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ON THE RATE OF FALL-OFF OF EIGENFUNCTIONS OF A MODEL RANDOM HAMILTONIAN

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A B S T R A C T

The rate of fall-off in configuration space of eigenfunctions of a model random Hamiltonian is studied. It is proved that an exponential rate of fall-off does not follow from the "exponential growth of particular solutions" ([24],[25]), as sometimes conjectured ([15],[8]). A theorem concerning the fall-off of non-isolated point eigenstates of the Hamiltonian is then proved, based upon an argument of Agmon ([17]).

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Equilibrium statistical mechanical properties - specially regarding low-temperature behaviour - of random systems, which have been studied rigorously in some models ([1],[2],[3],[4],[5]), are not particularly sensitive to the "fine structure" of the spectrum of the Hamiltonian, which is known to be pure point, with probability one, for a large class of one-dimensional models ([6]). This is because thermodynamic quantities such as, e.g., specific heat, depend only on the "integrated density of states", which is expected to be continuous (this fact is known at least for a class of one-dimensional models [7],[8]). In contrast, this does not seem to be the case for nonequilibrium (transport) properties. Although no rigorous derivation (from first principles) of Mott's formula for the hopping conductivity of amorphous semi-conductors exists, in all model derivations we know of (see [22] and references given there) an important role is played by the assumption that states in the so-called "mobility edge" belong to the point spectrum and have exponential fall-off in configuration space. In this paper we present some rigorous results concerning this assumption in model one-dimensional random systems.

In this paper we shall study the random tight-binding electron model in one dimension ([9],[8]). It is also a special case of the one-dimensional version of the model studied in ([1], [2]). We give here a brief description of the model, following [2]. Let $I \equiv \{\varepsilon_1 = 0, \varepsilon_2, \ldots, \varepsilon_r\}$, $2 \leq r < \infty$, be a sequence of distinct positive real numbers, assign to ε_i, $1 \leq i \leq r$, a measure $p_i > 0$, such that $\sum_{i=0}^{r} p_i = 1$, and let Ω be the cartesian product of copies of I indexed by the points of Z. Assign to Ω the product (probability) measure, denoted by P. becomes thus a compact topological space, with family B of Borel sets, and (Ω, B, P) a probability space. On Ω we define the independent, identically distributed positive random variables:

$$\nu_n(\omega) \equiv \omega_n \quad \text{if} \quad \omega \equiv (\omega_i)_{i \in Z} \in \Omega \tag{1}$$

For each $\omega \in \Omega$, the Hamiltônian is defined as the bounded positive self-adjoint operator on $H \equiv \ell^2(Z)$ by

$$H^\omega = H_o + V^\omega \tag{2}$$

where

$$H_o = -\frac{1}{2}\Delta_d \tag{3}$$

where, for all $u \equiv (u_n)_{n \in Z} \in H$,

$$(\Delta_d u)_n \equiv u_{n+1} + u_{n-1} - 2u_n \qquad n \in Z \tag{4}$$

is the difference Laplacian operator, and

$$(V^\omega u)_n = \nu_n(\omega) u_n \qquad n \in Z, \ \omega \in \Omega \tag{5}$$

We also introduce the Hilbert space $\tilde{H} \equiv L^2(B, dk)$, where B is the first Brillouin zone, $B \equiv (-\pi, \pi)$ (in fact, B is a circle, the points $-\pi$ and π being identified). $\tilde{H}$ is isomorphic to H, the isomorphism being given by Fourier transformation. On $\tilde{H}$, H_o takes the form

$$(H_o \tilde{u})(k) = \omega(k)\, \tilde{u}(k) \qquad k \in B, \quad \tilde{u} \in \tilde{H} \tag{6a}$$

where

$$\omega(k) \equiv 1 - \cos k \qquad k \in B \tag{6b}$$

By ([10], example 1.9, pg.518), $\Sigma_{H_o} = [0,2]$[1] is absolutely continuous. For each $\omega \in \Omega$, $\Sigma_{H\omega} \subseteq [0, a_\omega]$, where

$$a_\omega \leq a \equiv 2 + \sup_{i \in [1,r]} \varepsilon_i \tag{7}$$

[1] For any linear operator A, Σ_A denotes spectrum of A and $\Sigma_A^{p.p.}$ its pure point part.

Let $\{E^{\omega}(\lambda), \lambda \in \mathbb{R}_+ \equiv [0,\infty)\}$ denote the spectral family associated to the self-adjoint operator H^{ω}. The following fundamental result applies to a large class of random Hamiltonians in one dimension (including the present one):

<u>Theorem 1</u> ([6])

Let

$$A \equiv \{\omega : \Sigma_{H^{\omega}} = \overline{\Sigma_{H^{\omega}}^{p.p.}}\} \tag{8}$$

where the bar denotes closure (in the topology of $\mathbb{R}$). Then

$$P(A) = 1 \tag{9}$$ ■

<u>Remark 1</u> : We shall denote by $\{E_{\alpha}^{\omega}\}_{\alpha \in I}$ the eigenvalues of H^{ω}, and by $\{\phi_{\alpha}^{s_{\alpha},\omega}(E_{\alpha}^{\omega})\}$ the corresponding eigenfunctions, where I is some index set, and s_{α} labels the multiplicities. In this notation, theorem 1 asserts that, with probability one, $\{\phi_{\alpha}^{s_{\alpha},\omega}(E_{\alpha}^{\omega})\}$ is a basis of $\mathcal{H}$. ■

<u>Remark 2</u>: Theorem 1 does not assert whether the states $\phi_{\alpha}^{s_{\alpha},\omega}(E_{\alpha}^{\omega})$ are in any sense "localised". In fact, they are localized in the sense of Anderson ([9]): this is a direct corollary of theorem 1 and is proven for completeness in appendix B, although it follows essentially from remarks in [8]. ■

A logical question which now poses itself is to know <u>what type</u> of point spectrum is involved in theorem 1. Let

$$(e_n)_{n'} = \delta_{n,n'} \qquad n,n' \in \mathbb{Z}$$

be the standard basis of $\mathcal{H}$. We shall occasion to consider the subspace $\mathcal{H}_+$ of $\mathcal{H}$ consisting of all $\phi \in \mathcal{H}$ of the form $\phi = \sum_{n \geq 0} c_n e_n$, $\sum_{n \geq 0} |c_n|^2 < \infty$ and we shall denote by H_+^{ω} the (bounded, positive, self-adjoint) restriction of H^{ω} to $\mathcal{H}_+$ (defined to be zero on $\mathcal{H}_+^{\perp}$).

Theorem 2

There exists a fixed closed set S and a number $0 < b \leq a$ such that $[0,b] \subseteq S$ and

$$P(\{\omega : \Sigma_{H^\omega} = \overline{\Sigma^{p.p.}_{H^\omega}} = S\}) = 1 \tag{10}$$

Proof: It follows from (1) and (5) and the ergodicity of the two-sided shift in $\mathbb{Z}$ (see, e.g., [11], pg.18) that V is a metrically transitive potential (as defined in [12]). Hence, by a theorem of Pasthur ([13]) there exists a fixed closed set $S \subseteq [0,a]$ such that $P(\{\omega : \Sigma_{H^\omega} = S\}) = 1$. (10) follows then by Theorem 1. Let, now

$$\rho_i^\omega(\lambda) \equiv (E^\omega(\lambda)e_i, e_i) \qquad i \in \mathbb{Z} \quad \omega \in \Omega, \ \lambda \in \mathbb{R}$$

and, denoting by <.> expectation with respect to P,

$$\rho(\lambda) \equiv \langle \rho_0^\cdot(\lambda) \rangle \qquad \lambda \in \mathbb{R}$$

We shall also denote by $\rho^\omega_{i,+}(\cdot)$, $E^\omega_+(\ )$ and $\rho_+(\cdot)$ the analogous quantities for H^ω_+. Note that: a) $\rho(\cdot)$ and $\rho_+(\ )$ are the "integrated densities of states" of ([12],[2]) for the models described by H^ω and H^ω_+ respectively; b) $\rho^\omega_{i,+}(\cdot)$ is, for each $\omega \in \Omega$ and $i \in \mathbb{Z}_+$ a continuous function of $\lambda \in \mathbb{R}$ by ([8], Lemma 9,4) and c) $H^\omega_+ \leq H^\omega$ for each $\omega \in \Omega$. It follows from a), b) and c) that d) $\rho(\lambda) \leq \rho_+(\lambda)$ at each continuity point λ of ρ. By ([1]) or ([2]) (under the assumed conditions $p(\varepsilon_1 = 0) > 0$ and $r \geq 2$), ρ is not identically zero in some neighbourhood of zero, hence the same holds for ρ_+ by d). From this and the fact that ρ_+ is a continuous (and nondecreasing) function of λ, it follows that there exists an interval $[0,b]$, $0 < b \leq a$, such that each $\lambda \in [0,b]$ is an increasing point of ρ_+ (in the sense of [14], §82, pg. 238). Since, for each $\omega \in \Omega$, e_0 is a cyclic vector for H_+^ω (see theorem

A-1 of appendix A), $\lambda \in \mathbb{R}$ belongs to $\Sigma_{H_+^\omega}$ if and only if it is an increasing point of $\rho_{0,+}^{\omega}$ ([14], Chap. VI, pg.246). The last two assertions, coupled with the fact that $\rho_{0,+}^{\omega} = \rho_+$ for almost all $\omega \in \Omega$ ([12]), imply that $\Sigma_{H_+^\omega} \supseteq [0,b]$ for almost all $\omega \in \Omega$, whence $\Sigma_{H^\omega} \supseteq$ $\supseteq [0,b]$ for almost all $\omega \in \Omega$, and therefore $S \supseteq [0,b]$. ∎

What can be said of the rate of fall-off of the eigenfunctions in configuration space ? Thouless[1] ([15], see also [16] and [8]) conjectured the following behaviour of the eigenstates $\{\varphi_\alpha^\omega (E_\alpha^\omega)\}$ (in the notation of remark 1, we omit the index S_α and consider just H_+^ω as in appendix A):

$$\varphi_\alpha^\omega (E_\alpha^\omega) \underset{n\to\infty}{\sim} \exp\left[-\gamma_1^\omega (E_\alpha^\omega)\, |n - n_\alpha^\omega|\right] \qquad (11)$$

where n_α^ω is some point of $\mathbb{Z}_+$, and $\gamma_1^\omega(\cdot)$ is the function characterising the "exponential growth of particular solutions", defined in (A-3a) of appendix A. By (A-4), $\gamma_1^\omega(E) = \gamma_2^\omega(E) > 0$ (where $\gamma_2^\omega(\ )$ is defined in (A-3b)), for $E \notin A_\omega$ where A_ω is a set of zero Lebesgue measure, with probability one (in ω). This set A_ω might consist, however, of just the eigenvalues of H_+^ω. This is physically reasonable, because we might expect that eigenfunctions of the restriction of the Hamiltonian to a box, growing exponentially with the size of the box, correspond to eigenvalues which "in the limit" do not belong to the spectrum. The latter must therefore be contained in the complement of this "limiting set". We make this idea precise in

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1 In fact, Thouless' conjecture was formulated for <u>finite</u> systems, but the uniform rate of fall-off (independent of the size of the system) depended crucially on the property $\gamma_1^\omega(E_\alpha^\omega) > 0$ (with probability one in ω), where γ_1 is the "coefficient of exponential growth" for the <u>infinite</u> system.

<u>Theorem 3</u>

In the notation of remark 1,

$$P(\{\omega : \ \gamma_1^{\omega}(E_{\alpha}^{\omega}) = 0 \quad \forall \alpha \in I\}) = 1$$

<u>Proof</u> - By theorem 1 there exists $\Omega_0 \subseteq \Omega$ such that $P(\Omega_0)=1$ and for all $\omega \in \Omega_0$ and all $n \in \mathbb{Z}_+$, $\varrho_n^{\omega}(\cdot)$ is pure point with respect to Lebesgue measure, in the notation of (A-1) of appendix A. By (A-3), (A-4) and (A-5) there exists $\Omega_1 \subseteq \Omega$ such that $P(\Omega_1)=1$ and for all $\omega \in \Omega_1$,

$$|P_i^{\omega}(E)|^2 + |P_{i+1}^{\omega}(E)|^2 \xrightarrow[i\to\infty]{} \infty$$

for all $E \notin A_{\omega}$ where A_{ω} is a set of zero Lebesgue measure. Suppose that for some $\omega_1 \in \Omega_0 \cap \Omega_1$ and some $\alpha_1 \in I$, $\gamma_1^{\omega_1}(E_{\alpha_1}^{\omega_1}) \neq 0$, that is, that there exist some subsequence $\{i_K\}_{K\in\mathbb{Z}_+} \subseteq \mathbb{Z}_+$ such that

$$|P_{i_K}^{\omega_1}(E_{\alpha_1}^{\omega_1})|^2 + |P_{i_K+1}^{\omega_1}(E_{\alpha_1}^{\omega_1})|^2 \xrightarrow[K\to\infty]{} \infty \tag{12}$$

Taking $S = \{E_{\alpha_1}^{\omega_1}\}$ as the Borel set in (A-2), we obtain

$$|(\varphi_{\alpha_1}^{\omega_1} | e_{i_K})|^2 + |(\varphi_{\alpha_1}^{\omega_1} | e_{i_K+1})|^2 =$$
$$= |(\varphi_{\alpha_1}^{\omega_1} | e_0)|^2 \cdot \left[|P_{i_K}^{\omega_1}(E_{\alpha_1}^{\omega_1})|^2 + |P_{i_K+1}^{\omega_1}(E_{\alpha_1}^{\omega_1})|^2 \right] \tag{13}$$

By (12) and (13),

$$(\varphi_{\alpha_1}^{\omega_1} | e_0) = 0 \tag{14}$$

It follows from (14) and (A-2) (with $S = \{E_{\alpha_1}^{\omega_1}\}$) that

$$(\varphi_{\alpha_1}^{\omega_1} | e_i) = 0 \quad \forall i \in \mathbb{Z}_+ \tag{15}$$

Since $\{e_i\}_{i\in\mathbb{Z}_+}$ is a basis of $\mathcal{H}=\ell^2(\mathbb{Z}_+)$, it follows from (15) that $\varphi_{\alpha_1}^{\omega_1}=0$. Therefore (12) cannot hold, and hence, by (A-3a), for each $\omega\in\Omega_0\cap\Omega_1$ and each $\alpha\in I$, $\gamma_1^{\omega}(E_\alpha^{\omega})=0$. ∎

We shall now prove a theorem about the rate of fall-off. In ordinary ("one-body") quantum mechanics with a potential $V(\vec{x})\underset{|\vec{x}|\to\infty}{\to}0$ in some sense (for instance, if V is relatively compact with respect to $H_0=-\Delta$), the following bound holds for an eigenfunction $\psi_{E_i}\in L^2$ corresponding to an eigenvalue $E_i<0$: for all $\varepsilon>0$ there exists $C_\varepsilon<\infty$ such that:

$$|\psi_{E_i}(\vec{x})| \leq C_\varepsilon \exp\left[-(1-\varepsilon)\sqrt{-E_i}\,|\vec{x}|\right] \tag{16}$$

From this it follows in particular a uniform exponential bound for any compact subset $C\subset(-\infty,0)$ (i.e., not including the point E=0: this is the only possible limit point of the set of eigenvalues, under similar assumptions on V):

$$\sup_{E_i\in C} |\psi_{E_i}(\vec{x})| \leq A \exp\left[-d\,|\vec{x}|\right]$$

where $d>0$ and $A<\infty$ are constants depending only on C (17)

In the present case, V^{ω} is not, except perhaps for ω in a set of zero probability, relatively compact with respect to H_0, and by theorems 1 and 2 there exist $0<b$ and $\Omega_0\subset\Omega$ with $P(\Omega_0)=0$ such that

$$J\equiv[0,b]=\overline{\Sigma_{H^\omega}^{p.p.}} \quad \forall\omega\notin\Omega_0 \tag{18}$$

Since each point of $\Sigma_{H^\omega}^{p.p.}\cap J$ is a limit point of J, one might expect that no bound of the form (17) be possible, for C any compact subinterval $[c,d]$ of J, with $0<c<d<b$. We now prove this, following closely a beautiful argument of Agmon ([17]), or rather its version given in theorem XIII-33 of ([18]). We shall say that a sequence $f\equiv\{f_n\}_{n\in\mathbb{Z}}\in\mathcal{S}$ iff for all $p\in\mathbb{Z}_+$,

$\|f\|_p \equiv \sup_{n\in\mathbb{Z}} \left[(1+|n|^2)^p |f_n|\right] < \infty$. It follows that $f \in \mathcal{S}$ iff $\tilde{f} \in C_0^\infty(B)$. We shall call a linear functional on $\mathcal{S}$, continuous in the topology defined by each of the norms $\|\ \|_p$, a tempered distribution. If T is a tempered distribution, its Fourier transform is a functional $\tilde{T}$ on $C_0^\infty(B)$ defined in the standard way. We shall also need the "weighted spaces":

$$\ell_\delta^2(\mathbb{Z}) = \{ f : \mathbb{Z} \to \mathbb{C} : \|f\|_\delta^2 \equiv \sum_{n\in\mathbb{Z}} (1+|n|^2)^\delta |f_n|^2 < \infty \} \tag{19}$$

<u>Lemma</u> - If $f \in \ell^1(\mathbb{Z})$, there exist tempered distributions $\tilde{T}$ satisfying

$$(e^{i(k-k_0)} - 1)\,\tilde{T} = \tilde{f} \tag{20}$$

where $k_0 \in (-\pi, \pi)$. Further, there exists among them one and only one such that

$$T_n \underset{|n|\to\infty}{\longrightarrow} 0 \tag{21}$$

<u>Proof</u> - By Fourier transformation, it follows that $T \equiv \{T_n\}$, defined by

$$T_n = -\sum_{m \geq n} f_m\, e^{ik_0(m-n)} \tag{22}$$

also satisfies (20). Since $f \in \ell^1(\mathbb{Z})$, it also satisfies (21). To prove unicity, let $\mathcal{U}$ be a tempered distribution satisfying

$$(e^{i(k-k_0)} - 1)\,\tilde{\mathcal{U}} = 0 \tag{23}$$

Since the zeroes of $(e^{i(k-k_0)}-1)$ are simple, it follows from (23) that $\tilde{\mathcal{U}}$ is a measure supported by the points $\{k_0+2\pi n, n\in\mathbb{Z}\}$ and, as $k_0\in(-\pi,\pi)$, it follows from (23) that as a functional on $C_0^\infty(B)$, $\tilde{\mathcal{U}} = c_0\,\delta(k_0)$ and $\mathcal{U}_n \xrightarrow[|n|\to\infty]{} 0$ requires $c_0=0$. ∎

Theorem 4

Let J be defined by (18), with $b\le 2$. In the notation of remark 1, for each $\omega\notin\Omega_0$, each interval $C\equiv[c,d]$ with $0<c<d<b$, and each $\varepsilon>0$,

$$\sup_{E_\alpha^\omega\in C} \|\varphi_\alpha^{s_\alpha,\omega}(E_\alpha^\omega)\|_{1+\varepsilon} = \infty \tag{25}$$

Proof - Suppose the contrary, i.e., that there exists $\varepsilon>0$, $\omega\notin\Omega_0$ and an interval $C=[c,d]$, with $0<c<d<b$, such that

$$\sup_{E_\alpha^\omega\in C} \|\varphi_\alpha^{s_\alpha,\omega}(E_\alpha^\omega)\|_{1+\varepsilon} < \infty \tag{26}$$

We omit henceforth the index ω, assuming it is fixed not in Ω_0, and the indices α, s_α. Let

$$\psi(E) \equiv V\,\varphi(E)$$

Then, since V is bounded,

$$\sup_{E\in C} \|\psi(E)\|_{1+\varepsilon} < \infty \tag{27}$$

Since $\varphi(E)$ is an eigenfunction of H of eigenvalue E,

$$(H_0-E)\,\varphi(E) = -\,\psi(E)$$

and hence

$$\tilde{\varphi}(k,E) = -(\omega(k)-E)^{-1}\,\tilde{\psi}(k,E) \qquad k\in B \tag{28}$$

By (27), for each $E\in J$, $\tilde{\psi}(\cdot,E)$ is a continuous function on B, hence it has a restriction to the "hypersurface" defined by

$$\omega(k) - E = 0 \qquad E \in (0,2) \tag{29}$$

(see, e.g. [19], sect. IX-9). In the present case, the hypersurface consists of two points $\pm K(E)$, where $k(E) \in (0,\pi)$, and the restrictions must be zero, otherwise $\tilde{\varphi}$ could not belong to $\tilde{\mathcal{H}}$:

$$\tilde{\varphi}(\pm k(E), E) = 0 \tag{30}$$

We have

$$\omega(k) - E = 1 - \cos k - E = \cos k(E) - \cos k =$$

$$= -2 \sin \frac{k + k(E)}{2} \sin \frac{k - k(E)}{2}$$

Hence,

$$\frac{\tilde{\varphi}(k,E)}{\omega(k) - E} = \frac{\tilde{\varphi}(k,E)}{-2 \sin \frac{k+k(E)}{2} \sin \frac{k-k(E)}{2}} =$$

$$= \frac{1}{2 \sin k(E)} \left[\frac{\cos \frac{k+k(E)}{2} \tilde{\varphi}(k,E)}{\sin \frac{k+k(E)}{2}} - \frac{\cos \frac{k-k(E)}{2} \tilde{\varphi}(k,E)}{\sin \frac{k-k(E)}{2}} \right] \tag{31}$$

Under assumption (27), it is easy to verify that the functions $h_\pm$, defined by

$$\tilde{h}_\pm(k,E) = \cos \frac{k \pm k(E)}{2} \tilde{\varphi}(k,E)$$

satisfy

$$\tilde{h}_\pm(\pm k(E), E) = 0 \tag{32a}$$

and

$$\sup_{E \in C} \| h_\pm(E) \|_{1+\varepsilon} < \infty \tag{32b}$$

We shall prove that the functions $g_\pm$ defined by

$$\tilde{g}_\pm(k,E) = \frac{\tilde{h}_\pm(k,E)}{\sin \frac{k \pm k(E)}{2}} \tag{33}$$

satisfy

$$\sup_{E \in C} \| g_\pm(E) \|_{\varepsilon/2} < \infty \tag{34}$$

We give the proof for g_, the other is simular. g_ satisfies

$$\left(e^{i(k-k(E))} - 1\right) \tilde{g}_-(k,E) = 2i\, e^{i\frac{k-k(E)}{2}} \tilde{h}_-(k,E) \equiv$$
$$\equiv \tilde{r}_-(k,E) \tag{35}$$

By (32a), (32b) and (35)

$$\tilde{r}_-(k(E),E) = 0 \tag{36a}$$

and

$$\sup_{E \in C} \| r(E) \|_{1+\varepsilon} < \infty \tag{36b}$$

By (36b) and the lemma, there exists one and only one tempered distribution g_ satisfying (35) and such that $(g_-)_n \xrightarrow[|n| \to \infty]{} 0$, given explicitly by (22) (with $f_m = (r_-(E))_m$) and (34) follows from this explicit formula and (36a) by a proof identical to ([19] sect. IX-9, pg.83). It now follows from (28), (31), (33) and (34) that

$$\sup_{E \in C} \| \varphi(E) \|_{\varepsilon/2} < \infty \qquad (37)$$

To obtain (37), we used the fact that $C = [c, d]$ with $0 < c < d < 2$, hence $k(E) \neq 0, \pi$, and $\sin k(E)$ is bounded away from zero in the denominator in (31). By (37), any eigenfunction $\varphi(E)$ of H may be written

$$\varphi_n(E) = (1 + |n|^2)^{-\varepsilon/2} \, \eta_n(E)$$

where

$$\sup_{E \in C} \| \eta(E) \| < \infty$$

The operator of multiplication by $(1 + |n|^2)^{-\varepsilon/2}$ is easily seen to be compact (it transforms any weakly convergent sequence into a strongly convergent one). Hence, by the Rellich compactness theorem as in ([17]) the set $\{E_\alpha \in C\}$ must consist of a finite number of eigenvalues, each of finite multiplicity, which by (18) it does not. Therefore (21) must be false, and (20) follows. ▦

<u>Remark 3</u> - We note that in theorem 4 E must belong to the interior of Σ_{H_0} for the proof of (25) to go through (if $H_0 = -\Delta$ as in [19], sect IX-9, E must be in $(0, \infty)$). For that reason it was important to prove that $J \cap \Sigma_{H_0}$ is not empty (theorem 2). In fact, from our experience with the case of a finite number of impurities in an infinite crystal ("zero concentration") ([20]), we expect that, with probability one, $\Sigma_{H_0} \subset \Sigma_{H^\omega}$, and that the part of Σ_{H^ω} in the complement of Σ_{H_0} consist of discrete eigenvalues with finite multiplicites (the latter part corresponds in our model to the "mobility edge" - in a model of decoupled bands such as [2], these eigenvalues would lie in the "gap"). ▦

<u>Remark 4</u> - There exists a solvable model ([21],[4]) where the whole spectrum consists, with probability one, of the closure of the set of all rational numbers in a fixed interval $[0,a]$, $a>0$

(that is, there are no isolated eigenvalues with probability one-see the previous remark). ▦

Remark 5 - Theorem 4 illuminates theorem 3 insofar as it shows that, because of the existence of a non-isolated point spectrum, the function γ_1 in (11) should have rather unusual properties. As an example, suppose $\Sigma_H = \overline{\Sigma_H^{p.p.}} = [0,a]$, $a>0$, and $\Sigma_H^{p.p.} = \{$all rational numbers in $[0,a]\}$ (see remark 4). A possible type of behaviour allowed by theorem 4 would be

$$(\varphi_\alpha(E_\alpha))_n \underset{n\to\infty}{\sim} \exp\left[-\lambda(E_\alpha)\ |n-n_\alpha|\right]$$

where n_α is fixed, and

$$\lambda(E_\alpha) = \begin{cases} 0 & \text{if } E_\alpha \text{ irrational} \\ 1/n & \text{if } E_\alpha = m/n,\ m,n, \text{ integers, relatively prime} \end{cases}$$

In the above example, "many" states are localised, but $\|\varphi_\alpha(E_\alpha)\|_{1+\varepsilon}$ is not uniformly bounded in any compact subset of $[0,a]$ with non-empty interior. The above function λ is not so pathological:it is even continuous almost everywhere. ▦

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APPENDIX A

In this appendix, we collect some important spectral properties of the Hamiltonian of the model used in the text. Let $\{P_k^\omega(\lambda),\ k\in\mathbb{Z}_+,\ \lambda\in\mathbb{R}\}$ denote the system of real-valued polynomials ("of the first kind" in the terminology of [23]) associated to H^ω, satisfying

$$\begin{cases} \lambda P_k^\omega(\lambda) = \frac{1}{2} P_{k-1}^\omega(\lambda) - \left[1 + \frac{\nu_k(\omega)}{2}\right] P_k^\omega(\lambda) + \frac{1}{2} P_{k+1}^\omega(\lambda) \\ P_{-1}^\omega(\lambda) = 0 \qquad P_0^\omega(\lambda) = 1 \qquad k\in\mathbb{Z}_+ \end{cases}$$

Let $(e_n)_{n'} = \delta_{n,n'},\ n,n'\in\mathbb{Z}_+$ be the standard basis of $\mathcal{H} = (\ell^2(\mathbb{Z}_+)$, and

$$\rho_i^\omega(\lambda) \equiv (E^\omega(\lambda)e_i, e_i) \qquad \lambda\in\mathbb{R},\ \omega\in\Omega,\ i\in\mathbb{Z}_+ \tag{A-1}$$

Theorem A-1 ([23], pg.145)

For each $\omega\in\Omega$, H^ω has simple spectrum, with cyclic vector e_0. $\{P_k^\omega\}$ form a total system in $L^2_{\rho_0^\omega}(\mathbb{R}) = \{f : \int_{-\infty}^{\infty} d\rho_0^\omega(\lambda)|f(\lambda)|^2<\infty\}$, orthonormal with respect to ρ_0^ω:

$$\int_{-\infty}^{\infty} d\rho_0^\omega(\lambda)\, P_i^\omega(\lambda)\, P_j^\omega(\lambda) = \delta_{i,j} \qquad i,j\in\mathbb{Z}_+,\ \omega\in\Omega$$

Further, the Radon-Nikodym derivative of ρ_i^ω with respect to ρ_0^ω is $|P_i^\omega(\ )|^2$, that is, for all Borel subsets S of R,

$$\int_{\lambda\in S} d\rho_i^\omega(\lambda) = \int_{\lambda\in S} d\rho_0^\omega(\lambda)\, |P_i^\omega(\lambda)|^2 \quad \forall i\in\mathbb{Z}_+ \tag{A-2}$$

We now define, for each $\omega\in\Omega$ and $\lambda\in\mathbb{R}$,

$$2\gamma_1^\omega(\lambda) \equiv \limsup_{n\to\infty} \frac{1}{n}\left\{\log\left[P_{n+1}^\omega(\lambda)^2 + P_n^\omega(\lambda)^2\right]\right\} \tag{A-3a}$$

and, for all $\omega \in \Omega$ and $\lambda \in \mathbb{R}$ such that the limit exists:

$$2\gamma_2^{\omega}(\lambda) \equiv \lim_{n\to\infty} \frac{1}{n} \left\{ \log \left[P_{n+1}^{\omega}(\lambda)^2 + P_n^{\omega}(\lambda)^2 \right] \right\} \tag{A-3b}$$

<u>Theorem A-2</u>

If $\int_{-\infty}^{\infty} |c|\, d\nu(c)$ and the support of ν is not a single point, and $C = \{\omega$: there exists a set $A_\omega \subset \mathbb{R}$ of Lebesgue measure zero such that $\forall \lambda \notin A_\omega$, $\gamma_2^{\omega}(\lambda)$ exists and $\gamma_2^{\omega}(\lambda) > 0 \}$ (A-4a)
then

$$P(C) = 1 \tag{A-4b}$$

<u>Proof</u> - This theorem follows from an extension of a theorem of Matsuda and Ishii ([24]) due to Yoshioka ([25]), after an application of Fubini's theorem along the lines of appendix 1 of ([8]). ∎

<u>APPENDIX B</u>

In this appendix we show briefly (see remark 2) that Anderson's criterion of localisation ((B-1) below) holds (in the strongest possible form) for our model, as a corollary of theorem 1 (see also [8], pg.126).

Let V_N denote the unique solution of the Schroedinger equation

$$i\,\frac{du_n^{\omega}(t)}{dt} = H^{\omega} u_n^{\omega}(t) \quad n \in \mathbb{Z} \quad t \in [0,\infty),\ \omega \in \Omega$$

with boundary condiction (B-1a)

$u_n^{\omega}(0) = \delta_{n,N}$ for some $N \in \mathbb{Z}$ and all $\omega \in \Omega$ (B-1b)

(that is, initially localised). Anderson's ([9]) definition of localisation may now be precisely stated:

$$P\left(\{\omega : \limsup_{t\to\infty} |(e_N \mid v_N^\omega(t))| = c > 0\}\right) = 1 \quad \text{(B-2)}$$

When $V^\omega \equiv 0$, the (unique) solution of (B-1a) with boundary condition (B-1b) is given by $v_N(t) = e^{-2it} e^{i(n-N)\pi/2} J_{n-N}(t)$, which does not satisfy (B-2), but exhibits instead, as expected, the typical $0(|t|^{-1/2})$ decay due to the spreading of the wave-packet.

We have now the following corollary of theorem 1:

Theorem B-1

(B-2) holds with c=1.

Proof - $(e_N \mid v_N^\omega(t)) = (e_N \mid e^{-itH^\omega} e_N) = \int_{-\infty}^{\infty} e^{-it\lambda}\, d\rho_N^\omega(\lambda)$

Let $B_N \equiv \{\omega : \limsup_{t\to\infty} |(e_N \mid v_N^\omega(t))| = 1\}$ (B-3)

and $A_N \equiv \{\omega ;\ \rho_N^\omega$ is pure point with respect to Lebesgue measure on $\mathbb{R}\}$ (B-4)

It then follows from a standard theorem on almost-periodic characteristic functions ([26]) that

$$A_N \subseteq B_N \qquad (\forall N \in \mathbb{Z}_+)$$

Hence $P(B_N) \geq P(A_N)$. By theorem 1, $P(A_N) = 1$ for all $N \in \mathbb{Z}_+$, hence $P(B_N) = 1$. $\blacksquare$

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