INSTITUTO DE FÍSICA

preprint

IFUSP/P-173

ON THE COULOMB SOLUTION OF YANG-MILLS EQUATIONS

J.Frenkel

Instituto de Fisica, Universidade de São Paulo
São Paulo, Brasil

UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE FÍSICA
Caixa Postal - 20.516
Cidade Universitária
São Paulo - BRASIL

175837

ON THE COULOMB SOLUTION OF YANG-MILLS EQUATIONS

J. Frenkel

Instituto de Física, Universidade de São Paulo
São Paulo, Brasil

Abstract: We show that the Coulomb solution of Yang-Mills equations with external sources represents the maximum energy configuration, when there are no magnetic fields.

1979

As is well known, Yang-Mills fields [1] and quarks form the basic constituents of Quantum Chromodynamics (QCD), a quantum field theory of strong interactions [2]. The solution of the classical equations represents an important step toward the understanding of QCD. For this reason, several authors [3 - 7] have recently considered a simplified model of QCD, namely, the interaction of Yang-Mills fields with an external source. As a result, it has been established that there exist non-Coulombic solutions with lower energy than the Coulomb energy.

In this note, we will consider a general SU(N) Yang-Mills theory with an external source. We will show that, for configurations without magnetic fields, Gauss' law, when expressed in terms of appropriate gauge invariant electric fields and sources, has a form similar to that in QED. As a consequence, it follows in a very simple way, that the Coulomb solution has necessarily higher energy than any other solution.

The classical equations of the theory are

$$\partial_\mu F^a_{\mu\nu} + g f^{abc} A^b_\mu F^c_{\mu\nu} = \delta_{\nu 0}\, q^a \tag{1a}$$

where $q^a(x)$ denotes the external source density and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{1b}$$

Here, A^a_μ represent the Yang-Mills fields, which transform as an adjoint representation of SU(N), and f^{abc} are the completely antisymmetric structure constants of the theory.

Taking $\nu = 0$ in the above equation, we obtain the non-abelian version of Gauss' law

$$\partial_i E^a_i + g f^{abc} A^b_i E^c_i \equiv (D_i E_i)^a = q^a \tag{2}$$

where the electric field E^a_i is given by F^a_{i0}. Because of the non-abelian nature of the covariant derivative D_i, this equation is, in general, much more difficult to solve than in the corresponding abelian case.

In order to get a better understanding of the meaning of $\mathcal{D}_i$, we will consider configurations with vanishing magnetic fields $B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a$. Note that under a gauge transformation

$$A_\mu'^a \lambda^a = U^{-1} A_\mu^a \lambda^a U + \frac{i}{g} U^{-1}\partial_\mu U \tag{3a}$$

where λ is a matrix representation of the group generators, we have

$$B_i'^a \lambda^a = U^{-1} B_i^a \lambda^a U \tag{3b}$$

so that configurations with no magnetic fields have a gauge-invariant meaning. A general solution with zero magnetic fields has the form

$$A_i^a \lambda^a = \frac{i}{g} O^{-1}\partial_i O \tag{4}$$

where O is an unitary matrix. In this case, the $\nu = i$ components of (1) reduce to

$$\partial_0 E_i^a - g f^{abc} A_0^b E_i^c \equiv (\mathcal{D}_0 E_i)^a = 0 \tag{5}$$

which shows that the electric field "precesses" around A_0 with angular velocity proportional to g in such a way that its modulus $(E_i^a E_i^a)^{1/2}$ remains constant in time.

We will now consider more specifically the adjoint representation where $(\lambda^a)_{bc} = -i f^{abc}$. This has the important property of making the covariant derivative $\mathcal{D}_i$ of a vector (in the internal symmetry space) V^a proportional to the usual derivative

$$\partial_i V^a + g f^{abc} A_i^b V^c \equiv (\mathcal{D}_i V)^a = [O^{-1}\partial_i (OV)]^a \tag{6}$$

In this representation O is given by an orthogonal matrix, and V on the right side of this equation is to be understood as a column vector. We can then write Gauss' law (2) as follows

$$\partial_i (O E_i) = O q \tag{7}$$

Let us now consider the invariance properties of this equation. Under the gauge transformation (3a), with U in the adjoint representation, and using (4), we have

$$A'^{a}_{i}\lambda^{a} = \frac{i}{g}\,(O')^{-1}\,\partial_i\,O' \tag{8a}$$

where $O' = O\,U$. Furthermore, we obtain

$$E'^{a}_{i}\lambda^{a} = U^{-1}\,E^{a}_{i}\lambda^{a}\,U \tag{8b}$$

$$q'^{a}\lambda^{a} = U^{-1}\,q^{a}\lambda^{a}\,U \tag{8c}$$

It is straightforward to check that the electric fields $\mathcal{E}_i$ and the charge density ρ defined by

$$\mathcal{E}_i = O E_i \tag{9a}$$

$$\rho = O q \tag{9b}$$

are gauge invariant quantities in the sense that $O E_i = O' E'_i$ and $O q = O' q'$. In terms of these quantities we can then write Gauss' law (7) in the following form

$$\vec{\nabla}\cdot\vec{\mathcal{E}} = \rho \tag{10}$$

which is similar to that encounterd in QED.

Of course there are important differences. Equation (10) represents, in fact, a non-linear equation since $E_i = F_{io}$ is itself a function of O . Furthermore, for consistency, we need q to be covariantly conserved: $D_o\, q = 0$.(*) Together with (5) these represent a complicated non-linear system of equations. This system will admit (3 - 7), for suitably choosen sources q , finite energy solutions with lower energy than the corresponding Coulomb solution. Fortunately, for our purposes, we will only need the functional form of (10).

We will also use the non-abelian version of Faraday's law which, for vanishing magnetic fields, reads

$$\varepsilon_{ijk}\, D_j\, E_k = 0 \tag{11a}$$

Using (6) we can write (11a) in terms of the invariant field $\vec{\mathcal{E}}$ as

$$\vec{\nabla}\times\vec{\mathcal{E}} = 0 \tag{11b}$$

(*) This equation implies that $|q|$ is time independent.

From equations (10) and (11) we obtain that $\vec{\mathcal{E}}$ is given by the following integral equation $[\, x=(\vec{x},t); \quad x'=(\vec{x}',t)\,]$

$$\vec{\mathcal{E}}(x) = -\frac{1}{4\pi}\vec{\nabla}\int d^3x' \frac{\rho(x')}{|\vec{x}-\vec{x}'|} \tag{12}$$

which is the analogue of the corresponding formula in QED.

Now we can easily obtain the expression for the energy of the electric fields. We find ($\tilde{\rho}$ denotes the transpose of ρ)

$$H = \frac{1}{8\pi}\int d^3x\, d^3x' \frac{\tilde{\rho}(x)\cdot\rho(x')}{|\vec{x}-\vec{x}'|} \tag{13}$$

This formula represents our central result. From equation (5) it is clear that H is conserved in time, as it should be. Furthermore the effective charge density ρ is, as we have seen, invariant under gauge transformations, which guarantees the gauge invariance of the energy. Note that, due to the orthogonality of the matrix O, ρ and q have the same modulus. Using the Schwartz inequality we see that the maximum value of H is given by

$$H_c = \frac{1}{8\pi}\int d^3x\, d^3x' \frac{|q(x)|\,|q(x')|}{|\vec{x}-\vec{x}'|} \tag{14}$$

In order to understand the physical content of equations (13) and (14), it will be convenient to consider the case when the external source density points into a fixed direction in the internal space, say the first

$$q^a(x) = \delta^{a1}\,|q(x)| \tag{15}$$

We can always attain such a configuration because the Yang-Mills equations are gauge covariant under the gauge transformation (3a) and (8c). Of course, this choice does not affect gauge invariant quantities, but has the advantage of making the physical content more transparent. We can then write (14) as

$$H_c = \frac{1}{8\pi}\int d^3x\, d^3x' \frac{\tilde{q}(x)\cdot q(x')}{|\vec{x}-\vec{x}'|} \tag{16}$$

Equation (16) can be obtained from (13) if $\mathcal{O}$ is independent of $\vec{x}$. Using (4), we see that this requires $A_i^a = 0$ and then Gauss' law implies that $A_0^a = \nabla^{-2} q^a$. The resulting configuration corresponds precisely to the Coulomb solution.

With the exception of this solution, we observe that the effect of the non-abelian fields, manifested through the non-trivial dependence on x of $\mathcal{O}$, is to rotate the initially parallel sources $q(x)$ into $\rho(x) = \mathcal{O}(x)\, q(x)$. In general, the angles of rotation (in the internal space) will be different at distinct points in space. So, the effective sources ρ will point in different directions, $\rho(x)$ and $\rho(x')$ being, in general, non-parallel. Consequently, the energies of all non-trivial solutions will be necessarily lower than the one corresponding to the Coulomb configuration.

I would like to thank J.C.Taylor for a helpful correspondence, and M.L.Frenkel for useful discussions.

References

1- C.N.Yang and R.L.Mills, Phys.Rev. 96 (1954) 191.
2- See review by W.Marciano and H.Pagels, Phys.Reports 36 (1978) 137.
3- J.Mandula, Phys.Lett. 67B (1977) 175.
4- P.Sikivie and N.Weiss, Phys.Rev.Lett. 40 (1978) 1411 and SLAC preprint 2147.
5- P.Pinla and P.Presnajder, Nucl.Phys. B142 (1978) 229.
6- K.Cahill, Phys.Rev.Lett. 41 (1978) 599.
7- R.Jackiw, L.Jacobs and C.Rebbi, CTP preprint 768 (1979).
After this work was completed, I learnt from Prof. Taylor about related works done at Oxford by R.Hughes, Y.Leroyer and A.Raychaudhuri.