IFUSP/P-143

ON THE EQUIVALENCE OF MASSIVE QED WITH RENORMALIZABLE AND IN UNITARY GAUGE

by

E.Abdalla

Instituto de Física - Universidade de São Paulo

ON THE EQUIVALENCE OF MASSIVE QED WITH RENORMALIZABLE AND IN UNITARY GAUGE

E.ABDALLA*+

Instituto de Física da Universidade de São Paulo

ABSTRACT

In the framework of BPHZ renormalization procedure, we discuss the equivalence between 4-dimensional renormalizable massive quantum electrodynamics (Stueckelberg lagrangian), and massive QED in the unitary gauge.

* Work supported by "Fundação de Amparo à Pesquisa do Estado de São Paulo" - FAPESP.

+ Present adress: II Institut für Theoretische Physik der Universität Hamburg.

MAR/78

I. INTRODUCTION

Massive QED can be described by the Proca-Wentzel[1] Lagrangian:

$$\mathcal{L}_R = \frac{1}{2}(1+d)i\bar{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - (M-c)\bar{\Psi}\Psi - \frac{(1+b)}{2}\partial_\mu A_\nu \partial^\mu A^\nu +$$

$$+ \frac{1}{2}(m^2+a)A_\mu A^\mu + \frac{1}{2}(1+b)(\partial_\mu A^\mu)^2 \tag{1}$$

However, this is a non-renormalizable theory, due to the ultra-violet behavior of the vector meson propagator:

$$\tilde{\Delta}_{F_{\mu\nu}}(k) = i\,\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2 + i\varepsilon} \tag{2}$$

As in massless QED, we introduce a term proportional to $(\partial_\mu A^\mu)^2$, which improves the ultra-violet behavior. In this way we are led to the Stueckelberg[1,2] Lagrangian:

$$\mathcal{L}_{st} = \mathcal{L}_R - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 \tag{3}$$

which describes the interaction of a vector meson with a fermion, in an indefinite metric Hilbert space.

We define the physical subspace by the same procedure as used by Gupta-Bleuler:

$$\partial_\mu A^{\mu +}|\psi\rangle = 0 \tag{4}$$

If we now make the following formal gauge transformation:

$$U^{\mu} = A^{\mu} + \partial^{\mu}\Lambda \tag{5a}$$

$$\varphi(x) = \exp\left[-ie\,\Lambda(x)\right]\psi(x) \tag{5b}$$

$$\Lambda(x) = \frac{1}{m^2}\,\partial_{\mu}A^{\mu}(x) \tag{5c}$$

we obtain the separation of the dynamics:

$$\mathcal{L}_{St}(A_{\mu},\psi) = \mathcal{L}_{Pr}(U_{\mu},\varphi) + \frac{1}{2\alpha}\left(\partial_{\mu}A^{\mu}\right)^2 + \text{surface terms} \tag{6}$$

so that the dynamics of the physical fields is separated from that of the ghost $\partial_{\mu}A^{\mu}$ (which has negative metric).

However, all this separation is only formal, first because the renormalization of the Proca lagrangian requires an infinite number of counterterms(3); and also $\exp\left[-ie\,\Lambda(x)\right]$ is not well defined(2), in such a way that if $\psi(x)$ defines a operator valued distribution, $\varphi(x)$ does not(2).

In the present paper we proove the equivalence between the theory in the Stueckelberg's lagrangian and that one with Proca's lagrangian with counterterms (which we shall call unitary gauge), taking into account the problem of renormalization. In 2-dimensions this has been done in Ref. (3).

Our paper is divided as follows:

Having stated the problem in section II, we adjust the parameters (which appear in unitary gauge's lagrangian) in section III so that Green's functions in the renormalizable case be independent of m_0^2 $(=\alpha m^2)$ on the mass shell.

In section IV we prove that parameters can be fixed in such a way that Green's functions be equivalent in both theories, in such a way that they differ only by a renormalization.

In section V we give an explicit example which shows that the parameters become infinite in the limit $m_o^2 \to \infty$.

II. STATEMENT OF THE PROBLEM

With the Proca lagrangian the photon propagator turns out to be

$$\Delta_{F_{\mu\nu}} = - \frac{i}{k^2 - m^2} \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right] \tag{7}$$

The interaction is given by

$$e \bar{\Psi} \gamma_\mu \Psi A^\mu \tag{8}$$

yielding a superficial degree of divergence

$$\delta_\mu(\gamma) = 4 - \frac{3F}{2} - 2B + N_\gamma(v) \tag{9}$$

$N_\gamma(v)$ = nº of vertices in γ

Since $\delta_\mu(\gamma)$ depends on $N_\gamma(v)$, any Green's function will eventually turn out to be divergent. Consequently we have a non-renormalizable theory, and an infinite number of counterterms is generated. In order to define a unique theory an infinite number of renormalization conditions is criterion: the Green's functions of the unitary gauge (non-renormalizable theory) must be equivalent to those of the renormalizable gauges.

Having in mind this aim, we define the following λ-dependent Lagrangian (λ-Lagrangian):

$$
\begin{aligned}
\mathcal{L}_{ef}(\lambda) = {} & \frac{i}{2}(1+d)\, N_4\left[\bar{\Psi}\overleftrightarrow{\not\partial}\Psi\right] - (M-c)\, N_4\left[\bar{\Psi}\Psi\right] - \frac{1}{4} N_4\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}(m^2+a)\, N_4\left[A_\mu A^\mu\right] - \\
& - \frac{1}{2}\,\frac{m^2+a}{m_0^2}\, N_4\left[(\partial_\mu A^\mu)^2\right] + \frac{1}{4} b\left[(1-\lambda)\, N_6\left[F_{\mu\nu}F^{\mu\nu}\right] + \lambda\, N_4\left[F_{\mu\nu}F^{\mu\nu}\right]\right] + (e+f)\times \\
& \times\left[(1-\lambda)\, N_5\left[\bar{\Psi}\not{A}\Psi\right] + \lambda\, N_4\left[\bar{\Psi}\not{A}\Psi\right]\right] + \mathcal{L}_g = \mathcal{L}_0 + \mathcal{L}_{int}
\end{aligned}
\tag{10}
$$

with $\mathcal{L}_g = \sum f^{\mathcal{P}}_{mn\,p p_1 q}\, N_\delta\left[\Omega^{\mathcal{P}}_{mn\,pp_1q}\right]$

where $\Omega^{\mathcal{P}}_{mn\,pp_1q}$ is a complete, linearly independent set of formally gauge-invariant counterterms. The prescription to find the finite part is the BPHZ renormalizable procedure, with degree:

$$
\delta_\lambda(\gamma) = 4 - \frac{3}{2} F_\gamma - B_\gamma - 2\bar{B}_\gamma + \sum_{V_a \in \gamma} (\delta_a - 4)
$$

$\bar{B}$ = nº of external boson lines atached to vertices $N_5\left[\bar{\Psi}\not{A}\Psi\right]$, or $N_{\delta_a}\left[\Omega_a\right]$ but not to A_μ from $(\partial_\mu - ie A_\mu)$, B=nº of the other enternal boson lines. in such a way that

$$
R(g) = \text{Finite part of } \langle 0|T\, \psi(x_1)\ldots\psi(x_N)\,\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_N)\, A_{\mu_1}(z_1)\ldots A_{\mu_L}(z_L)\, e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle
$$

$\Omega^{\mathcal{P}}_{mn\,pp_1q}$ can be parametrized as follows:

$$
\Omega^{\mathcal{P}}_{mn\,pp_1q} = m_{(\mu)(\nu)(\lambda)(\sigma)}\, \partial^{(i_1)} F_{\mu_1\nu_1}\ldots\partial^{(i_n)} F_{\mu_n\nu_n}\left(\bar{\Psi}_{\alpha_1}\,\bar{\theta}_{\lambda'_1}\,\theta_{\lambda_1}\,\psi_{\beta_1}\right)\cdots
$$

where $$\Theta_{\lambda_j} = \prod_{\ell=1}^{k_j} \left(\partial_{\eta_\ell} - ie\, A_{\eta_\ell} \right)$$

$$\bar{\Theta}_{\lambda_{j'}} = \prod_{\ell'=1}^{k_{j'}} \left(\overleftarrow{\partial}_{\eta_{\ell'}} + ie\, A_{\eta_{\ell'}} \right)$$

$$\sum_1^m k_j = p \quad , \quad \sum_1^m k_{j'} = p_1 \quad , \quad \sum_1^n i_\ell = q$$

$\partial^{(m)}$ is the same as $\partial_{\sigma_1} \cdots \partial_{\sigma_m}$

$\mathcal{P}$ refers to any of the possible permutations of k_j, i_ℓ, $j_{\ell'}$, $k_{j'}$, Θ_{λ_j} , $\bar{\Theta}_{\lambda_{j'}}$, such that m,n,p,p_1,q stay invariant, besides integration by parts.

$\mathcal{M}_{(\mu)(\nu)(\lambda)(\sigma)}$ is a matrix to contract the Lorentz indices, and finally

$$\delta = q + 2p + 2p_1 + 2n + 3m$$

We call attention to the fact that if λ=0 and $m_o^2 = \infty$ the λ-lagrangian becomes the Proca-Lagrangian (plus,of course, an infinite number of counterterms). If $f_y(\lambda=1)=0$, we have for λ=1 the Stueckelberg lagrangian.

The problem now is to show that Green's functions are independent of λ.

III. DEPENDENCE ON m_o^2

Now we shall fix the counterterms in such a way that the usual relation is ensured(1,4):

$$\frac{\partial G}{\partial m_o^2} = \frac{1}{m^2 + a} \Delta_o G \tag{11}$$

where $\Delta_o G$ vanishes in the mass-shell.

For simplicity, we define

$$\Omega^{\mathcal{P}}_{mn\,pp_1q} = \Omega_y \quad , \quad y > 6 \tag{12a}$$

$$f_1 = a \quad , \quad f_2 = b \quad , \quad f_3 = c \quad , \quad f_4 = d \quad , \quad f_5 = f$$

$$f_6 = -b - \frac{m^2 + a}{m_0^2} \quad , \text{ and } \quad f_y = f^{\mathcal{P}}_{mn\,pp_1q} \quad \text{for} \quad y > 6 \tag{12b}$$

We define furthermore:

$$\begin{aligned}
\Delta_1 &= \frac{i}{2}\int d^4x \; N_4\,[A_\mu A^\mu](x) \\
\Delta_2 &= \frac{i}{4}\int d^4x \; N_4\,[F_{\mu\nu} F^{\mu\nu}](x) \\
\bar{\Delta}_2 &= \frac{i}{4}\int d^4x \; N_6\,[F_{\mu\nu} F^{\mu\nu}](x) \\
\Delta_3 &= i\int d^4x \; N_4\,[\bar{\Psi}\Psi](x) \\
\Delta_4 &= -\frac{1}{2}\int d^4x \; N_4\,[\bar{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi](x) \\
\Delta_5 &= i\int d^4x \; N_4\,[\bar{\Psi}\gamma^\mu \Psi A_\mu](x) \\
\bar{\Delta}_5 &= i\int d^4x \; N_6\,[\bar{\Psi}\gamma^\mu \Psi A_\mu](x) \\
\Delta_6 &= \frac{i}{2}\int d^4x \; N_4\,(\partial_\mu A^\mu)^2(x)
\end{aligned} \tag{13}$$

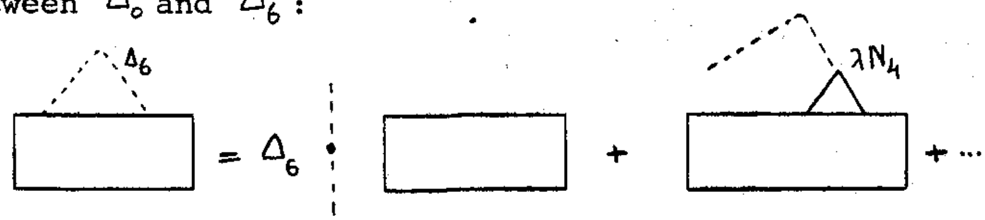
$$\begin{aligned}
\Delta_0 G = i\int d^4x \Big\{ &\sum_{i\neq j} \partial_{y_i}\Delta_F(x-z_i,m_0^2)\,\partial_{y_j}\Delta_F(x-z_j,m_0^2)\,\langle 0|T X_{\hat{ij}}|0\rangle + \\
+ i\,\frac{e+f}{1+d} &\sum_{i=1}^{L}\sum_{j=1}^{N} \partial_{y_i}\Delta_F(x-z_i,m_0^2)\,[\Delta_F(x-x_j,m_0^2) - \Delta_F(x-y_j,m_0^2)\langle 0|T X_{\hat{i}}|0\rangle + \\
+ i\left(\frac{e+f}{1+d}\right)^2 &\sum_{i\neq j}^{N} [\Delta_F(x-x_i)\,\Delta_F(x-x_j) + \Delta_F(x-y_i)\,\Delta_F(x-y_j)]\,G + 2\left(\frac{e+f}{1+d}\right)^2 \sum_{ij}\Delta_F(x-x_i)\,\Delta_F(x-y_j)\,G \Big\}
\end{aligned} \tag{14}$$

From the Gell-Man Low formula we have:

$$\frac{\partial G}{\partial m_0^2} = \sum_{y\neq 2,5} \frac{\partial f_y}{\partial m_0^2}\,\Delta_y G + \frac{\partial b}{\partial m_0^2}\left[(1-\lambda)\bar{\Delta}_2 + \lambda\Delta_2\right] G$$

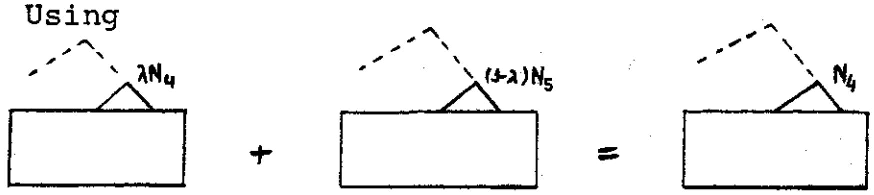
$$+ \frac{\partial f}{\partial m_0^2}\left[(1-\lambda)\bar{\Delta}_5 + \lambda\Delta_5\right] G \tag{15}$$

In order to use this formula let us establish a relation between Δ_o and Δ_6 :

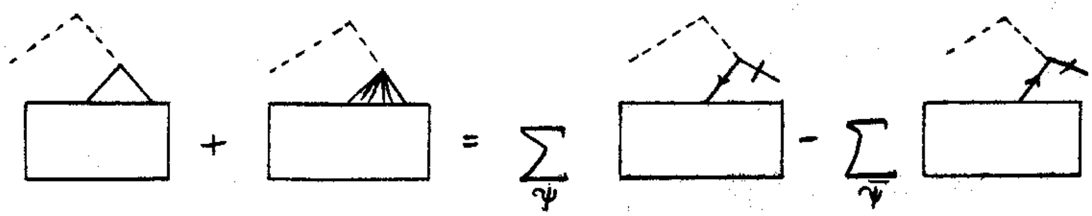


(16)

Using

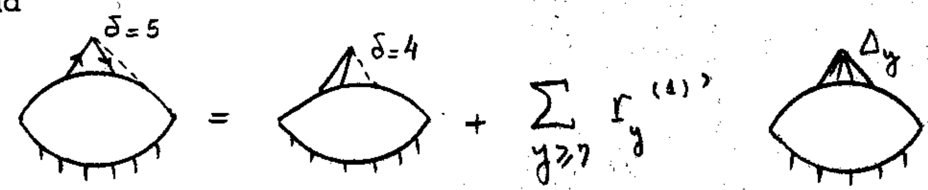


(17)



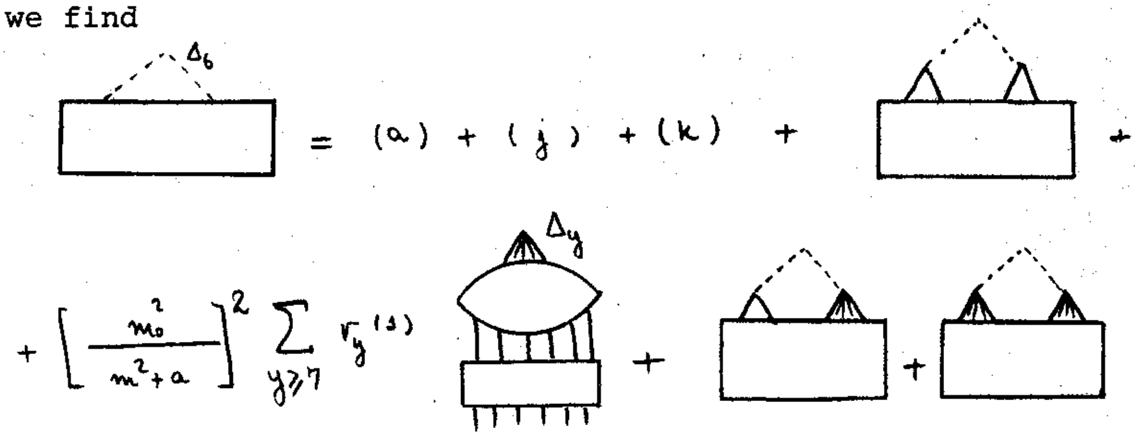
(18)

and



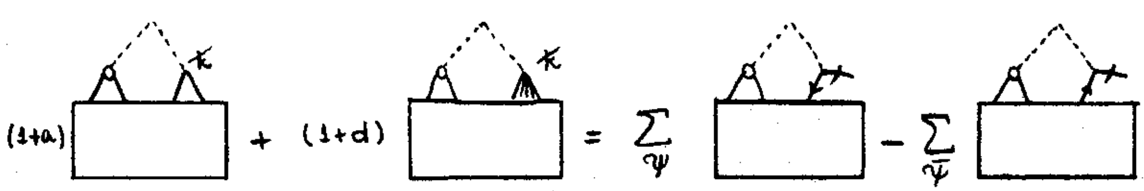
(19)

we find



(20)

Due to gauge invariance of $\mathcal{L}_g$:



(21)

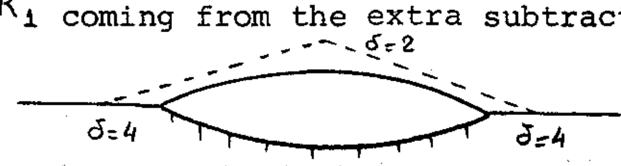
where  is equal to . or 

Iterating the above process for the other vertex we find:

$$\Delta_0 G = \left[\frac{m^2+a}{m_0^2}\right]^2 \Delta_6 G + 2i\,(c-M)\,R_1 - (c-M)^2 R_2$$

$$- \sum r_y^{(1)} \Delta_y G \qquad (22)$$

with R_1 coming from the extra subtraction of



and R_2 from the 2 extra subtractions from:

They can be written as $\sum r_y^{(2)} \Delta_y G$ so that

$$\Delta_0 G = \sum_{y \geq 3} r_y \Delta_y G \qquad (23)$$

Now inserting

$$\bar{\Delta}_5 - \Delta_5 = \sum \xi_{5y} \Delta_y$$

$$\bar{\Delta}_2 - \Delta_2 = \sum \xi_{2y} \Delta_y \qquad (24)$$

in (15) and comparing with (23) we obtain

$$a = a_0(\lambda) \qquad (25a)$$

$$f = f_0(\lambda) + \frac{1}{m^2+a} \int_0^{m_0^2} r_5 \, (m_0'^2) \, dm_0'^2 \tag{26}$$

$$f_y = f_y^0(\lambda) + \frac{1}{m^2+a} \int_0^{m_0^2} \Big[(1-\lambda)\, r_5 \, \xi_{5y} + r_y \Big] \, dm_0'^2 \tag{27}$$

So that (11) holds.

IV) THE EQUIVALENCE

In this section we shall proove that the Green's functions calculated with the λ-lagrangian (10) are λ-independent, i.e. we shall proove that [3]

$$\frac{\partial \mathcal{G}}{\partial \lambda} = 0 \tag{28}$$

From the Gell-Man-Low formula:

$$\frac{\partial \mathcal{G}}{\partial \lambda} = \Big\{ \sum_{y \neq 2,5} \frac{\partial f_y}{\partial \lambda} \Delta_y + (e+f)(\Delta_5 - \bar{\Delta}_5) + b\,(\Delta_2 - \bar{\Delta}_2) + $$

$$+ \frac{\partial f}{\partial \lambda} \Big[(1-\lambda)\, \bar{\Delta}_5 + \lambda\, \Delta_5 \Big] + \frac{\partial b}{\partial \lambda} \Big[(1-\lambda)\, \bar{\Delta}_2 + \lambda\, \Delta_2 \Big] \Big\} \mathcal{G} \tag{29}$$

Inserting (24) in (29)

$$\frac{\partial \mathcal{G}}{\partial \lambda} = \sum_y \Big[\frac{\partial f_y}{\partial \lambda} + \frac{\partial f}{\partial \lambda}(1-\lambda)\,\xi_{5y} + \frac{\partial b}{\partial \lambda}(1-\lambda)\,\xi_{2y} - (e+f)\,\xi_{5y}$$

$$- b\,\xi \quad \Big] \Delta \; \mathcal{G} \tag{30}$$

Now it is our aim to find $\tilde{\Delta}_y \mathcal{G}$ gauge independent, that is:

$$\left(\frac{\partial}{\partial m_o^2} - \frac{1}{m^2 + a} \Delta_o \right) \tilde{\Delta}_y \mathcal{G} = 0 \tag{31}$$

We can construct $\tilde{\Delta}_y$ by the following procedure: gauge invariant $\tilde{\Delta}_y$ are constructed, taking linear combinations of the Δ_y's .

$$\tilde{\Delta}_y^{(n)} = \Delta_y + \sum_{j=1}^{n} \sum_{y'} \beta_{yy'}^{(j)} \Delta_{y'} \tag{32}$$

$$\beta_{yy'}^{(n)} = \int_0^{m_o^2} \alpha_{yy'}^{(n)} \, d m_o'^2 \tag{33}$$

$$\alpha_{yy'}^{(n)} = \sum_{y''} \beta_{yy''}^{(n-1)} \alpha_{y''y'}^{(n-1)} \tag{34}$$

$$\left(\frac{\partial}{\partial m_o^2} - \frac{1}{m^2 + a} \Delta_o \right) \Delta_y \mathcal{G} = - \sum \alpha_{yy'}^{(1)} \Delta_{y'} \mathcal{G} \tag{35}$$

Then we can write:

$$\Delta_y = \sum \omega_{yy'} \tilde{\Delta}_{y'} \quad \text{such that}$$

$$\frac{\partial \mathcal{G}}{\partial \lambda} = \sum_{yy'} \left\{ \frac{\partial f_y}{\partial \lambda} + \frac{\partial f}{\partial \lambda}(1-\lambda) \xi_{5y} + \frac{\partial b}{\partial \lambda}(1-\lambda) \xi_{2y} - (e+f)\xi_{5y} - b \xi_{2y} \right\} \omega_{yy'} \tilde{\Delta}_{y'} \mathcal{G} \tag{36}$$

where the coeficient of $\tilde{\Delta}_{y'}$ is independent of m_o^2. We put $m_o^2=0$, and impose $\partial \mathcal{G} = 0$. ($\partial \mathcal{G} = 0$ holds independently of m_o^2, since the coeficient of $\tilde{\Delta}_{y'}$ is m_0^2 independent). By the fact that det $[\omega] \neq 0$:

$$\frac{\partial f_y^{(0)}}{\partial \lambda} + (1-\lambda)\frac{\partial f_{(0)}}{\partial \lambda}\xi_{5y} + (1-\lambda)\frac{\partial b^{(0)}}{\partial \lambda}\xi_{2y} - (e + f_{(0)})\xi_{5y} - \tag{37}$$

$$- b_{(0)}\xi_{2y} = 0$$

$f_y^{(0)}$ can be calculated in perturbation theory, using:

$$f_y^{(0)}(\lambda = 1) = 0 \ , \quad y > 6$$

and the normalization conditons of QED for $y < 6$. Equation (37) implies independence of the Green's functions with respect to λ, eq.(28).

V) AN EXAMPLE

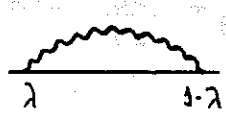
In this chapter we take the explicit Green's function $\mathcal{G}^{(2,0)}$ and proove that

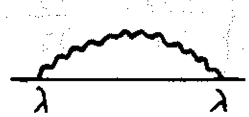
1. $\mathcal{G}^{(2,0)}(\lambda = 1) = \mathcal{G}^{(2,0)}(\lambda)$

2. The counterterms diverge for λ=1 as it should be, because in this case, the graphs are explicitly finite, but Green's function must be infinite, because in the case λ =0 the Green's functions are infinite.

Let us take

$\mathcal{G} =$ 

$- 2$ 

$+$ 

$+$

$f_0 + f_1\lambda' + f_2\lambda^2 + f_3\lambda^3$

An straightforward calculation shows that:

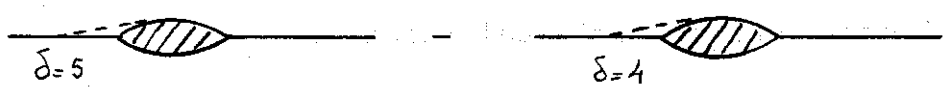
$$\mathcal{G}(\lambda) = \mathcal{G}(1) \tag{38}$$

where

$$f_y = f_y^{(0)}(\lambda) + \frac{1}{m^2+a}\int_0^{m_0^2} r_y\,(m_0'^2)\; dm_0'^2 \tag{39}$$

$$r_y = r_y^{(1)} + r_y^{(2)} \tag{40}$$

$r_y^{(2)}$ comes from anisotropies, and $r_y^{(1)}$ from:



(41)

we note that

$$r_3^{(1)} = r_4^{(1)} = 0 \tag{42}$$

$$r_7^{(2)} = r_8^{(2)} = 0 \tag{43}$$

$$r_3 = M^3 \int_0^1 \frac{y\,dy}{y m_0^2 + (1-y) M^2} \tag{44}$$

$$r_4 = -3M^2 \int \frac{d^4k}{(k^2-m_0^2)(k^2-M^2)} - 2M^4 \int \frac{d^4k}{(k^2-m_0^2)^2(k^2-M^2)^3} \tag{45}$$

$$r_7 = (1-\lambda^2) M \int \frac{d^4k\,(k^2+M^2)}{(k^2-m_0^2)(k^2-M^2)^2}\left[1 + \frac{M^2}{k^2-M^2}\right] \tag{46}$$

For $m_0^2 \gg M^2$ we find:

$$r_3 \simeq \frac{M^3}{m_0^2} \tag{47}$$

$$r_4 \simeq -\frac{3M^2}{m_0^2} \tag{48}$$

$$r_7 \simeq \frac{\pi}{m_0^2} \tag{49}$$

which diverge logaritmically when integrated.

Acknowledgement the author wishes to thank Dr. M. Gomes for discussions, and specially Dr. R. Koberle for the idea of the problem, the discussions and critical reading.

REFERENCES

(1) J.H.Lowenstein - Lectures on renormalizable theory - Maryland University - 1972.

(2) K.Symanzik - Islamabad Lectures on Lagrangian quantum field theory, DESY report .

(3) J.Frenkel, M.Gomes, R.Köberle - Rev.Bras.Física 6, 365 (1976).

(4) J.H.Lowenstein, B.Schroer - PRD 6, 1553 (1972).

FOOTNOTES

[1] The gauge independence of $\tilde{\Delta}_y$ will be used to put $m_0^2 = 0$ in equation (37). This is important in order not to have contradiction between

$$\frac{\partial G}{\partial m_0^2} = \frac{1}{m^2 + a} \Delta_0 G \qquad \text{and} \qquad \frac{\partial G}{\partial \lambda} = 0$$