

Application of the Method of Effective Boundary Conditions for Calculating the Critical Dimensions of Reactors

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When the size of a reactor greatly exceeds the neutron moderation length $\sqrt{\tau}$, the density distribution of the thermal neutrons $N(r)$ in the homogeneous multiplying medium may be described by the one-group equation

$$\Delta N(r) + \alpha^2 N(r) = 0 \quad (1)$$

where the "Laplacian" α^2 equals

$$\alpha^2 = \frac{k-1}{L^2 + k\tau} \quad (1')$$

k is the multiplication factor, L the reactor diffusion length.

Equation 1 is most readily obtained when the thermal neutron density equation is written thus:

$$\Delta N(r) - \frac{N(r)}{L^2} + \frac{k}{L^2} \int W(r,r') N(r') dr' = 0 \quad (2)$$

where $W(r,r')$ is the probability that the fast neutron produced at point r' in the capture of the thermal neutron will slow down to thermal energy at point r . Density $N(r)$ substantially differs within the range of the order of the system's dimensions, whereas $W(r,r')$ differs within the range of the order $\sqrt{\tau}$. Therefore, $N(r)$ the integral in Equation 2 may be regarded as a slow changing function and may be expanded into the powers $|r - r'|$:

$$N(r') = N(r) + \sum_{i=1}^3 (x_i' - x_i) \frac{\partial N}{\partial x_i} + \frac{1}{2} \sum_{i,k=1}^3 (x_i' - x_i)(x_k' - x_k) \frac{\partial^2 N}{\partial x_i \partial x_k}$$

Upon substitution of this expansion in Equation 2, and integration, we obtain Equation 1 if we define

$$\tau = \frac{1}{6} \bar{r}^2 = \frac{1}{2} \bar{x}^2 = \frac{1}{2} \int W(r,r') (x' - x)^2 dr$$

and consider that

$$\int W(r,r') dr' = 1$$

The deviation from Equation 1, as may be seen from the above, will be substantial only near the boundary between the multiplying medium and the reflector. It is therefore worth while to examine separately the layer whose thickness is of the order

of $\sqrt{\tau}$ near the boundary between the multiplying medium and the reflector, and from this examination to establish a certain effective boundary condition so that Equation 1, together with this boundary condition, would yield the correct values of the critical size and correct distribution of neutron density for the systems in question. Since the thickness of the layer is considerably smaller than the dimensions of the multiplying medium, the boundary condition may be deduced by examining neutron density distribution for the case of a plane boundary, using the two-group theory.*

Thus, the problem may be formulated as follows: by the two-group treatment we solve the problem of the neutron density distribution near the boundary between the infinite multiplying medium (with the multiplication factor $k = 1$, and the other properties the same as in the real system) and the reflector, and obtain the boundary condition

$$\lambda = - \left(\frac{1}{N} \frac{\partial N}{\partial x} \right)_{x=0} \quad (3)$$

where N is the neutron density in the multiplying medium far from the boundary $x = 0$. Then the thermal neutron density distribution in the reactor and the critical size will be found by solving Equation 1 with the boundary condition

$$\lambda = - \frac{1}{N} \frac{\partial N}{\partial \nu} \quad (3')$$

where ν is the normal to the multiplying medium surface.

Let us compute λ for plane geometry by the two-group treatment. Assuming that the infinite multiplying medium extends from $x < 0$, and the reflector from $x > 0$, the two-group equation is written thus:

In the multiplying medium

$$\Delta N_1 - \frac{N_1}{L_1^2} = - \frac{n_1}{L_1^2}, \quad \Delta n_1 - \frac{n_1}{\tau_1} = - \frac{kN_1}{\tau_1} \quad (4)$$

In the reflector

$$\Delta N_2 - \frac{N_2}{L_2^2} = - \frac{n_2}{L_1^2} \frac{B}{\rho}, \quad \Delta n_2 - \frac{n_2}{\tau_2} = 0 \quad (5)$$

* In the present case of large systems, the two-group treatment does not introduce any appreciable errors and is quite justified.

where N and n are respectively the densities of the slow and fast neutrons; L_1^2, L_2^2 are the squares of the diffusion lengths; τ_1, τ_2 are the squares of the slowing down lengths; $p = D_2/D_1$ is the ratio of the thermal neutron diffusion coefficients. In Equation 5

$$B = \frac{l_s}{l_{s2}} \cdot \frac{\xi_2}{\xi_1} \quad (6)$$

where l_s is the scattering mean free path averaged by the energy logarithm, and ξ is a similarly averaged mean logarithm energy loss. Equations 4 and 5 should be supplemented by boundary conditions: on the boundary of the multiplying medium and the reflector

$$\begin{aligned} N_1 &= N_2, & n_1 &= n_2 \\ N_1' &= pN_2' - tN_1, & n_1' &= sn_2' \end{aligned} \quad (7)$$

($s = \bar{l}_{s2}/\bar{l}_{s1}$ is the ratio of the averaged transport mean free paths); and on the boundary of the reflector, or more precisely on its extrapolated boundary, at $x = \Delta$:

$$N_2 = n_2 = 0 \quad (8)$$

The boundary conditions of Equation 7 take into account that a thin absorbing wall may be placed between the multiplying medium and the reflector. Its influence is considered by introducing t

$$t = \frac{3d}{l_{t1}l_{co}}$$

where d is the thickness of the wall, and l_{co} is the neutron capture mean free path in it.

The solutions of Equations 5 and 6 in the present case have the forms:

$$\begin{aligned} N_1 &= 1 - \lambda x + be^{\beta x} \\ N_2 &= A_1 e^{-\frac{x}{L_2}} + B_1 e^{-\frac{x}{\sqrt{\tau_2}}} + A_2 e^{\frac{x}{L_2}} + B_2 e^{\frac{x}{\sqrt{\tau_2}}} \\ n_1 &= 1 - \lambda x - b_2 e^{\beta x}, \quad n_2 = B_1 e^{-\frac{x}{\sqrt{\tau_2}}} + B_2 e^{\frac{x}{\sqrt{\tau_2}}} \end{aligned} \quad (9)$$

Here

$$\beta_2 = \frac{1}{L_1^2} + \frac{1}{\tau_1} \quad (10)$$

$$\gamma_2 = \beta^2 L_1^2 - 1, \quad \gamma_3 = \frac{BL_2^2}{pL_1^2} \left(1 - \frac{l_2^2}{\tau_2}\right)^{-1} \quad (11)$$

and $A_1, B_1, A_2, B_2, \lambda, b$ are the coefficients to be deduced from the boundary conditions of Equations 7 and 8. It is clear that λ introduced in Equation 9 coincides with λ obtained from Equation 3 as the functions N satisfies asymptotically, at $x < 0$ and $|x| \ll 1/\beta$, Equation 1 with $k = 1$ and

$$\lambda = -\frac{1}{N} \frac{\partial N}{\partial x}$$

From Equation 7, 8 and 9 one can readily obtain:

$$\lambda = s\mu' - \gamma_2(\beta + s\mu')b \quad (12)$$

where

$$b = \frac{-p\zeta' - p\gamma_3(\mu' - \zeta') + s\mu' - t}{\beta(1 + \gamma_2) + p\zeta' + \gamma_2 s\mu' - p\gamma_2 \gamma_3(\mu' - \zeta') + t} \quad (13)$$

$$\zeta' = \zeta \coth \zeta \Delta, \quad \mu' = \mu \coth \mu \Delta,$$

$$\mu = \frac{1}{\sqrt{\tau}}, \quad \zeta = \frac{1}{L_2} \quad (14)$$

Upon deriving the expression for λ it may be substituted in the boundary conditions for Equation 1 which will have the form:

(1) In the case of plane geometry:

$$\lambda = -\frac{1}{N} \frac{\partial N}{\partial x} \quad (15)$$

(2) In the case of spherical geometry:

$$\lambda = -\left(\frac{1}{N} \frac{\partial(rN)}{\partial r}\right)_{r=R} \quad (16)$$

where R is the sphere radius.

(3) In the case of cylindrical geometry:

$$\lambda = -\left(\frac{1}{N\sqrt{r}} \frac{\partial(\sqrt{rN})}{\partial r}\right)_{r=R} \quad (17)$$

where R is the cylinder radius.

In the case of spherical and cylindrical geometry, the use of the functions $rN(r)$ and $\sqrt{r}N(r)$ instead of $N(r)$ permits approximate allowance for the surface curve. The use of Equations 15, 16 and 17 leads to the following equations for determining the critical size:

(1) Slab reactor of thickness $2H$ with the same reflectors on its bases:

$$\alpha \tan \alpha H = \lambda \quad (18)$$

(2) Spherical reactor with radius R

$$\alpha \cot \alpha R = -\lambda \quad (19)$$

(3) Cylindrical reactor with radius R

$$\frac{\alpha J_1(\alpha R)}{J_0(\alpha R)} = \lambda + \frac{1}{2R} \quad (20)$$

where J_0 and J_1 are zero and first order Bessel functions.

Of particularly great interest is the use of the effective boundary conditions method in examining problems of the critical size of a cylindrical reactor surrounded by a reflector on all sides. In this case an analytical solution of the problem is rather difficult, as one cannot separate the variables, whereas the effective boundary conditions method yields quick results.

In the first approximation one may consider that the boundary condition on the cylinder side surface does not depend on density distribution along the height, and vice versa. From Equations 18, 20 we have†

† If the properties of the reflector on the cylinder bases differ, then Equation 22 should be replaced by

$$\tan \alpha_2 H = -\frac{(\lambda_{1z} + \lambda_{2z})\alpha_2}{\lambda_{1z}\lambda_{2z} - \lambda_2^2}$$

where λ_{1z} and λ_{2z} are the boundary conditions on two bases.

$$\frac{\alpha_z J_1(\alpha_z R)}{J_0(\alpha_z R)} = \lambda_r + \frac{1}{2R} \quad (21)$$

$$\alpha_z \tan \alpha_z H = \lambda_z, \quad \alpha_r^2 + \alpha_z^2 = \alpha^2 \quad (22)$$

In these equations λ_r and λ_z are given (they are determined by the properties of the reflector on the side surface and bases, and are computed by Equation 12); α^2 is also given. Thus, Equations 21 and 22 are a system of three equations containing the four unknown quantities R , H , α_z , α_r . From them, one may obtain R as a function of H .

The solution of the problem by this method may be appreciably more accurate if one considers the dependence of the side surface boundary conditions on neutron density distribution along the height (that is, on α_z), and vice versa: For this purpose, instead of the semi-infinite multiplying medium, considered in deducing Equations 12, 13 and 14, we shall examine the slab, which is infinite in the direction perpendicular to the boundary (axis X), but limited on one of the directions lying in the dividing plane (axis Z). Then, over sufficiently large distances from the boundary along axis X ($|x| \gg 1/\beta$) and over the entire region along the axis (with exception of a small part adjoining the boundary along axis Z) the neutron density dependence on Z may be regarded proportional to $\cos \alpha_z z$. One can readily see that as a result of the action of the Laplace operator on N and n in the left-hand side of Equations 4 and 5, $\Delta = \Delta_x + \Delta_z$ will be replaced by $\Delta_x + \alpha_z^2$. Consequently, the solution of the problem of determining λ_r is reduced to the one we have already discussed, and Equations 12-14 will be applicable if the following change is made in them:

$$\zeta \rightarrow \sqrt{\zeta^2 + \alpha_z^2}; \quad \mu \rightarrow \sqrt{\mu^2 + \alpha_z^2}; \quad \beta \rightarrow \sqrt{\beta^2 + \alpha_z^2} \quad (23)$$

The quantity λ_r computed by Equations 12-14 and 23

thus becomes a function of α_z . Substituting it in Equation 21 one may with the given R obtain α_r as a function of α_z . On the other hand, by similar argumentation, one may determine λ_z as a function of α_r and using Equation 22, deduce α_z as a function of α_r with H given. In plane α_r, α_z plot two functions $\alpha_r = \alpha_r(\alpha_z)$ and $\alpha_z = \alpha_z(\alpha_r)$ and determine the point of their intersection. This point will yield definite values of α_r , α_z and of $\alpha^2 = \alpha_r^2 + \alpha_z^2$. Thereby the system with the given R and H will be critical if its Laplacian α^2 deduced from Equation 1'† coincides with the one derived from the above examination. If this result is not obtained, then R or H should be changed and the above process should be repeated until the required value of the Laplacian is obtained.

The error of the effective boundary conditions method may be found by comparing its results with those of the exact solutions of the two-group equations. Such solutions may be obtained, for example, for a cylindrical reactor without a reflector on its bases. A comparison shows that in this case when $\alpha_r \sim 0.025$ and $\sqrt{\tau} \sim 10$, the error is about 5 to 10 per cent in α_r ; and when $\alpha_r \sim 0.015$ about 2 per cent in α_r .

The accuracy of the effective boundary conditions method for the problem of the critical size of a cylinder reactor surrounded by a reflector on all sides may be evaluated by a comparison with the exact computation by the numerical solution of the two-group equations. This comparison shows that the effective boundary conditions method yields quite satisfactory accuracy also for this problem even in the case of relatively small reactors ($\alpha \sim 0.025$).

† For greater accuracy, α^2 is best to be computed not by Equation 1' but by the transcendental equation $1 + \alpha^2 L^2 = ke^{-\alpha^2 \tau}$.