) - \exp\left[i\omega_n^R (t-T) - \omega_n^I (t-T)\right]\right\} \tag{V.4}$$

where $\omega_n = \omega_n^R + i\omega_n^I \quad (\omega_n^I > 0)$ are the solutions of (III.6) and the coefficients A_n can be infered from (V.2).

The extension of the results (V.3) and (V.4) for any external force can be easily done.

In the limit $T \to \infty$, equation (V.3) reduces, for large enough times, to

$$m\ddot{z}(t) = F_0 \tag{V.5}$$

As had been realized by many authors before, within different contexts(10), solution (V.5) is an apparently puzzling one. The charged particle accelerates like a neutral one. All the power supplied by the external force is being used in increasing its kinetic energy. On the other hand, since the particle is accelerated, Maxwell's equations predicts that it is radiating and consequently loosing energy.

The solution of this problem, first suggested by Schott(9) and then studied by many researchers can be traced to the Q contribution to the internal energy of the particle. As we can check, by using (IV.8), (IV.10) and (IV.11), what happens in this case is that

$$\frac{dQ}{dt} - P_{rad} = 0 \tag{V.6}$$

In other words, all the radiated power is being supplied by Schott's energy.

B) The harmonic force

Up to now we have restricted ourselves to time dependent external forces. It is still possible to get explicit solutions for the case in which the particle is under the action of an harmonic type force

$$F_{ext}(t) = -K\, z(t) \tag{V.7}$$

In this case the equation of motion (II.27) becomes

$$\left(m_0 - \frac{m_e}{3}\right)\ddot{z}(t) + \int_0^\infty d\tau\, H(\tau)\, \ddot{z}(t-\tau) + K\, z(t) = 0 \tag{V.8}$$

We shall be mainly interested in a solution describing a steady state of motion which, under certain circunstances, can take place when the particle is excited by the harmonic force (V.7).

In this way if one looks for a solution of the form $\ddot{z}(t) = c\, \exp(i\omega t)$, the allowed values of ω can be obtained from the equation

$$m_0 - \frac{m_e}{3} + M(\omega) - \frac{K}{\omega^2} = 0 \tag{V.9}$$

A necessary condition for having a real ω as a solution of (V.9) is that the charge distribution should have at least one Fourier component vanishing[8]. This restricts the class of distributions to those which, roughly speaking, are characterized as sharply varying. Examples of such distributions are the uniformly charged sphere and the spherical shell. The latest is defined by the charge distribution:

$$\rho(\mathbf{x},t) = \frac{q}{4\pi a^2}\,\delta\left(|\mathbf{x}-\mathbf{z}(t)| - a\right) \tag{V.10}$$

For such a charge density the electromagnetic mass, $H(\tau)$ and $M(\omega)$ as defined in (II.15), (II.26) and (III.3) are given, respectively, by

$$m_{\ell} = \frac{q^2}{2a} \tag{V.11}$$

$$H(\tau) = \begin{cases} \dfrac{q^2}{3a^2} & 0 < \tau < 2a \\ 0 & \tau > 2a \end{cases} \tag{V.12}$$

$$M(\omega) = \frac{2}{3}\,\frac{q^2}{a}\left[\frac{\exp(-2i\omega a) - 1}{2i\omega a}\right] \tag{V.13}$$

We would like to mention that for time dependent external forces, we can get explicit solutions, as illustrated in part (A), by just studying the poles of (III.2) with $M(\omega)$ given by (V.13) (something which can be done at least numerically).

By substituting (V.13) in (V.9) we shall get

$$m_0 - \frac{m_e}{3} - \frac{K}{\omega^2} - \frac{q^2}{3\omega a^2}\sin(2\omega a) - \frac{iq^2}{3\omega a^2}\left[\cos(2\omega a) - 1\right] = 0 \qquad (V.14)$$

The imaginary part of (V.14) implies that the allowed real ω_n are those such that

$$\omega_n = \pm \frac{n\pi}{a} \qquad n = 1, 2, 3 \ldots \qquad (V.15)$$

Whereas the real part implies (taking into account (V.15))

$$K = \left(m_0 - \frac{m_e}{3}\right)\omega_n^2 \qquad (V.16)$$

This result shows that a stationary solution is possible only if K is given by (V.16), which is a very restrictive condition. Assuming that the particle is oscilating with one of the frequencies given by (V.15) and that (V.16) is satisfied, then the motion of the particle is a periodic one. The explicit solution is

$$\dot{z}(t) = V_0 \cos \omega_n t \qquad (V.17)$$

Let us see how the energy balance equation (IV.13) works when the motion is a stationary one.

If one uses $H(\tau)$ as given by (V.12) we obtain for W, as defined in (IV.6)

$$W = \frac{q^2}{6a^2}\int_0^{2a} d\tau\, \dot{z}^2(t-\tau) = \frac{m_e}{3} V_0^2 \qquad (V.18)$$

and for I, defined by (IV.8),

$$I(t) = \frac{q^2}{3a^2}\,\dot{z}(t)\cdot\int_0^{2a} d\tau\,\ddot{z}(t-\tau) = 0 \qquad (V.19)$$

Therefore the radiated power (IV.10) will be

$$P_{rad} = 0 \qquad (V.20)$$

This result associated with the energy balance equation (IV.13) implies that the conserved quantity is

$$E = \frac{1}{2} m \dot{z}^2(t) + \frac{1}{2} K z^2(t) - Q(t) \qquad (V.21)$$

The two first terms in the right hand side can be easily recognized as the kinetic and potential energy of the particle. By taking into account the radiation reaction, however, the sum of these two terms are not constant. What remains constant is the total energy (V.21), which besides these terms includes the Schott's energy (in this case $Q(t) = \frac{m_e}{3} V_0^2 \cos 2\omega_n t$).

The self-oscilation as studied by Bohm and Weinstein[8] can be easily realized from our expressions (V.8) and (V.21) by making K=0.

As a final remark we would like to point out that for the sphericall shell, we can write the equation of motion under the form of a differential-difference equation. That can be done by substituting (V.6) into (II.27). In this way we get, after the integration in the τ variable

$$\left(m_0 - \frac{m_e}{3}\right)\ddot{z}(t) - \frac{q^2}{3a^2}\left[\dot{z}(t) - \dot{z}(t-2a)\right] = F_{ext}(t) \qquad (V.22)$$

This equation, aside from the term $\frac{m e}{3} \ddot{z}(t)$, was got in a different (but equivalent) way by H.Levine, E.J. Moniz and D.H.Sharp[12]. Many aspects of the solution has been analysed in detail by them. The only relevant point, in which our conclusions will differ from theirs, is the condition for non-existence of unphysical solutions, equation (III.9a).

VI. CONCLUSIONS

Throughout this paper we have analysed new aspects of the classical motion of extended particles or aspects which have not received the attention they deserve.

If one starts from a covariant equation of motion then, in the nonrelativistic limit, such an equation gives rise to new (non linear) terms. By casting such an equation in a convenient form we have shown how the dropping of all non-linear terms can be justified.

We have called the attention to the fact that it is more fruitfull in getting solutions of the equation of motion, as well as in analysing many features of the motion of extended charges, if one uses an integral equation, rather than the higher orders differential equation. We have shown that these procedures are equivalent.

It is pointed out that the best way of tackling many aspects of the nonrelativistic motion is by using the method of Green's function. As shown here all one needs to do, in order to get explicit solutions for a very general class of models is to solve an eigenvalue equation (the eigenvalues correspond to the location of the poles of the Fourier tranform of the Green's function).

As a byproduct of our work the extension of many results, obtained up to now only for specific charge distributions (3,12) is easily achieved.

One of the virtues of starting with a relativistically covariant treatment from the very begining is that one fixes the 4/3 puzzle at the equation of motion level. We have pointed out that a consequence of taking the nonrelativistic limit of a covariant equation, is that it leads us to a new picture on the question of under which conditions the equation of motion will exhibit unphysical solutions. The new condition (III.9a) implies that even for positive mechanical mass (which is still a necessary condition) the classical motion of extended charges is stable and causal as long as the total charge does not exceeds a critical value. In this way, the existence of unphysical solutions is intimately related to the electromagnetic interaction of that particle.

A consistent equation for the energy balance is obtained as the limit of the fourth component of the covariant equation. The Schott's energy and the power radiated by extended particles have been identified.

As a final remark we would like to say that these results have been obtained within a phenomenological approach (as far as the structure of the particle is concerned). A more consistent way of handling the question of the motion of extended charges, would be one in which one associates extended particles to solutions of nonlinear classical field theoretical equation (solitons). This approach, which has a field theoretical method underlying it, is presently being pursued by us.

AKNOWLEDGMENTS

We would like to thank Dr. M.S.Hussein for a critical reading of the manuscript.

REFERENCES

(1) E.J.Moniz and D.H.Sharp; Phys.Rev. D10, 1133 (1974), Phys. Rev. D15, 2850 (1977)

(2) J.D.Jackson "Classical Electrodynamics" (Wiley New York, second edition 1975)

(3) For a review of theories of extended particles see T.Erber, Forschr.Phys. 9, 343 (1961). More recent references on this subject can be found in (1)

(4) See for example the excelent review by F.Rohrlich; "The first theory of on elementary particle: History and present state of the theory of the electron" (Siracuse University preprint 1972)

(5) F.Rohrlich; "Classical charged particles". Addison-Wesley (1965), and references therein.

(6) D.J.Kaup; Phys.Rev. 152, 1130 (1966)

(7) J.S.Nodvik; Ann.Phys. 28, 225 (1964)

(8) D.Bohm and M.Weinstein; Phys.Rev. 74, 1789 (1948)

(9) G.A.Schott; "Electromagnetic Radiation", Cambridge Univ. Press, London and New York, 1912

(10) T.Fulton and F.Rohrlich; Ann.Phys. (N.Y.) 9, 499 (1960)

(11) C.Teiltelboim; Phys.Rev. D1, 1572 (1970)

(12) H.Levine, E.J.Moniz and D.H.Sharp; Am.J.Phys. 45, 75 (1977)