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SOLITONS AS NEWTONIAN PARTICLES

by

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ABSTRACT

The effect of external electromagnetic fields on non relativistic solitons is studied. Although the solitons are distorted by external fields, they still exhibit a Newtonian behavior. We give explicit examples of such a phenomenon, presenting solutions which exhibit Newtonian behavior for simple external fields. Furthermore, general results like charge and flux quantization are shown.

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I. INTRODUCTION

There are solutions to non linear field theoretical equations (herewith designed by solitons) which exhibit properties similar to extended particles. For instance, these solutions move with constant velocity, when they are not under the action of external fields, so they look very much like extended free particles(1). We shall show that for some simple examples it is not inappropriate to associate such solutions to particles.

Since Newton's law has to be obeyed by classical non-relativistic particles under the action of external fields, one has to check if solitons exhibit such a behavior under the action of external fields - that is, if solitons are newtonian particles. This is the question that we address ourselves in this paper. Previous works on this subject can be found in refs. (2) and (3). In ref. (3), it was shown that the sine-Gordon solitons do not exhibit a newtonian behavior.

We intend to show in this paper that a class of non-relativistic solitons do exhibit a newtonian behavior.

In section II we set the basic equations which describe the interaction of a soliton with external fields $\vec{E}$ and $\vec{B}$. This set of equations is then shown to be equivalent to that of the motion of a charged fluid under the action of external fields $\vec{E}$ and $\vec{B}$. We propose also an alternative way of obtaining Dirac's charge quantization condition on classical grounds.

We show in section III that if one is able to find solutions to the equations of motion describing the particle at rest, then it is possible to find solutions which describes a particle exhibiting a newtonian behavior. This is made explicitly for the simple examples of time dependent electric field, constant $\vec{E}$ and $\vec{B}$ fields and for harmonic oscillator like potentials.

In section IV we get a better picture of the classical motion of these solitons by looking explicitly at exactly soluble examples in 3-dimensions. The external fields do distort the soliton that is driven from a spherical soliton to a cigar like soliton (for $\vec{B}$ constant) or to a thinner soliton for other field configurations. Other possible "excited modes" solitons, in which the particle has an oscillatory shape, are explicitly exhibited. We also show that it is possible to blow up the soliton under the action of external fields in certain circunstances.

Conclusions are drawn in section V.

II. BASIC FRAMEWORK AND GENERAL RESULTS

Let $\phi_0(\vec{x},t)$ represent a soliton like solution to a non relativistic non linear equation:

$$i\frac{\partial}{\partial t}\phi_0(\vec{x},t) + \frac{\vec{\nabla}^2}{2m}\phi_0(\vec{x},t) = -G'|\phi_0| = \phi_0 F(\phi_0\phi_0^*) \tag{2.1}$$

having zero momentum.

The interaction of the soliton with external electromagnetic fields will proceed via the minimal substitution scheme:

$$\partial_\mu \rightarrow \partial_\mu - iqA_\mu \tag{2.2}$$

Where q in (2.2) is the charge of the particle, for dimensional reasons[4].

In this way we will be interested in the solution to the equation of motion:

$$i\frac{\partial}{\partial t}\phi - qV\phi + \frac{(\vec{\nabla}-iq\vec{A})^2}{2m}\phi = \phi F(\phi\phi^*) \tag{2.3}$$

The charge, linear momentum and energy associated with solution of (2.3) are given respectively by

$$Q = \int d^3\vec{x}\ \phi^*\phi \tag{2.4}$$

$$\vec{P} = \int d^3\vec{x}\ \phi^*(-i\vec{\nabla})\phi \tag{2.5}$$

$$E = \int d^3\vec{x}\left\{\phi^*(i\partial_t)\phi - F(\phi,\phi^*)\phi\phi^* + G(\phi^*,\phi)\right\} \tag{2.6}$$

where $G(\phi^*,\phi) = \int_0^{|\phi|^2} F(s)ds$.

Before continuing we let us a comment on the system of unit which we employ. The interpretation of solitons as classical particles requires the introduction of a fundamental constant with dimension of action (this fact has nothing to do with external fields applied to solitons, it appears at the level of free solitons). We will choose a system of units in which this fundamental constant and the velocity of light are taken equal to 1 and dimensionless.

First of all one notes that the equation of motion of a soliton under the action of external fields $\vec{E}$ and $\vec{B}$ is analogous to the motion of a charged fluid under the action of the same fields $\vec{E}$ and $\vec{B}$. In order to see this, let us write ϕ under the form:

$$\phi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}\ e^{is(\vec{x},t)} \tag{2.7}$$

where ρ and s are real functions. By substituting (2.7) into (2.3) we will get for the real and imaginary parts the equations:

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}_0\left[\rho\left(\frac{\vec{\nabla}s-q\vec{A}}{m}\right)\right] = 0 \tag{2.8}$$

$$\frac{\partial}{\partial t}s + \frac{1}{2m}(\vec{\nabla}s-q\vec{A})^2 - qV + H(\rho) = 0 \tag{2.9}$$

where $H(\rho)$ is given by:

$$H(\rho) = \frac{(\vec{\nabla}\rho)^2}{8m\rho^2} - \frac{\nabla^2\rho}{4m\rho} + F(\rho) \tag{2.10}$$

By taking the gradient of (2.9) and defining $\vec{V}$ as

$$\vec{V} = \frac{\vec{\nabla}s - q\vec{A}}{m} \tag{2.11}$$

we obtain for (2.9) and (2.8)

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \tag{2.12}$$

$$\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} = \frac{q}{m}\vec{E} + \frac{q}{m}\vec{V}\wedge\vec{B} - \frac{\vec{\nabla}H(\rho)}{m} \tag{2.13}$$

which are the equations describing the motion of a charged fluid, whose internal energy is $H(\rho)$, under the action of external fields $\vec{E}$ and $\vec{B}$.

Equation (2.11) imposes an extra condition on $\vec{V}$ that

$$\text{rot}\,\vec{V} = -\frac{q}{m}\vec{B} \tag{2.14}$$

An interesting feature of equation of the equation (2.3) is that by assuming only $|\phi|^2$ to be even we can derive a "non-linear Ehrenfest Theorem", that is, by defining

$$\langle\vec{x}\rangle \equiv \int d^3\vec{x}\,\phi^*\vec{x}\phi \tag{2.15}$$

$$\langle\vec{p}\rangle \equiv \int d^3\vec{x}\,\phi(-i\vec{\nabla}-q\vec{A})\phi \tag{2.16}$$

the mean value of the Lorentz force

$$\langle q\vec{E} + q\vec{V}\wedge\vec{B}\rangle = \int d^3\vec{x}\,\phi^*\left(q\vec{E} + \frac{q}{m}(-i\vec{\nabla} - q\vec{A})\wedge\vec{B}\right)\phi \tag{2.17}$$

then it follows that

$$\frac{d\langle\vec{x}\rangle}{dt} = \frac{\langle\vec{p}\rangle}{m} \tag{2.18}$$

$$\frac{d\langle\vec{p}\rangle}{dt} = \langle q\vec{E} + q\vec{V}\times\vec{B}\rangle \tag{2.19}$$

Equations (2.18) and (2.19) allows us to interpret these solutions to (2.3) as describing deformable newtonian particles.

By studying the interaction of a soliton with a Dirac monopole we shall be able to get Dirac's quantization of the charge within a classical framework. We will just follow Dirac's argument. Under a gauge transformation:

$$\vec{A} \to \vec{A} + \vec{\nabla}\alpha \qquad \text{and} \qquad V \to V - \partial_t\alpha \tag{2.20}$$

$$\phi \to \phi\,\exp - i\,q\,\alpha \tag{2.21}$$

It is possible to move the Dirac string Γ to another string Γ' by making a gauge transformation for which we have just to require that α in (2.20) and (2.21) be expressed by:

$$\alpha = -\frac{g}{4\pi}\,\Omega(x) \tag{2.22}$$

Where g is the monopole intensity and $\Omega(\vec{x})$ in (2.22) is the solid angle that an observer in the point $\vec{x}$ sees the two strings Γ and Γ'. The requirement that ϕ be single valued under change in the solid angle by an interger n times 4π is equivalente to

$$q.g = 2\pi\, n \tag{2.23}$$

which is Dirac's quantization condition. It is not so unexpected that we can derive the quantization condition without using any quantum mechanics[1].

III. GENERAL RESULTS FOR SIMPLE EXTERNAL FIELDS

We will study the motion of classical solitons under the action of time dependent electric fields, constant $\vec{E}$ and $\vec{B}$ fields and finally under the action of harmonic oscillator like potential. These are the simplest examples of soluble problems in non relativistic classical mechanics.

III.a - Time dependent electric field

In this case the potencial V(x,t) in (2.3) is given as

$$V(x,t) = -\vec{E}(t)\cdot\vec{x} \qquad (3.1)$$

If we assume that a solution of (2.3) in the absence of external fields is of the form:

$$\phi_0(\vec{x},t) = e^{i\omega t}\ \rho(\vec{x}) \qquad (3.2)$$

with ω a real constant and $\rho(\vec{x})$ a real function, then it can be easily checked that for V given by (3.1) there is a solution of (2.3) under the form:

$$\phi(\vec{x},t) = e^{i\omega t - i\int_0^t \frac{\vec{p}^{\,2}_{clas}(t')dt'}{2m}}\ e^{i\vec{p}_{clas}(t)\cdot\vec{x}}\ \rho(\vec{x}-\vec{x}_{clas}(t)) \qquad (3.3)$$

where $\vec{p}_{clas}(t)$ is the classical momentum of the particle under the action of the electric field

$$\vec{p}_{clas}(t) = q\int_0^t \vec{E}(t')dt' \qquad (3.4)$$

and $\vec{x}_{clas}$ is a solution to Newton's equation for a particle under the action of this electric field. As we can see from (2.4) the solution (3.3) describes, in this way, an extended particle whose "distribution of charge" follows the classical trajectory and whose linear momentum is given by (3.4) if ρ is even. This soliton behaves like a classical particle without deformations.

This feature (the particle moving without deformation) is a peculiarity of the constant space electric field here proposed. If the external field depends on $\vec{x}$ then one can expect deformations of the soliton. Even in the presence of a uniform magnetic field, the soliton is expected to deform as it will be shown by using explicit examples in the next section.

The question that we will address ourselves is: admitting that a solution of equation (2.3) is known, describing a particle, yet deformed, does it move like a newtonian particle? We will show in the case of simple fields that the answer is yes. We postpone to the next section the explicit solutions describing a particle at rest[5].

Let us analyse the case in which $\vec{E}$ and $\vec{B}$ are constant. We know from our previous analysis that the electric field does not change the shape of the solution. Let us then search for a solution describing the soliton at rest under the action of the magnetic field. We call such a solution $\psi_0(\xi,t)$:

$$i\partial_t\psi_0(t,\xi) + \frac{(\vec{\nabla}_\xi - iqA(\xi,t))^2}{2m}\psi_0(t,\xi) = \psi_0(\xi,t)\,F(\psi_0(\xi,t)) \qquad (3.5)$$

where $\vec{A} = -\frac{1}{2}\vec{\xi}\wedge\vec{B}$.

It is simple to check that an undistorted free particle like solution is no longer a solution to (3.5) and consequently one expects that the magnetic field will deform the soliton. However, this deformed soliton will move in a accordance with Newton's law. For a soliton under the action of constant electric and magnetic fields the equation (2.3) becomes

$$i \frac{\partial}{\partial t} \phi(\vec{x},t) - qV(\vec{x},t)\phi(\vec{x},t) + \frac{(\vec{\nabla} + \frac{iq}{2}\vec{x}\wedge B)^2}{2m} \phi(\vec{x},t) = \phi F(|\phi|^2) \qquad (3.6)$$

where v is given by (3.1).

A solution to (3.6) can be written as:

$$\phi(\vec{r},t) = \exp(i\ \vec{p}.\vec{r} - \vec{f}(t)\ \phi_0(\vec{r} - \vec{x}_{cla}(t);t) \qquad (3.7)$$

where $\vec{p} = m\ \dot{\vec{x}}_{cla}(t) - \frac{q}{2}\ \vec{x}_{cla}(t) \wedge \vec{B}$, $\qquad (3.8)$

$$f(t) = \int_0^t dt' \left\{ \frac{m}{2}\left[\dot{\vec{x}}_{cla}(t')\right]^2 + \frac{q}{2}(\vec{x}_{cla} \wedge \dot{\vec{x}}_{cla})(t).\vec{B} \right\} ,$$

$\vec{x}_{cla}(t)$ is the classicle trajectory of a particle under the action of these fields, and $\phi_0(\xi,t)$ is a solution to (3.5).

The physical meaning of the solution (3.7) is simple: the deformed particles moves in accordance with Newton's law - that is, the momentum of the particle is given by Newton's law (provided that ϕ_0 is at rest (4)) and its charge distribution follows the classical path as a whole.

The last example that we will consider is the motion of a soliton under the action of an harmonic oscillator - like potential, that is given by:

$$qV(\vec{x}) = \frac{Kx^2}{2} \qquad \text{and} \qquad \vec{A} = 0 \qquad (3.9)$$

Let $\psi_0(\xi,t)$ be a solution describing the soliton at rest under the action of such a external field:

$$i\ \partial_t \psi_0 + \frac{\vec{\nabla}^2_\xi \psi_0}{2m} - \frac{K}{2}\ \xi^2 \psi_0(\xi,t) = \psi_0\ F(|\psi_0|^2) \qquad (3.10)$$

If we have a solution of (3.10), it is possible to find a solution which obeys Newton's law. This solution is:

$$\phi(\vec{r},t) = \exp\{i(\vec{P}_{cla}.\vec{r} - f(t))\}\ \psi_0(\vec{r} - \vec{x}_{cla}(t),t) \qquad (3.11)$$

where $\vec{P}_{cla}$ in (3.11) is the momentum of the soliton obeying Newton's law:

$$\dot{\vec{P}}_{cla} = -\,k\ \vec{x}_{cla}(t) \qquad (3.12.a)$$

and $\quad f(t) = \frac{m}{2}\int_0^t (\dot{\vec{x}}_{cla}(t'))^2\ dt' \qquad (3.12.b)$

In this example again, the particle, yet deformed, moves in accordance with Newton's law.

We have been able, in this way, to find a set of transformations [(3.3),(3.7),(3.11)] which separates the motion of the center of charge from the internal motion for some simple examples of external fields. For more complex examples we have been unable, as yet, to find such transformations.

Explicit examples of non topological solitons in three dimensions are very rare. Fortunately in the only example which we know in three dimensions we are able to find explicit solutions for solitons under the action of the external fields discussed in this section.

IV. EXPLICIT EXAMPLES WITHIN THE I.B. BIRULA-J.MYCIELSKI MODEL

Motivated by the formulation of a Non Linear Quantum Mechanics preserving some basic properties of the linear theory I. B. Birula and J. Mycielski[7] were led to a model whose lagrangian is given by:

$$L[\phi,\dot{\phi},\vec{\nabla}\phi] = i\phi^{*}\partial_t\phi + \phi^{*}\frac{\vec{\nabla}^2\phi}{2m} + \frac{1}{2m\ell^2}\phi^{*}\phi[\ell n(\phi^{*}\phi a^3) - 1] \tag{4.1}$$

The Euler-Lagrange equation for ϕ which result from (4.1) is:

$$i\partial_t\phi + \frac{\vec{\nabla}^2\phi}{2m} + \frac{1}{2m\ell^2}\phi\,\ell n\,(\phi\phi^{*}a^3) = 0 \tag{4.2}$$

The solutions to equation (4.2) in ref. (7) have a probabilistic interpretation which we abandon in this paper and instead we interpret, in the language of Field Theory, as a classical particle - that is, a soliton. This viewpoint has been taken and persued in ref. (8).

A solution of (4.2) describing a soliton at rest is of the form

$$\phi(\vec{r},t) = A(\omega)\ e^{i\omega t}\ \exp\left\{-\frac{1}{2\ell^2}\vec{r}\cdot\vec{r}\right\} \tag{4.3}$$

where $|A(\omega)|^2 = \frac{1}{a^3}\exp(3-2m\ell^2\omega)$.

The coupling of the solitons to classical external fields is implemented via the minimal substitution (2.2) which for this particular model leads to the following equation of motion:

$$i\frac{\partial\phi}{\partial t} - qV(\vec{x},t)\phi + \frac{(\vec{\nabla}-iq\vec{A})^2}{2m}\phi + \frac{\phi}{2m\ell^2}\ell n(\phi^{*}\phi a^3) = 0 \tag{4.4}$$

First of all, let us search for a solution to (4.4) describing a particle at rest under the action of a constant magnetic field. In this situation we have that $\vec{A} = -\frac{\vec{r}}{2}\wedge\vec{B}, \vec{B}=b(\vec{e}_z)$, and $V=0$. Searching such a solution will be possible by making use of the ansatz solution

$$\psi_0(\vec{r},t) = A(\omega)\exp\left(\frac{-z^2}{\ell^2 2}\right)\psi'_0\,(x,y,t) \tag{4.5}$$

where

$$|A(\omega)|^2 = a^{-1}\exp(1-2m\ell^2) \tag{4.6}$$

and

$$\psi'_0(x,y,t) = C[\det A(t)]^{1/4}\exp\left[i\phi(t) - \frac{1}{2\ell^2}\sum_{K=1}^{2}\sum_{j=1}^{2}\vec{x}_k(A(t)+iB(t))_{kj}x_j\right] \tag{4.7}$$

with A and B (2x2) real symmetric matrices.

Substituting (4.5)-(4.7) into (4.4) we are led to the following set of equations[7],

$$2m\ell^2\dot{\phi} + \mathrm{tr}A - \ell n[C^2a^2(\det A)^{1/2}] = 0 \tag{4.8}$$

$$\mathrm{tr}B - \frac{m\ell^2}{2}\frac{d}{dt}[\ell n\,\det A] = 0 \tag{4.9}$$

$$m\ell^2\dot{A} = AB + BA - \frac{qb\ell^2}{2}[A,\Omega] \tag{4.10}$$

$$m\ell^2\dot{B} = B^2 - A^2 + A - \frac{qb\ell^2}{2}\,[B,\Omega] - \frac{\ell^4 q^2 b^2}{4}\,\Omega^2 \qquad (4.11)$$

where $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We have reproduced this system of equations in order to call the attention to the fact that the classical motion of solitons might be richer than its classical point like counterpart. This is due to the fact that in the case of extended particle the system exhibits "excitation modes". Let us see how this happens.

One can find a solution(7) to (4.8)-(4.11) in which the only effect on the particle will be to deform it. This solution is characterized by:

$$A = \frac{1 + \sqrt{1 + q^2 b^2 \ell^4}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (4.12)$$

$B=0$ and $\phi(t) = \beta t$, where β is constant and determined from (4.8). More explicitly one yets:

$$\psi_0(x,t) = C \left(\frac{1 + \sqrt{1 + q^2 b^2 \ell^4}}{2}\right)^{1/2} A(\omega) \exp i t \beta \exp\left[-\frac{1}{2\ell^2} \cdot \frac{1 + \sqrt{1 + q^2 b^2 \ell^4}}{2}(x^2 + y^2) - \frac{z^2}{2\ell^2}\right] \qquad (4.13)$$

As can be seen from (4.13) the soliton has changed its shape in the orthogonal directions to the external field $\vec{B}$. From a spherically symmetric soliton, it tends , for very strong fields, to a thin cigar. From (3.7) one can see that this cigar moves as a classical particle.

There is, however, another solution to (4.8)-(4.11) in which the cigar size oscillates. This kind of solution is described by the ansatz:

$$A(t) = \alpha(t)\,1$$
$$B(t) = \beta(t)\,1 \qquad (4.14)$$

By substituting (4.14) into (4.10) we get:

$$m\ell^2\dot{\alpha} = 2\alpha\beta.$$

Using that the energy of this solution is constant we get that

$$\varepsilon = -\ell n\alpha + \alpha + \beta^2\alpha^{-1} + \frac{q^2 b^2 \ell^4}{4}\alpha^{-1} \qquad (4.16)$$

is constant.

From (4.16) it follows that $\alpha_{min} \leq \alpha \leq \alpha_{max}$. By getting β in terms of ε and α from (4.16) and substituting in (4.15) we get:

$$m\ell^2\dot{\alpha} = 2\alpha\left(\varepsilon\alpha + \alpha\ell n\alpha - \alpha^2 - \frac{q^2 b^2 \ell^4}{4}\right)^{1/2} \qquad (4.17)$$

The solutions to the equation (4.17) are periodic with other periods given by:

$$T = \int_{\alpha_{min}}^{\alpha_{max}} \frac{m\ell^2\, d\alpha}{\alpha\left(\varepsilon\alpha + \alpha\ell n\alpha - \alpha^2 - \frac{q^2 b^2 \ell^4}{4}\right)^{1/2}} \qquad (4.18)$$

In other words, there are soliton solutions whose shapes change in a periodic way. Their periods are given in (4.18). These periodic cigars will, from (3.7), move like a newtonian particle.

Besides these periodic solutions, we can find others excition modes in which the solitons rotate around an axes that goes through by the center of charge and is paralel to the external magnetic field. If we choose a new coordinate system rotating with frequency ω around the field direction we can write that:

$$\dot{M} = (\dot{M})_{rot} + \omega\,[\Omega,M] \tag{4.19}$$

where $\dot{M}$ is the derivative in our original coordinate system and $(\dot{M})_{rot}$ is the derivative in the rotating frame.

Using (4.19) we have that (4.10) and (4.11) are transformed to

$$m\ell^2(\dot{A})_{rot} = AB + BA - \left(m\ell^2\omega - \frac{qb\ell^2}{2}\right)[\Omega,A] \tag{4.20}$$

$$m\ell^2(\dot{B})_{rot} = B^2 - A^2 + A - \left(m\ell^2\omega - \frac{qb\ell^2}{2}\right)[\Omega,B] - \frac{\ell^4 q^2b^2}{4}\Omega^2 \tag{4.21}$$

Let us look for time independent solutions in the in the frame. Our search for such a solutions is easier if we parametrize A and B in the following manner:

$$A = \begin{pmatrix} \alpha+\alpha_1 & \alpha_2 \\ \alpha_2 & \alpha-\alpha_1 \end{pmatrix} \tag{4.22}$$

$$B = \begin{pmatrix} \beta+\beta_1 & \beta_2 \\ \beta_2 & \beta-\beta_1 \end{pmatrix} \tag{4.23}$$

where $\vec{\alpha} = (\alpha_1\alpha_2)$ and $\vec{\beta} = (\beta_1,\beta_2)$ are two dimensional vectors and α and β are scalar parameters.

Substituting (4.22) - (4.23) into (4.20)-(4.21) we get that:

$$\alpha\vec{\beta} + \beta\vec{\alpha} + \left(m\ell^2\omega - \frac{qb\ell^2}{2}\right)\vec{e}_z \wedge \vec{\alpha} = 0 \tag{4.24}$$

$$\left(\frac{1}{2} - \alpha\right)\vec{\alpha} + \beta\vec{\beta} + \left(m\ell^2\omega - \frac{qb\ell^2}{2}\right)\vec{e}_z \wedge \vec{\beta} = 0 \tag{4.25}$$

$$\beta^2 + \vec{\beta}^2 - \alpha^2 - \vec{\alpha}^2 + \alpha + \frac{q^2b^2\ell^4}{4} = 0 \tag{4.26}$$

$$\alpha\beta + \vec{\alpha}.\vec{\beta} = 0 \tag{4.27}$$

A solution to the above set of equations is given by[7]:

$$\alpha = \frac{1}{4}\left(\sqrt{1 + 16\left(m\ell^2\omega - \frac{qb\ell^2}{2}\right)^2} + 1\right) \tag{4.28}$$

$$\beta = 0$$

$$\vec{\beta} = -\alpha^{-1}\left(m\ell^2\omega - \frac{qb\ell^2}{2}\right)\vec{e}_z \wedge \vec{\alpha}$$

$$\vec{\alpha}^2 = \alpha$$

and the direction of $\vec{\alpha}$ is arbitrary.

If we want the solution (4.28) to have finite energy charge and momentum, we have restriction on ω:

$$\left(m\ell^2\omega - \frac{q\ell^2 b}{2}\right)^2 > \frac{q^2b^2\ell^4}{4} \tag{4.29}$$

Making use of (3.7) we can get from (4.28) a solution that moves like an extended newtonian particle that is spinning around its center of charge.

One might wonder if this is not an artifact of the external field we are considering. We shall see now that for a harmonic oscillator like potential the same type of behavior happens.

For this kind of potential the equation of motion is:

$$i\partial_t\psi + \frac{\vec{\nabla}^2\psi}{2m} - \frac{k\vec{\Omega}^2}{2}\psi + \psi\,\ell n(\psi\,\psi\,\vec{a}) = 0 \tag{4.30}$$

We shall look for a solution of (4.30) describing the soliton at rest. The ansatz for this solution is:

$$\psi_0(\vec{x},t) = C\,[\det A]^{1/4}\ \exp\{i\phi(t) - \frac{1}{2\ell^2}\sum_{k=1}^{3}\sum_{j=1}^{3} x_k(A(t)+iB(t))_{kj}\ x_j \tag{4.31}$$

where now A and B are 3x3 symmetric matrices.

By applying the ansatz solution (4.31) to (4.30) we shall get the following set of coupled equations:

$$-2m\ell^2\dot{\phi} - \mathrm{tr}A + \ln[\,c^2a^3(\det A)^{1/2}] = 0 \tag{4.32}$$

$$-\mathrm{tr}B + \frac{1}{4}\,m\ell^2\,\frac{d}{dt}\,[\ln\det A] = 0 \tag{4.33}$$

$$m\ell^2\dot{A} = AB + BA \tag{4.34}$$

$$m\ell^2\dot{B} = B^2 - A^2 + A + m\ell^4 k\ 1 \tag{4.35}$$

In this case one can also find a solution in which the particle is squeezed by the external field. The solution is

$$\begin{aligned} A &= \alpha_0\ 1 \\ B &= 0 \\ \phi(t) &= t\Big(\frac{3\alpha_0+\ln\ c^2a^3}{2m\ell^2}\Big) \end{aligned} \tag{4.36}$$

$$\text{where}\quad \alpha_0 = \frac{1+\sqrt{1+4m\ell^2k}}{2} \tag{4.37}$$

Explicitly one has

$$\psi_0(\vec{r},t) = C\,\alpha_0^{3/4}\ \exp\ i\ t\ \Big(\frac{3\alpha_0+\ln\ c^2a^3}{2m\ell^2}\Big)\ \exp -\frac{\alpha_0\vec{r}^2}{2\ell^2} \tag{4.38}$$

From (4.38) one can see that the soliton has been deformed due to the action of the external field. The rate of

the radius of the soliton without extend field (R_0) and with the external field (R) is

$$\frac{R_0}{R} = \sqrt{\alpha_0} = \left[\frac{1+\sqrt{1+4m\ell^2k}}{2}\right]^{1/2} \tag{4.39}$$

In this example the particle maintains its spherical shape but becomes smaller if $K > 0$. For $K < 0$ the radius of the particle will increase its size until R assumes the critical size:

$$R_{cri} = \sqrt{2}\quad R_0 \tag{4.40}$$

This critical value is associated to a critical value of K given by:

$$K_{cri} = \frac{-1}{4m\ell^2} \tag{4.41}$$

These results are simple to understand on physical grounds. K positive means that the charge is positive (negative) and the field points inwards (outwards) and consequently one does not expect implosion of the particle as a result of the interaction responsible for holding the matter within the particle, but rather a small decrease of the radius of the particle due to the attraction of the field. K negative means repulsion (since it corresponds to the opposite situation described above) and if the external field is too strong ($K<K_{cri}$) the particle does not stand this external field and it will "explode" - that is, there is no soliton like solution for too intense repulsive field, as expected also on physical grounds.

Using (3.11) and (4.38) one can find a soliton like solution that behaves like a newtonian particle:

$$\phi(\vec{r},t) = C\,\alpha_0^{3/4}\,\exp\left\{i\phi(t) - \frac{\alpha_0}{2\ell^2}\,(\vec{r}-\vec{x}_{cla}(t))^2 + ip_{cla}\,\vec{r} - if(t)\right\} \tag{4.42}$$

where p_{cla} and $f(t)$ are given by (3.12a) and (3.12b) respectively.

We would like to stress that when $K<K_{cri}$ the "extended particle" given by (4.41) looses its newtonian features.

In this example again one can exhibit excitation modes - that is, the particle moves with a radius oscillating in time. We just look for solutions such that

$$\begin{aligned} A(t) &= \alpha(t)\,1 \\ B(t) &= \beta(t)\,1 \end{aligned} \tag{4.43}$$

From (4.33) it follows that

$$m\ell^2\dot{\alpha} = 2\alpha\beta \tag{4.44}$$

The conservation of energy implies that

$$\varepsilon \text{ is constant} = \alpha + \beta^2\alpha^{-1} + km\ell^4\alpha^{-1} - \ell n\alpha \tag{4.45}$$

Using (4.44) and (4.45) one gets that:

$$m\ell^2\dot{\alpha} = 2\alpha\,(\varepsilon\alpha - \alpha^2 + \alpha\ell n\alpha - km\ell^4)^{1/2} \tag{4.46}$$

Equation (4.46) exhibits solutions in which the soliton can be visualized as having an oscillating radius whose period is given by:

$$T = \int_{\alpha_{min}}^{\alpha_{max}} \frac{m\ell^2 d\alpha}{\alpha(\varepsilon\alpha - \alpha^2 + \alpha\ell n\alpha - km\ell^4)^{1/2}} \tag{4.47}$$

$$\text{where } \varepsilon = \alpha_{\substack{max\\min}} + km\ell^4\,\alpha^{-1}_{\substack{max\\min}} - \ell n\,\alpha_{\substack{max\\min}} \tag{4.48}$$

From this and using the result (3.11), one can see that this oscillating soliton will move again as a newtonian particle.

One can also find solutions to (4.31)-(4.34) which describe a soliton moving along the classical path and spinning around its center of charge. Lets find some of these "excitation modes" spinning around de z axis.

First of all we make the ansatz that:

$$A = \begin{pmatrix} A' & 0 \\ 0 & \alpha_0 \end{pmatrix} \tag{4.49}$$

$$B = \begin{pmatrix} B' & 0 \\ 0 & \beta_0 \end{pmatrix} \tag{4.50}$$

$$\text{where} \quad \alpha_0 = \frac{1 + \sqrt{1 + 4m\ell^2 k}}{2} \tag{4.51}$$

and A' and B' are 2 x 2 real symmetric matrices the substitution of (4.49)-(4.51) into (4.33)-(4.34) lead us to find that:

$$m\ell^2\dot{A}' = A'B' + B'A' \tag{4.52}$$

$$m\ell^2\dot{B}' = B'^2 - A'^2 + A' + m\ell^4\,k\,1 \tag{4.53}$$

Now we choose an coordinate system that rotates around the z axis with frequency ω. We consider now the relation:

$$\dot{M} = (\dot{M})_{new} + \omega\,[\Omega, M] \tag{4.54}$$

We parametrize A' and B' in the following any:

$$A = \begin{pmatrix} \alpha+\alpha_1 & \alpha_2 \\ \alpha_2 & \alpha-\alpha_1 \end{pmatrix} \tag{4.55}$$

$$B = \begin{pmatrix} \beta+\beta_1 & \beta_2 \\ \beta_2 & \beta-\beta_1 \end{pmatrix} \tag{4.56}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\beta_1, \beta_2)$ are two dimensional vectors and α and β we scalar parameters.

If we look for time independent solutions in the rotating frame and substitute (4.54)-(4.56) into (4.52)-(4.53) we shall get that:

$$\alpha\vec{\beta} + \beta\vec{\alpha} + m\ell^2\omega \; \vec{e}_z \wedge \vec{\alpha} = 0 \tag{4.57}$$

$$(\frac{1}{2} - \alpha) \; \vec{\alpha} + \beta\vec{\beta} + m\ell^2\omega \; \vec{e}_z \wedge \vec{\beta} = 0 \tag{4.58}$$

$$\beta^2 + \vec{\beta}^2 - \alpha^2 - \vec{\alpha}^2 + \alpha + m\ell^4 k = 0 \tag{4.59}$$

$$\alpha\beta + \vec{\alpha}\,\vec{\beta} = 0 \tag{4.60}$$

The solution to the system of equations (4.57)-(4.60) is given by:

$$\alpha = \frac{1}{4} \left(1 + \sqrt{1 + 16(\omega m\ell^2)^2}\right) \tag{4.61}$$

$$\beta = 0 \tag{4.62}$$

$$\vec{\beta} = -\frac{\omega m\ell^2}{\alpha} \, e_z \wedge \vec{\alpha} \tag{4.63}$$

and $\vec{\alpha}$ is an arbitrary vector whose modulus is

$$\vec{\alpha}^2 = \alpha^2 - 2\alpha \; \left[m^2\ell^4\omega^2 - \gamma\right] \tag{4.64}$$

In order to have a solution with finite charge energy and momentum, the following relation has to be verified:

$$\alpha > 2(m^2\ell^4\omega^2 - \gamma) \tag{4.65}$$

If we apply the transformation (3.11) to the solution given above, the new solution that we get moves like a Newtonian particle which is spinning around its center of charge.

The next question would be to investigate the stability of the solution we have. This would be relevant in order to see if all "excitation modes" are stable or if they will decay after some time. This is a very interesting and difficult question. In the appendix we show that the solution in which the particle is rigid (solution (4.42)) is stable.

V. CONCLUSIONS

We have tried to unveal some aspects of the classical motions of solitons. Some features of the motion are really intriguing. Since solitons under the action of external fields have a great similarity with the motion of a fluid, the fact that such a fluid exhibits "excitation modes" is not surprising. We have exhibited some of these "excited modes" explicity here.

One does not know if all these modes are stable. We have shown, however, that the "quasi-rigid" mode is stable. In this mode, the particle is distorted by external fields without exhibiting any internal motion.

The fact that the soliton, not being rigid, should be deformed by external fields varying in space is expected on physical grounds. We have shown, however, that no matter how complex these deformations are the soliton still moves as a Newtonian particle - that is, the distribution (or center) of charge follows the classical path and the momentum of the soliton is given classically. This was shown explicitly for simple configurations of external fields, and all the solutions which we have shown have finite charge, energy and momentum.

Another feature exhibited here in a simple 3-dimensional example is that the soliton might be deformed in an irreversible way by strong and fast changing fields. We have shown explicitly that there is no longer soliton like solution for fields under these circunstances.

We believe that our results are in contradiction with those of ref. (2), and this is somewhat puzzling in view of the fact that the non-relativistic limit of sine-Gordon model is a non-linear Schrödinger model which, as we have shown here, exhibits a Newtonian behavior.

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APPENDIX

In this appendix we shall proof that the solution (4.42) is stable. First of all we add a small fluctuation η to the solution (4.42)(ϕ):

$$\psi(\vec{x},t) = \phi(\vec{x},t) + \eta(\vec{x},t) \qquad (A.1)$$

Substituting ψ into (4.30) and keeping only the linear terms in η we get that:

$$i\partial_t\eta + \frac{\vec{\nabla}^2\eta}{2m} - \frac{1}{2}k\vec{x}^2\eta + \frac{\eta^*\phi}{\phi^*} + \eta + \eta\,\ell n(\phi^*\phi a^3) = 0 \qquad (A.2)$$

We define:

$$\eta(\vec{r},t) = \exp\{i(\vec{P}_{cla}\ \vec{r} - f(t))\}\, h(\vec{x} - \vec{x}_{cla}\,;t) \qquad (A.3)$$

where $\vec{P}_{cla}$ and $f(t)$ are given by (3.12.a) and (3.12.b).

Using (A.3) and (4.41) and defining $\vec{\xi} = \vec{x} - \vec{x}_{cla}$ we get that

$$i\,\partial_t h + \frac{\vec{\nabla}^2_{\xi} h}{2m} - \frac{K}{2}\,\frac{\xi^2 h}{2m\ell^2} + \frac{h(1+\gamma)}{2m\ell^2} = -\,\frac{h^*}{2m\ell^2} \qquad (A.4)$$

where ∂_t does not act on ξ, $K = 2m\ell^2 k + \frac{2\alpha_0}{\ell^2}$, and

$r = \ell n\, C^2 a^3$.

The solutions of (A.4) are:

$$h_{\{k_i\}}(\vec{\xi},t) = f_{\{k_i\}}(t) \prod_{j=1}^{3} h_{k_j}\left[(\xi_j - \xi_{0j})(\frac{K}{2\ell^2})^{1/4}\right] \qquad (A.5)$$

where $\{k_i\}$ is a set of 3 , non negative integers and h_k is the $k^{\underline{th}}$ eigenfunction of 1-dimensional harmonic oscillator, and

$$f_{\{k_i\}} = A_{\{k_i\}} \quad e^{i\delta t} + B_{\{k_i\}} \quad e^{-i\delta t} \qquad (A.6)$$

with $A_{\{k_i\}}$ and $B_{\{k_i\}}$ constants and δ is real and given by

$$\delta^2 = \frac{(1 + \gamma - \sum_{j=1}^{3} (2k_j+1) \alpha_0^2\ell^2)^2 - 1}{4m^2\ell^4} \quad . \qquad (A.7)$$

From (A.5) we can see that (4.14) is stable in the infinitesimal sense.

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(4) One notes that in Field Theory the charge of a particle is related to the coupling constant, which appears in the minimal substitution, by $q=eh$.

(5) We say that a soliton is at rest if its linear momentum is zero.

(6) It is assumed throughout this article that we have chosen the c constants such that $Q(\phi)=1$.

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