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ABSTRACT

We propose an exact, factorized, S-Matrix for the two-dimensional supersymmetric CP^{N-1} model, which is compatible with the $\frac{1}{N}$ expansion.

The two-dimensional supersymmetric CP^{N-1} model (SCP^{N-1}) was introduced by the authors of refs. (1,2), whose motivation was the understanding of some fenomena expected to be present also in four-dimensional QCD. The basic advantage is the available $\frac{1}{N}$ - expansion in the SCP^{N-1} case. This model is the natural geometrical generalization of the CP^{N-1} model, whose exact solubility should be expected[3]. In fact recently we have shown that the usual CP^{N-1} model, minimally coupled to massless fermions, posses a factorized S-matrix[4]. One may now hope that the S-matrix of the supersymmetric extension is equally factorized.

In this paper we will construct a factorized S-matrix, which by standart arguments will be shown to belong to the SCP^{N-1} model. In order to present our arguments we will first recall some proprieties of this and related models.

The CP^{N-1} model and, by supersymmetrization, the SCP^{N-1} model have a close relationship to exactly soluble models whose S-matrix are explicitly known[5-7]. Indeed there is a remarkable analogy between the SCP^{N-1} and the supersymmetric generalization[8,9] of the non-linear σ and Gross-Neveu models. I.e., in supersymmetrizing the non-linear σ model[8,9] one gets an expression that can be interpreted equally well as the supersymmetric Gross-Neveu model, since supersymmetry couples both of them[*1]. In the same way supersymmetry couples the CP^{N-1} and the chiral Gross-Neveu[10,11] models so that what we have been calling the supersymmetric CP^{N-1} model can be viewed as the supersymmetric chiral Gross-Neveu model. The last observation leads one to suspect that the SCP^{N-1} model has a factorized S-matrix since it is known that the chiral Gross-Neveu model does[11,12] and one does not expect to destroy this propriety by supersymmetrizing. Another interesting property of this model[1] is the existence of an extended (O(2)) supersymmetry on the Lagrangean level supported by some arguments given in ref. 2 about the origin of the (topological) central charges.

To formulate the 2→2 scattering problem subject to the factorization constraints[13] one will need, in principle, the explicit knowledge of the super algebra. This was for instance the way the S-matrix of the supersymmetric non linear σ model was gotten[7]. However, it is possible to solve the factorization equations (see below) without knowing any supersymmetry relations between the scattering amplitudes, the only drawback being the appearance of two free parameters. But these can be fixed using the available $\frac{1}{N}$ - perturbative expressions[*2].

The $\frac{1}{N}$ - Feynman rules of this model were obtained in refs (1,2) whereto we refer the reader. To establish some notation we will write the defining functional integral

$$Z=\int |dZ|\,|dZ^*|\,|d\psi|\,|d\overline{\psi}|\,|d\omega|\,|dA_\mu|\,|d\sigma|\,|d\pi|\,|d\xi|\,|d\overline{\xi}|\ \exp\frac{i}{2\alpha}\int d^2x\sum_{j=1}^{N}\Big[(\partial_\mu-iA_\mu)Z_j^*(\partial^\mu+iA^\mu)Z_j+\omega(Z_j^*Z_j-1)+\overline{\psi}_j i(\partial_\mu-iA_\mu)\gamma^\mu\psi_j - \frac{1}{\sqrt{2}}\overline{\psi}_j(\sigma+i\pi\gamma_5)\psi_j+\overline{\xi}\,\psi_j Z_j^*+\overline{\psi}_j Z_j\xi\Big]$$

Using these rules one observes that all particle-antiparticle reflection amplitudes vanish in order $\frac{1}{N}$ and the absence of particle production to order $\frac{1}{N^2}$. Thus in constructing the exact S-matrix one will, as in the chiral Gross-Neveu model[11,12], start with vanishing reflection amplitudes from the very beginning[*3]. Transmission amplitudes are now defined introducing the simbols $(\overline{b}_\alpha(\theta_i))$ $b_\alpha(\theta_i)$ and $(f_\alpha(\theta_i))$ $f_\alpha(\theta_i)$ to denote (anti) bosons and (anti) fermions respectively, where the variables θ_i are related to energy-momentum by $P_0^i = m\,\mathrm{ch}\theta^i$, $P_1^i = m\,\mathrm{sh}\theta_i$ and $P_1 P_2 = m\,\mathrm{ch}\theta$ $(\theta = \theta_1 - \theta_2)$

$$_{out}\langle b_\beta(\theta_1) b_\delta(\theta_2) | b_\alpha(\theta_1) b_\gamma(\theta_2)\rangle^{in} = v_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + v_2(\theta)\delta_{\alpha\delta}\,\delta_{\gamma\beta}$$

$$_{out}\langle f_\beta(\theta_1) f_\delta(\theta_2) | f_\alpha(\theta_1) f_\gamma(\theta_2)\rangle^{in} = u_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + u_2(\theta)\delta_{\alpha\delta}\,\delta_{\gamma\beta}$$

$$_{out}\langle f_\beta(\theta_1) b_\delta(\theta_2) | f_\alpha(\theta_1) b_\gamma(\theta_2)\rangle^{in} = c_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + c_2(\theta)\delta_{\gamma\delta}\,\delta_{\gamma\beta}$$

$$_{out}\langle b_\beta(\theta_1) f_\delta(\theta_2) | f_\alpha(\theta_1) b_\gamma(\theta_2)\rangle^{in} = d_1(\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + d_2(\theta)\delta_{\gamma\delta}\,\delta_{\gamma\beta}$$

$$_{out}\langle b_\beta(\theta_1) \bar{b}_\delta(\theta_2) | b_\alpha(\theta_1) \bar{b}_\gamma(\theta_2)\rangle^{in} = v_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + v_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$_{out}\langle f_\beta(\theta_1) \bar{f}_\delta(\theta_2) | f_\alpha(\theta_1) \bar{f}_\gamma(\theta_2)\rangle^{in} = u_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + u_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$_{out}\langle b_\beta(\theta_1) \bar{b}_\delta(\theta_2) | f_\alpha(\theta_1) \bar{f}_\gamma(\theta_2)\rangle^{in} = d_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + d_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$_{out}\langle b_\beta(\theta_1) \bar{f}_\delta(\theta_2) | b_\alpha(\theta_1) \bar{f}_\gamma(\theta_2)\rangle^{in} = c_1(i\pi-\theta)\delta_{\alpha\beta}\,\delta_{\gamma\delta} + c_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta}$$

Its is straightfoward to get the following amplitudes with the $\frac{1}{N}$ - Feynman rules

$$v_1(\theta) = 1 - \frac{i\pi}{N}\coth\frac{\theta}{2};\quad v_2 = -\frac{2i\pi}{N\theta};\quad u_2 = \frac{2i\pi}{N\theta};\quad c_2 = -\frac{2i\pi}{N\theta};\quad d_1 = -\frac{\pi i}{N\,\mathrm{sh}\frac{\theta}{2}};$$

$$d_2 = 0 + O\left(\frac{1}{N^2}\right).$$

The computation of the transmission amplitudes $c_1(\theta)$ and $u_1(\theta)$ present a problem comming a $\frac{0}{0}$ - indetermination. This arises from the singular behavior of the mixed A_μ-π propagator at zero momentum transfer. We circunvent this problem by inserting an infinitesimal momentum transfer δ at one vertex, making use of the identity $\bar{u}(P_2)\varepsilon^{\mu\nu}(P_2-P_1)_\nu\gamma_5 u(P_1) = \frac{(P_1-P_2)^2}{2m}\bar{u}(P_2)\gamma^\mu u(P_1)$ and finally letting δ go to zero. With this prescription we get

$$c_1(\theta) = 1 + O\left(\frac{1}{N^2}\right);\quad u_1(\theta) = 1 + \frac{i\pi\coth\theta/2}{N}.$$

The calculation of exact scattering amplitudes from the bootstrap approach was done[5-7,14] and reviewed[13] many times so we will limit ourselves to state the result. Paying careful attention to signs due the statistics and fixing two free parameters by comparison with the $\frac{1}{N}$ - perturbative expressions one gets the following exact amplitudes $(\lambda = \frac{2}{N})$

$$c_1(i\pi\phi) = \prod_{\ell=0}^{\infty} \frac{\Gamma(\frac{\phi}{2}+\frac{\lambda}{2}+\ell)\Gamma(1-\frac{\phi}{2}+\ell)}{\Gamma(1-\frac{\phi}{2}+\frac{\lambda}{2}+\ell)\Gamma(\frac{\phi}{2}+\ell)} \prod_{\ell=0}^{\infty} \frac{\Gamma(\frac{\phi}{2}-\frac{\lambda}{2}+\ell)\ (1-\frac{\phi}{2}+\ell)}{\Gamma(\frac{1}{2}-\frac{\phi}{2}-\frac{\lambda}{2}+\ell)\Gamma(\frac{\phi}{2}+\ell)} \tag{1a}$$

$$v_1(i\pi\phi) = \frac{\sin\frac{\pi}{2}(\phi-\lambda)}{\sin\frac{\pi\phi}{2}}\ c_1(i\pi\phi) \tag{1b}$$

$$u_1(i\pi\phi) = \frac{\sin\frac{\pi}{2}(\phi+\lambda)}{\sin\frac{\pi\phi}{2}}\ c_1(i\pi\phi) \tag{1c}$$

$$d_1(i\pi\phi) = -\frac{\sin\frac{\pi\lambda}{2}}{\sin\frac{\pi\phi}{2}}\ c_1(i\pi\phi) \tag{1d}$$

$$v_2(\theta) = -\lambda\frac{i\pi}{\theta}\, v_1(\theta)\ ;\ u_2(\theta) = \frac{\lambda i\pi}{\theta}\, u_1(\theta)\ ;\ c_2(\theta) = -\frac{\lambda i\pi}{\theta}\, c_1(\theta)\ ;\ d_2(\theta) = -\frac{\lambda i\pi}{\theta}\, d_1(\theta) \tag{1e}$$

Let us now discuss some interesting proprieties of the solution (1a - e). The pole contained in $c_1(\theta)$ generates the well known spectrum[15]

$$m_n = m\,\frac{\sin\frac{n\pi\lambda}{2}}{\sin\frac{\pi\lambda}{2}} \qquad n = 1,2,\dots,\ N-1 .$$

In particular we have $m_{N-1} = m$. The fact that the bound state of N-1 particles has the same mass as the original particles is one ingredient in the proof that the symmetry of our solution (1.a-e) is SU(N). In fact there are two bound states of (N-1) particles with mass m in the $\overline{N}$ channel and one verifies that they have identical scattering amplitudes as the original antiparticles $\overline{b}$ and $\overline{f}$. This entails the following identification of antiparticles with bound states of (N-1) particles: the antiboson is identified with the bound state of (N-1) fermions and the antifermion is the bound state of (N-2) fermions with one boson. Symbolically we may write

$$\overline{b} = f_1 f_2 \, \cdots \, f_{N-1}$$

$$\overline{f} = f_1 \cdots f_{N-2} \, b_{N-1} + f_1 \cdots f_{N-3} b_{N-2} f_{N-1} + \cdots + b_1 f_2 \cdots f_{N-1}$$

This implies that our fundamental particles obey generalized statistics, which should be treated as in ref. 16.

For N = 2 equs (1a-e) prevent the existence of the elemetary O(3) triplet in the supersymmetric non linear σ model. This is also signaled by the vanishing of the (would be) fermion-boson reflection amplitude in the O(3) case[7]. Hence we expect the O(3) spectrum to consist only of SU(2) kinks with scattering amplitudes given by eq. (1a-e).*4

To exhibit the similarity of the supersymmetrization process with the one occurring in the non linear σ-model we rewrite $c_1(\theta)$ as

$$c_1(\theta) = u(\theta,\lambda) \; \hat{Y}_o(\theta,\lambda)$$

where

$$u(i\pi\phi,\lambda) = \frac{\Gamma(\frac{\phi}{2} - \frac{\lambda}{2}) \; \Gamma(1 - \frac{\phi}{2})}{\Gamma(\frac{1}{2} - \frac{\phi}{2} - \frac{\lambda}{2}) \; \Gamma(\frac{\phi}{2})}$$

$$\hat{Y}_o(i\pi\phi,\lambda) = \prod_{\ell=0}^{\infty} \frac{\Gamma(\frac{\phi}{2} + \frac{\lambda}{2} + \ell) \; \Gamma(1 - \frac{\phi}{2} + \ell)}{\Gamma(1 - \frac{\phi}{2} + \frac{\lambda}{2} + \ell) \; \Gamma(\frac{\phi}{2} + \ell)} \prod_{\ell=1}^{\infty} \frac{\Gamma(\frac{\phi}{2} - \frac{\lambda}{2} + \ell) \; \Gamma(1 - \frac{\phi}{2} + \ell)}{\Gamma(\frac{1}{2} - \frac{\phi}{2} - \frac{\lambda}{2} + \ell) \; \Gamma(\frac{\phi}{2} + \ell)}$$

Note that the pole is present in $u(\theta,\lambda)$ and $\hat{Y}_o(\theta,\lambda)$ is "minimal" in the sense of having no poles in the physical sheet. By the substitution $\lambda \to -\lambda$ the pole "moves" from $u(\phi,\lambda)$ to $\hat{Y}_o(\theta,\lambda)$ leaving $c_1(\theta)$ invariant in the same way as in the supersymmetric non linear σ model[7]. The analogy is now stressed by observing that the amplitude $u(\theta,\lambda)$ (with the pole) belongs to the chiral Gross-Neveu model[11,12] and $u(\phi,-\lambda)$ belongs to the CP^{N-1} model minimally coupled to massless fermions[4]. The function $\hat{Y}_o(\phi,\lambda)$ describes the supersymmetry.

Finally let us mention that it is possible to introduce a supersymmeteic Z(N) model that generalizes the one studied in ref (17). The fermion-boson transmission amplitude is the product of the minimal supersymmetry function $\hat{Y}_o$ times the amplitude of the ordinary Z(N) model[17]. This will be presented in a separate paper[18].

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FOOT NOTES

*1 As was explained in ref (7) this is even more transparent from the S-matrix point of view.

*2 In fact this also happens in the supersymmetric non linear σ model where one parameter has to be fixed[7].

*3 In the notation of ref (14) this would correspond to class II solution

*4 This expectation may possible be extended to the usual O(3) non linear σ-model.

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