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ABSTRACT

It is shown that results on mass generation in the Gross-Neveu model that were apparently in conflict lead to the same physics, being therefore equivalent. This is done by using renormalization group techniques.

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1. INTRODUCTION

In a recent paper (Ref.1), we have analysed the results of D.J.Gross - A.Neveu[2] and of H.D.I. Abarbanel[3] concerning the spontaneous mass generation in the Gross-Neveu model[2]. These results are apparently in conflict, as they arrive not only at different mass-gap equations, but at Callan-Symanzik β-functions[4] which differ in the number of zeros, a property that, normally, is physically meaningful. In ref.1 we were able to show, by using very simple and efficient techniques, that the results coincided up to a renormalization. It remained, therefore, to be shown that either the fact that the β-functions have a different number of zeros implied, here, no physical consequences, or else that some anomaly put us out of the range of Symanzik's theorem[5], invoked to perform one of the renormalizations.

In this work we conclude our analysis by showing that the first alternative is true. We do that by introducing a two-parameter class of renormalizations[6] that contains both the renormalizations used in ref.1 and by showing that it is not to be expected that the β-functions be simply related: in fact, the extra zero of one of them carries no physical meaning. It is also shown that the Green functions, expressed in terms of the generated mass, are completely equivalent.

In section 2 we briefly review the relevant results of ref.1 . In section 3 we discuss the equivalence of Green functions, and, in section 4, the relationship between the β-functions.

2. THE METHOD

The Gross-Neveu model is described by the lagrangian

$$\mathcal{L} = - \bar{\Psi}_0 \gamma \cdot \partial \Psi_0 + \frac{1}{2} g_0^2 (\bar{\Psi}_0 \Psi_0)^2 \tag{2.1}$$

where go is a dimensionless (in 2 dimensions) coupling constant. We use the imaginary metric and Dirac spinors which are, in fact , a multiplet of N fields, each with two components. For fermion Green functions one can also use the equivalent lagrangian [2]

$$\mathcal{L} = - \bar{\Psi}_0 \gamma \cdot \partial \Psi_0 - \frac{1}{2} \sigma_0^2 + g_0 \bar{\Psi}_0 \Psi_0 \sigma_0 . \tag{2.2}$$

To study spontaneous breakdown of symmetries it is convenient to add to (2.2) a driving term which breaks the symmetry, and , simultaneously, to shift the σ field by a constant v ,getting

$$\mathcal{L} = - Z \bar{\Psi} \gamma \cdot \partial \Psi - \frac{Z_\sigma}{2} (\sigma - v)^2 + Z_\lambda \frac{\lambda}{\sqrt{N}} \mu^{\frac{2-n}{2}} \bar{\Psi} \Psi (\sigma - v) + c\sigma \tag{2.3}$$

where the renormalization constants are defined by the following relations with the unrenormalized fields and coupling constants:

$$\Psi_0 = Z^{\frac{1}{2}} \Psi$$

$$\sigma_0 - v_0 = Z_\sigma^{\frac{1}{2}} (\sigma - v)$$

$$Z_\lambda^2 \lambda^2 = Z^2 Z_\sigma N g_0^2 \mu^{n-2} .$$

Here μ is a parameter with the dimension of mass and n is the continuous dimension in the sense of dimensional regularization [7]. We will look for solutions with spontaneoulsy broken symmetries by examining the possibility of having a nonvanishing vacuum expectation value of σ when c is put equal to zero after the computations are performed. This means we must put the sum of all tadpoles with a σ-leg equal to zero, thus getting an equation for v; if this equation has nonvanishing solutions for $c=0$, then we have spontaneous breakdown of symmetry (8). The calculation will be done in the limit $N\to\infty$ which, in our method, means including contributions up to one loop in the tadpole equation.

The Feynman rules are (see Fig.1)

a) $\dfrac{1}{(2\pi)^2 i}\,\dfrac{-i\gamma\cdot k + a}{k^2 + a^2}$

b) $\dfrac{1}{(2\pi)^2 i}$

c) $(2\pi)^2 i\,\dfrac{\lambda}{\sqrt{N}}\,\mu^{\frac{2-n}{2}}$

d) $(2\pi)^2 i\,(c+v)$

e) $(2\pi)^2 z\,\gamma\cdot k - (2\pi)^2 i\, z_\lambda a$.

f) $\dfrac{(2\pi)^2}{i}\, z_\sigma$

g) $(2\pi)^2 i\, z_\sigma v$

h) $(2\pi)^2 i\, z_\lambda \dfrac{\lambda}{\sqrt{N}}\,\mu^{\frac{2-n}{2}}$ 2.4

Fig. 1

where we introduced

$$a = \frac{\lambda}{\sqrt{N}}\,\mu^{\frac{2-n}{2}}\, v$$

and $z_i = Z_i - 1$.

At the one-loop level the tadpole equation, given in Fig. 2, is

$$c + v + v z_\sigma - \frac{\lambda}{\sqrt{N}}\,\mu^{\frac{2-n}{2}}\, N\, \mathrm{Tr} \int \frac{dq}{(2\pi)^2 i}\,\frac{-i\gamma\cdot q + a}{q^2 + a^2} = 0 \quad . \qquad 2.5$$

Fig. 2

Performing the integration, one has, for $v \neq 0$, the equation

$$1 + z_\sigma - \frac{\lambda^2}{(2\pi)} \frac{\Gamma(1-\frac{n}{2})}{\left(\frac{a^2}{\mu^2}\right)^{1-\frac{n}{2}}} = 0 \qquad 2.6$$

As soon as z_σ is determined by some renormalization prescription, (2.6) will give $\underline{a}$ as a function of the coupling constant λ . This is the mass-gap equation.

To determine z_σ we impose the condition

$$\Pi_\sigma(k^2 = \mu^2) + \frac{(2\pi)^2}{i} z_\sigma = 0 \qquad 2.7$$

where Π_σ is the two-point proper vertex of the σ -field (see Fig.3). The remaining renormalization constants are, up to this order, equal to unity.

Fig. 3

Now,

$$\Pi_\sigma(k^2) = -i\,\pi^{\frac{n}{2}} n(1-n)\, \lambda^2\, \Gamma(1-\tfrac{n}{2}) \int_0^1 dx \frac{1}{\left[\frac{k^2}{\mu^2} x(1-x) + \frac{a^2}{\mu^2}\right]^{1-\frac{n}{2}}} \qquad 2.8$$

If we put this into eq.(2.7), getting

$$z_\sigma = -\frac{\lambda^2}{4\pi^{2-\frac{n}{2}}} n(1-n)\, \Gamma(1-\tfrac{n}{2}) \int_0^1 dx \frac{1}{\left[x(1-x) + \frac{a^2}{\mu^2}\right]^{1-\frac{n}{2}}} \qquad 2.9$$

and then put (2.9) into (2.6), Abarbanel's mass-gap equation[3] obtains:

$$\sqrt{1+4\frac{a^2}{\mu^2}}\; \ln \frac{\sqrt{1+4\frac{a^2}{\mu^2}} - 1}{\sqrt{1+4\frac{a^2}{\mu^2}} + 1} = -\frac{2\pi}{\lambda^2} \qquad 2.10$$

If, instead, we put $a=0$ in (2.8) (or in (2.9)) and then compute (2.6), we get Gross-Neveu's mass-gap equation[2]

$$a^2 = \mu^2 \exp\left(-2\pi/\lambda^2\right) \tag{2.11}$$

That the second procedure is correct is shown by Symanzik's theorem[5].

So, two different renormalizations give two different mass-gap equations. This suggests that they are, in some sense , equivalent. However, if the β-functions corresponding to each renormalization are computed, it is seen that Abarbanel's one has a zero at $\lambda^2 = \pi$, which is absent in the Gross-Neveu's β .

3. GREEN FUNCTIONS

The most direct way to show the equivalence of both renormalization procedures is to analyse the Green functions . We consider here the four-point fermion function $G^{(4)}(p_1, p_2; p_3, p_4)$ depicted in Fig. 4 .

Fig. 4

The shaded bubble , in the diagrams stand for $\tilde{D}_\sigma$, the full σ propagator in the limit $N\to\infty$. We have

$$G^{(4)}(p_1, p_2; p_3, p_4) = -\frac{1}{(2\pi)^2}\left[\frac{(2\pi)^2 i\lambda}{\sqrt{N}}\right]^2 \left\{\tilde{D}_\sigma(u) + \tilde{D}_\sigma(s)\right\} \tag{3.1}$$

where

$$u = -(p_1 - p_4)^2 \quad ; \qquad s = -(p_2 - p_3)^2$$

We must now compute $\tilde{D}_\sigma$ in both renormalizations. In terms of the renormalized self-energy $\Pi_\sigma(k^2)$ one has

$$\tilde{D}_\sigma(k^2) = \frac{1}{(2\pi)^2 i} \frac{1}{1 - \frac{1}{(2\pi)^2 i} \Pi_\sigma(k^2)} \qquad 3.2$$

In the first renormalization

$$\Pi_\sigma(k^2) = i\, 2\pi \lambda^2 \left[B\left(\frac{a^2}{k^2}\right) - B\left(\frac{a^2}{\mu^2}\right) \right] \qquad 3.3$$

where

$$B(x^2) = \sqrt{1 + 4x^2}\, \ln \frac{\sqrt{1+4x^2} - 1}{\sqrt{1+4x^2} + 1} , \qquad 3.4$$

so that

$$\tilde{D}_\sigma(k^2) = \frac{1}{(2\pi)^2 i} \frac{1}{1 - \frac{\lambda^2}{2\pi}\left[B\left(\frac{a^2}{k^2}\right) - B\left(\frac{a^2}{\mu^2}\right)\right]} \qquad 3.5$$

Therefore

$$G^{(4)}(p_1, p_2; p_3, p_4) = \frac{2\pi i}{N} \left\{ \frac{}{-\frac{2\pi}{\lambda^2} + B\left(\frac{a^2}{u}\right) - B\left(\frac{a^2}{\mu^2}\right)} + (u \to s) \right\} \qquad 3.6$$

If we now use the mass-gap equation corresponding to this renormalization, that is, eq. (2.10), to eliminate λ, we get

$$G^{(4)}(p_1, p_2; p_3, p_4) = \frac{2\pi i}{N} \left\{ \frac{1}{B\left(\frac{a^2}{u}\right)} + \frac{1}{B\left(\frac{a^2}{s}\right)} \right\} \qquad 3.7$$

In the second (symmetric) renormalization one has

$$\Pi_\sigma(k^2) = -i\,2\pi\lambda^2 \left\{ \ln\frac{a^2}{\mu^2} - B\left(\frac{a^2}{k^2}\right) \right\} \qquad 3.8$$

and hence,

$$G^{(4)}(p_1, p_2; p_3, p_4) = \frac{2\pi i}{N} \left\{ \frac{1}{-\frac{2\pi}{\lambda^2} + B\left(\frac{a^2}{u}\right) - \ln\left(\frac{a^2}{\mu^2}\right)} + (u \rightarrow s) \right\} \qquad 3.9$$

Using now eq. (2.11) to eliminate λ , we end up exactly with the expression given by eq. (3.7) . Other Green functions are treated analogously.

4. THE QUESTION OF THE β-FUNCTIONS

It remains to be explained how is it possible that equivalent theories exhibit β-functions with a different number of zeros. To do that we introduce a two-parameter class of renormalizations which contains both Gross-Neveu's and Abarbanel's. See ref. (6) for the same renormalization, performed in a somewhat unconventional way.

The basic renormalization condition is given by eq.(2.7). Let us compute $\Pi_\sigma(k^2)$ with the Feynman rules of the symmetric theory, that is, by putting, in (2.4), a=0 . We complete our renormalization prescription by imposing the generalized condition

$$\frac{(2\pi)^2}{i} z_\sigma + \Pi_\sigma(k^2 = k_0^2) = 0 \qquad 4.1$$

where

$$k_0^2 = M^2 \left\{ \frac{\sqrt{1+4\frac{M^2}{\mu^2}} - 1}{\sqrt{1+4\frac{M^2}{\mu^2}} + 1} \right\}^{-\sqrt{1+4\frac{M^2}{\mu^2}}} \qquad 4.2$$

M being a new parameter with the dimension of mass.
By computing z_σ in this way and inserting the result in eq.(2.6), we obtain the mass-gap equation, already exhibited by Muta[6] .

$$\sqrt{1+4\frac{M^2}{\mu^2}}\ \ln \frac{\sqrt{1+4\frac{M^2}{\mu^2}} - 1}{\sqrt{1+4\frac{M^2}{\mu^2}} + 1} + \ln \frac{a^2}{M^2} = -\frac{2\pi}{\lambda^2} \qquad 4.3$$

If, in (4.1) and (4.2), one puts M=0 , the results of Gross-Neveu are obtained. If, at the same places, M is put equal to a, Abarbanel's results are obtained.

To understand the basic difference between the two renormalizations, consider the following renormalization group transformation:

$$\bar{\mu} = e^t \mu$$
$$\bar{M} = e^t M \qquad 4.4$$

The change in the coupling constant with t is given by

$$d\bar{\lambda}(t) = \frac{\partial \bar{\lambda}(t)}{\partial \bar{\mu}} d\bar{\mu} + \frac{\partial \bar{\lambda}(t)}{\partial \bar{M}} d\bar{M} \quad ,$$

that is

$$\frac{\partial \bar{\lambda}}{\partial t} = \frac{\partial \bar{\lambda}(t)}{\partial \bar{\mu}} \frac{\partial \bar{\mu}}{\partial t} + \frac{\partial \lambda(t)}{\partial \bar{M}} \frac{\partial \bar{M}}{\partial t} = \bar{\mu} \frac{\partial \bar{\lambda}}{\partial \bar{\mu}} + \bar{M} \frac{\partial \bar{\lambda}}{\partial \bar{M}}$$

and, as we would like that

$$\frac{\partial \bar{\lambda}}{\partial t} = \beta(\bar{\lambda}) \ . \ ,$$

we are led to the definition

$$\beta \equiv \mu \frac{\partial \lambda}{\partial \mu} + M \frac{\partial \lambda}{\partial M} \quad . \qquad 4.5$$

The relevant quantities, which are the fixed points of λ , are just the zeros of the so defined β-function.

By using the mass-gap equation it is easy to compute β , as defined by (4.5). It is given by

$$\beta(\lambda) = - \frac{\lambda^3}{2\pi} \quad . \qquad 4.6$$

The β-function of Gross-Neveu is closely related (in fact, equal!) to this one. For, though $M=0$, the renormalization group transformations can still be considered as being given by (4.4), $\bar{M}$ vanishing for any value of t . So, β describes the variation of λ under the same type of scaling, and must be the same function , which is indeed the case.

Consider, however, the particular choice of the renormalization parameters that gives Abarbanel's results. This means keeping μ free, and putting $M=a$. The situation is, now , completely different. As $\underline{a}$ is an invariant of the renormalization group[9], the renormalization group transformations cannot any more be taken as a special case of (4.4). In fact, they must be selected from the much larger scale transformations

$$\begin{aligned} \bar{\mu} &= e^{t} \mu \\ \bar{M} &= e^{\tau} M \end{aligned} \qquad 4.7$$

by fixing τ to be zero. In this case there are two independent β-functions, one of which coincides with Abarbanel's. If we write

$$d\,\bar{\lambda}(\bar{\mu},\bar{M}) = \beta_t\,dt + \beta_\tau\,d\tau \quad , \qquad 4.8$$

then

$$\beta_t = -\frac{\lambda^3}{2\pi} + 2\lambda\,\frac{a^2}{\mu^2}\,\frac{1}{1+4\frac{a^2}{\mu^2}} \qquad 4.9$$

which is Abarbanel's. It has a second zero at $\lambda^2=\pi$ (near which $a\to\infty$). The second β is

$$\beta_\tau = -\,2\lambda\,\frac{a^2}{\mu^2}\,\frac{1}{1+4\frac{a^2}{\mu^2}} \quad . \qquad 4.10$$

It should be clear by now that the β of Gross-Neveu and the one of Abarbanel are conceptually different entities, and that no simple relation between the two, or among their zeros, exists.

Finally, we will now show that, insofar as the transformation of a Green function under a change in renormalization can be entirely expressed in terms of masses, the particular form of a β function plays no role at all. We start by remarking that, in this model and approximation,

$$\lambda = \frac{\beta(\lambda)}{\gamma(\lambda)} \qquad 4.11$$

where

$$\gamma(\lambda) = \mu\,\frac{\partial}{\partial\mu}\,\ln\sqrt{Z_\sigma} \quad . \qquad 4.12$$

Therefore, if λ_0 is a nonvanishing zero of β, it is also a zero of γ. Now, the transformation formula for a vertex under a scaling of the momenta is[10]:

$$\Gamma^{(n)}(e^t p_i, \lambda, \mu) = \exp\left[(2-n)t - n\int_{\lambda}^{\bar{\lambda}(t)} dx\, \frac{\gamma(x)}{\beta(x)}\right] \Gamma^{(n)}(p_i, \bar{\lambda}(t), \mu) \qquad 4.13$$

which, by the use of (4.11), can be written as

$$\Gamma^{(n)}(e^t p_i, \lambda, \mu) = e^{(2-n)t}\left(\frac{\bar{\lambda}(t)}{\lambda}\right)^{-n} \Gamma^{(n)}(p_i, \bar{\lambda}(t), \mu) \qquad 4.14$$

In the Gross-Neveu model, it is known that the Green functions can be written in terms of the mass as the sole parameter. Hence,

$$\Gamma^{(n)}(e^t p_i, \lambda, \mu) = \Gamma^{(n)}(e^t p_i, a) \quad . \qquad 4.15$$

Now, $\Gamma^{(n)}(p_i, \bar{\lambda}(t), \mu)$ is a Green function renormalized at μ, with a different value of the coupling constant. In terms of the mass-gap equations, one keeps μ fixed and changes the value of λ. This means a different mass, $\bar{a}$. Hence, equation (4.14) becomes

$$\Gamma^{(n)}(e^t p_i, a) = e^{(2-n)t}\left(\frac{\lambda(t)}{\lambda}\right)^{-n} \Gamma^{(n)}(p_i, \bar{a}) \qquad 4.16$$

Now, in our case,

$$\Gamma^{(2)}(p_i, a) = i\gamma\cdot p_i + a \quad ,$$

so that, from (4.16), one has

$$e^{t}\, i\gamma\cdot p + a = \left(\frac{\bar{\lambda}(t)}{\lambda}\right)^{-2} (i\gamma\cdot p + \bar{a}) \qquad 4.17$$

and, for p=0 ,

$$a = \left(\frac{\bar{\lambda}(t)}{\lambda}\right)^{-2} \bar{a} \quad . \qquad 4.18$$

Putting (4.18) into (4.17) one gets

$$\frac{a}{\bar{a}} = e^{t} \quad . \qquad 4.19$$

Hence, equation (4.16) can be written

$$\Gamma^{(n)}\left(\frac{a}{\bar{a}} p_i , a\right) = \left(\frac{a}{\bar{a}}\right)^{2-\frac{n}{2}} \Gamma^{(n)}(p_i , \bar{a}) \quad , \qquad 4.20$$

that is, no vestige of β remained, and in no point of the derivation the existence or not of a second zero of β played any role.

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FIGURE CAPTIONS

Fig. 1 - Feynman rules for the Gross-Neveu model.

Fig. 2 - The tadpole equation.

Fig. 3 - The σ self-energy.

Fig. 4 - The 4-point Green function.

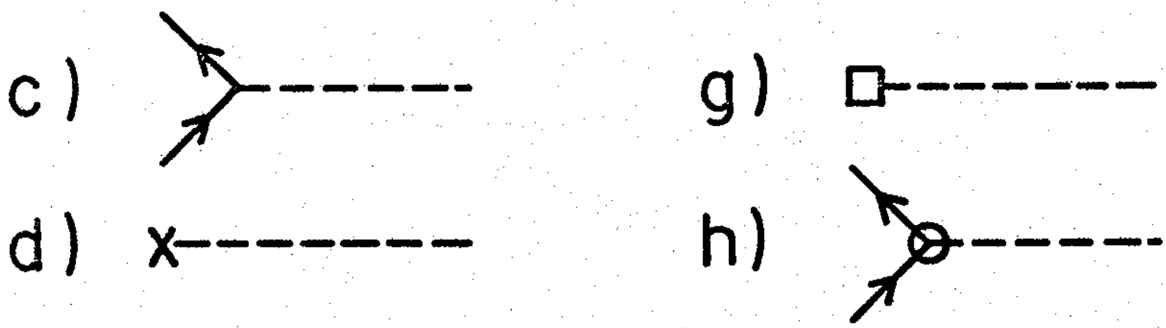

Fig.1

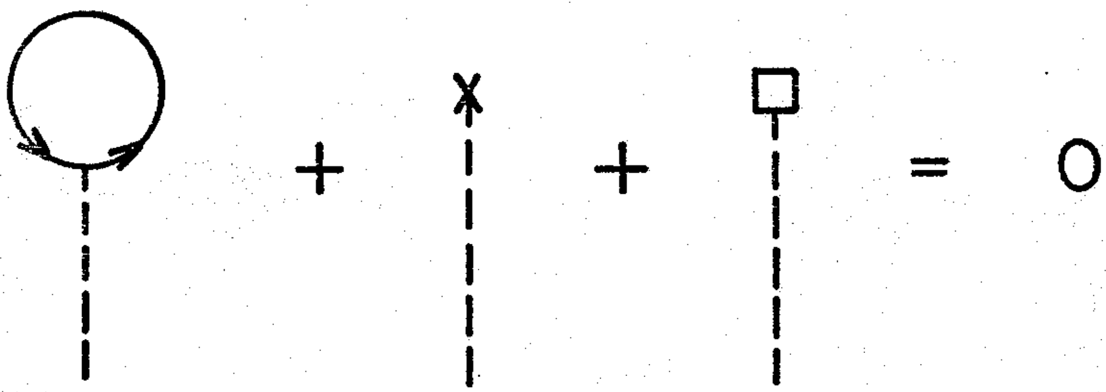

Fig.2

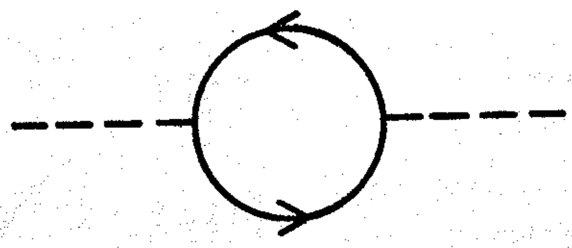

Fig.3

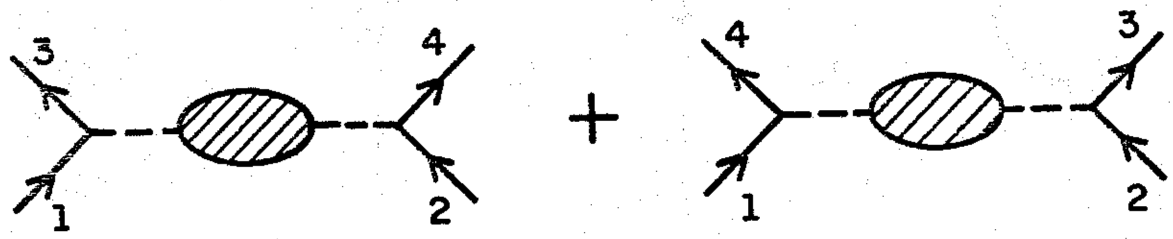

Fig.4