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THE MEASUREMENT OF THE ELECTRON DENSITY IN A TOKAMAK BY FARADAY ROTATION

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ABSTRACT

We describe a method for the measurement of the electron density in a Tokamak by measuring the Faraday rotation in a far-infrared beam propagating in the plasma. The propagation direction is considered tangential to a toroidal field line and it is shown that the effect on the polarization of the wave is reduced to pure optical activity related only to the toroidal field and to the electron density. Detailed calculations are presented for the TBR Tokamak at Instituto de Física (USP).

RESUMO

Neste trabalho é proposto um método para medir a densidade de elétrons num Tokamak por rotação de Faraday num feixe de radiação infravermelha. A propagação é analisada numa direção tangencial a uma linha de campo toroidal. É mostrado que o efeito sobre o estado de polarização da onda se reduz a atividade ótica pura dependente somente do campo toroidal e da densidade de elétrons. Resultados de cálculos detalhados são apresentados para o Tokamak TBR do Instituto de Física (USP).

1. INTRODUCTION

The measurement of the plasma electron density by Faraday rotation was made in a electrodeless linear machine[1]. In a tokamak, Craig[2] proposed the measurement of the electron density by measuring the change in the polarization state of a test wave propagating along a vertical direction (x-axis in Fig. 1). He shows that if the initial state is linearly polarized at an angle of 45° to the toroidal direction the change in the polarization is independent of the poloidal field and it is related only to the electron density. However, the change in the polarization in this method is not a pure Faraday rotation and the techniques for the measurement of an arbitrary change in the polarization state are not yet well developed.

In this work we analyse the far-infrared propagation in a horizontal direction (z-axis in Fig. 1). We show that at the usual conditions in tokamaks, the effect on the polarization state is a pure Faraday rotation related only to the toroidal magnetic field and to the electron density. Results from detailed calculations are presented for the TBR Tokamak[3]. The vessel of this tokamak has two aligned ports that are suitable for a measurement of the Faraday rotation of a laser probing beam.

2. LINEAR BIREFRINGENCE AND OPTICAL ACTIVITY IN A MAGNETIZED PLASMA

The effect of elliptical birefringence on the wave propagation can be described[4] as a superposition of the effects of linear birefringence and optical activity. According to cristal optics[4], the symmetric real part of the dielectric tensor describes the linear birefringence while the antisymmetric imaginary part describes the optical activity. The characteristic states and refractive indices can be obtained separately for linear birefringence and optical activity. In this section are presented the results obtained[5] from the dielectric tensor for a cold magnetized plasma. The reference system is defined in Fig. 2 where the wave propagation is in z-direction. The following dimensionless parameters are used

$$X = \frac{\omega_p^2}{\omega^2} \quad ; \quad A_x = 1 - X$$
$$Y = \frac{\omega_c}{\omega} = \frac{eB}{\omega m} \; , \quad Y_L = \frac{eB_L}{\omega m} \; , \quad \text{and} \quad Y_T = \frac{eB_T}{\omega m} \; , \tag{2.1}$$

where ω, ω_p, and ω_c are respectively the wave frequency, the electron plasma frequency, and the electron cyclotron frequency. B, B_T and B_L are the total magnetic field and its components transversal and longitudinal to the propagation direction.

The Poincaré sphere[6] is used to describe the polarization evolution of the wave in the plasma. The longitude on the Poincaré sphere is 2ψ, where ψ is the tilt angle of the polarization ellipse. The latitude 2ϕ is related to the eccentricity e of the ellipse by

$$\text{tg}\ \phi = \pm(1-e^2)^{1/2} \tag{2.4}$$

where the positive sign is chosen for right rotating ellipses (R). We will use the IEEE convention for the circularly

polarized waves L and R.

The characteristic states L_x and L_o for linear birefringence are represented on the equator of the Poincaré sphere (Fig. 3) and their longitudes are

$$2\psi_x = 2\beta \quad \text{and} \quad 2\psi_o = 2\psi_x + \pi \; , \tag{2.5}$$

where the angle β is indicated in Fig. 2. The corresponding refractive indices are given by[5]

$$\mu_o^2 = 1 - \frac{X}{1 - \frac{A_x Y_L^2}{A_x^2 - Y_T^2}} \quad \text{and} \quad \mu_x^2 = 1 - \frac{XA_x}{A_x - Y^2} \tag{2.6}$$

The characteristic states for the optical activity are the circularly polarized waves L and R. The refractive indices are given by[5]

$$\mu_{R,L}^2 = \frac{1}{1 + \frac{X}{A_x^2 - Y^2}\left(A_x - \frac{Y_T^2}{2A_x} \pm Y_L\right)} \tag{2.7}$$

where the upper sign is corresponding to R.

We will consider an arbitrary polarization state (P) propagating in an infinitesimal slab dz of the plasma. The effect of the linear birefringence is described on the Poincaré sphere as a counterclockwise rotation of P about L_x of an angle (see Fig. 3).

$$2d\alpha_b = \frac{\omega}{c}(\mu_o - \mu_x)dz = \frac{\omega}{c}\Delta\mu_b\, dz \tag{2.8}$$

For small X, we obtain from (2.6) by series expansion up to order X^2

$$\Delta\mu_b = \frac{XY_T^2}{2(1-Y^2)} \left[1 + \frac{X}{2(1-Y^2)} \left(3 - 2Y^2 + \frac{3}{2} Y_T^2\right)\right] \tag{2.9}$$

The effect of the optical activity is described as a counterclockwise rotation of P about R of an angle (Fig. 4)

$$2d\alpha_c = \frac{\omega}{c} (\mu_L - \mu_R)\, dz = \frac{\omega}{c} \Delta\mu_c\, dz \tag{2.10}$$

Up to orde X^2, we obtain for small X

$$\Delta\mu_c = \frac{XY_L}{(1-Y^2)} \left[1 + \frac{1}{2} \frac{X}{(1-Y^2)} \left(\frac{3}{2} Y_T^2 + 1\right)\right] \tag{2.11}$$

The effect of the elliptical birefringence can be obtained by the superposition method on the Poincaré sphere[4]. According to this method, the infinitesimal displacements corresponding to the linear birefringence and optical activity are supposed independent and summed.

In the next section, we will integrate separately the displacements due to the linear birefringence and the optical activity. The total displacement on the Poincaré sphere cannot be obtained by this procedure, except if one of the effects is negligible. But the results are useful in the comparison between the two effects.

3. THE MEASUREMENT OF THE ELECTRON DENSITY IN A TOKAMAK BY FARADAY ROTATION

We will consider the far-infrared propagation in a Tokamak (Fig. 1). The following conditions are usually fulfilled for $\lambda < 1$ mm:

$$X = 0.090\, n(10^{14}\text{cm}^{-3})\, \lambda^2(\text{mm}) \ll 1 \tag{3.1}$$

$$Y^2 = 0.0087 \left[B(T)\,\lambda(\text{mm})\right]^2 \ll 1 \tag{3.2}$$

$$B_{pm}^2 \ll B_t^2 \tag{3.3}$$

where B_t, B_{pm} and B are respectively the toroidal magnetic field, the maximum polidal field, and the total magnetic field.

The magnetic field transversal to the propagation direction (z-axis) can be written in the form

$$\vec{B}_T = \vec{B}_p + \vec{B}_v + \vec{B}_{tT} \tag{3.4}$$

where $\vec{B}_p$, $\vec{T}_{tT}$, and $\vec{B}_v$ are respectively the poloidal field, the transversal toroidal field, and the external vertical field. The vertical field can be approximately obtained using the cylindrical approximation and neglecting the external vertical field. These approximations are consistent with reproducing the actual vertical field. Assuming a uniform current distribution, we have

$$B_p = \frac{\mu_o I}{2\pi a^2} (R - R_m) \quad , \tag{3.5}$$

where I is the plasma current and a , R , and R_m are defined in Fig. 1. The toroidal field is given by

$$B_t = \frac{B_o R_o}{R} = B_o \cos\theta \ , \qquad (3.6)$$

where B_o is the maximum toroidal field along the path of the wave.

It results from Eq. (3.5) and (3.6)

$$\frac{B_p^2}{B_t^2} = \left(\frac{\mu_o I}{2\pi a B_o}\right)^2 \left(\frac{R-R_m}{a}\right)^2 \left(\frac{R}{R_o}\right)^2 \qquad (3.7)$$

where

$$\frac{R-R_m}{a} \leqslant 1 \qquad , \qquad \frac{R}{R_o} \simeq 1$$

and

$$\left(\frac{\mu_o I}{2\pi a B_o}\right) = 0.04 \left[\frac{I\,(KA)}{a\,(cm)\,B_o\,(KGauss)}\right]^2 \qquad (3.8)$$

is of the order 0.01 or smaller for the most of tokamaks. Thus, it results the condition (3.3).

We consider a linearly polarized wave as the initial state and calculate separately the effects of linear birefringence and optical activity. The total effect on the polarization state cannot be obtained by this procedure, except if one of the effects is negligible. It will be shown that the linear birefringence is negligible in the conditions that we assume for the wave propagation.

If the linear birefringence is absent, the optical activity is described on the Poincaré sphere as a counterclockwise rotation of P about R . The angle of rotation is obtained from Eqs. (2.10) and (2.11) in first approximation as

$$2\alpha_F = \frac{\omega}{c} \int XY_L \, dz \qquad (3.9)$$

The effect of the linear birefringence in a infinitesimal slab dz of the plasma is described as a counterclockwise rotation of P about L_x . The angle is obtained in first approximation from Eqs. (2.8) and (2.9) as

$$2d\alpha_b = \frac{\omega}{c} \ \frac{1}{2} \ XY_T^2 \, dz \qquad (3.10)$$

Because the state L_x is not fixed, this equation cannot be directly integrated, even if the absence of the optical activity is assumed. We define

$$2\alpha_b = \frac{\omega}{c} \int \frac{1}{2} XY_T^2 \, dz \qquad (3.11)$$

and we show later that the actual displacement corresponding to the linear birefringence is always smaller than $2\alpha_b$.

The longitudinal and the transversal magnetic fields are given by

$$B_L = B_t \cos\theta = B_o \cos^2\theta \qquad (3.12)$$

and

$$B_T = (B_t\sin^2\theta + B_p^2)^{1/2} = B_o\cos\theta\left(\sin^2\theta + \frac{B_p^2}{B_t^2}\right)^{1/2} \qquad (3.13)$$

It results from (2.1)

$$Y_L = Y_o\cos^2\theta$$

and

$$Y_T^2 = Y_o^2\cos^2\theta\left[\sin^2\theta + \frac{B_p^2}{B_t^2}\right] \quad , \qquad (3.14)$$

where

$$Y_o = \frac{eB_o}{\omega m}$$

The conditions (3.2) and (3.3) show that

$$\frac{1}{2}\,\frac{Y_T^2}{Y_L} = \frac{1}{2}\,Y_o\left[\sin^2\theta + \frac{B_p^2}{B_t^2}\right] << 1 \qquad (3.15)$$

if $\sin^2\theta$ is not too large. Comparing the integrals (3.9) and (3.11), it results

$$\alpha_b << \alpha_F \qquad (3.16)$$

The condition (3.15) will be worst at the plasma boundary where the plasma electron density (and so X) decrease to zero.

The longitude on the Poincaré sphere of the characteristic state L_o is obtained from the direction of the transversal magnetic field (Eq. (2.5)). The transversal field

is essentially the transversal toroidal field at the ends of the path of the wave and it will be the poloidal field in the intermediary region. Thus, the transversal field rotates by an angle π along the path of the wave and so, the state L_o rotates on the Poincaré sphere by an angle 2π .

In the following discussion, we assume the longitude of L_o at the initial instant as reference to measure the longitudes on the Poincaré sphere. Thus, the longitude $2\psi_o$ of the state L_o changes from 0 to 2π along the path of the wave. We also assume that the initial polarization of the test wave is $2\psi = 2\phi = 0$.

The change $2d\phi$ in the latitude of the test wave corresponding to an infinitesimal slab dz of the plasma can be obtained from Fig. 5. If ρ is the radius of rotation of P about L_o , the displacement of P corresponding to the linear birefringence is given by

$$dr = 2\rho d\alpha_b$$

and the corresponding change of the latitude is

$$2d\phi = \cos\gamma\ dr$$

If the latitude 2ϕ is not very large

$$\cos\gamma \cong 1 \qquad \text{and} \qquad \rho \cong \ell = \sin(2\psi_o - 2\psi)$$

and using these approximations

$$2d\phi = 2\sin(2\psi_o - 2\psi)\ d\alpha_b \quad . \qquad (3.17)$$

The longitude 2ψ changes essentially from 0 to $2\alpha_F$; thus, it results that $(2\psi_o-2\psi)$ changes from 0 to $2(\pi-\alpha_F)$. If $\alpha_F \gtrsim 2\pi$, it results from the oscilation of $\sin(2\psi_o-2\psi)$ that

$$\phi << \alpha_b \tag{3.18}$$

Actually this condition will be reasonable in any case. It results from Eqs. (3.16) and (3.18) that the angle ϕ is negligible. Thus, the effect on the polarization state is a displacement along the equator of the Poincaré sphere by an angle $2\alpha_F$. Or equivalently, the effect on the polarization is a pure Faraday rotation by an angle α_F .

The electron density can be written in the form

$$n(R_o,\theta) = n_o f(R_o,\theta) \tag{3.19}$$

where n_o is the electron density for $\theta=0$ and $R=R_o$. Using (3.1), (3.2), and (3.14), we obtain from (3.9)

$$\alpha_F = 26.3\ B_o(T)\lambda^2(mm)R_o(m)n_o(10^{14}cm^{-3})I_F(\theta_o) \tag{3.20}$$

where

$$I_F(\theta_o) = \int_{-\theta_o}^{\theta_o} f(R_o,\theta)\ d\theta$$

Similarly, we obtain from (3.11)

$$\alpha_b = 1.23\ B_o^2(T)\lambda^3(mm)R_o(m)n_o(10^{14}cm^{-3})I_b(\theta_o) \tag{3.21}$$

where

$$I_b(\theta_o) = \int_{-\theta_o}^{\theta_o} f(R_o,\theta) \left(\sin^2\theta + \frac{B_p^2}{B_t^2}\right) d\theta$$

Since $f(R_o,0) = 1$ and $f(R_o,\theta) = 0$, we can estimate the integrals I_F and I_b using a simple function

$$f(R_o,\theta) = \cos\frac{\pi}{2\theta_o}\theta$$

Neglecting the poloidal field and taking $\theta_o=40^o$ we have

$$I_F = 0.9 \quad \text{and} \quad I_b = 0.08 \tag{3.22}$$

Equations (3.21) and (3.18) show that linear birefringence is negligible. Estimating α_b for an unfavourable case,

$$\lambda = 1mm\ ,\quad B_o = 3T\ ,\quad R_o = 1m \quad \text{and} \quad n = 5\times10^{13}cm^{-3}$$

we have for the values (3.22),

$$\alpha_b \approx 0.4 \quad \text{and} \quad \alpha_F \approx 36$$

In this case $\alpha_F >> 2\pi$ and the condition (3.18) is very well satisfied, or

$$\phi << 0.4$$

Smaller values of the wavelength λ are available in the far-infrared region.

The density profile $n_o(R_o)$ in the Tokamak can be obtained from the measurements of α_F (for some values of R_o) by a convolution procedure. For each value of R_o, the integral $I_F(\theta_o)$ is not expected to depend strongly on the density profile (see Fig. 1). The values of α_F are expected to be strongly dependent on $n_o(R_o)$. It means that the convolution procedure to determine $n_o(R_o)$ from the values of α_F can be simple and accurate.

4. CALCULATIONS FOR TBR TOKAMAK

The electron density can be measured in the TBR Tokamak[3] by Faraday rotation in a far-infrared test beam. We consider the propagation through the two aligned diagnostic ports of the Tokamak ($R_o=R_m$ in Fig. 1). The main parameters of Tokamak are presented in Table 1 and for λ=1mm we have

$$X \approx 0.009$$

$$Y_o \approx 0.05$$

and

$$\frac{B_{pm}^2}{B_t^2} \approx 0.01$$

Thus, the conditions (3.1), (3.2) and (3.3) are fulfilled. It results from Eq. (3.21)

$$\alpha_b \simeq 0.0007 \text{ rd}$$

and we see from (3.18) that the linear birefringence is a negligible effect. Thus, the effect on the polarization state will only be pure optical activity. We consider a linearly polarized test wave at the initial instant and the Faraday rotation angle is obtained from (2.10) and (2.11)

$$d\alpha_F = \frac{1}{2}\,\frac{\omega}{c}\,\frac{XY_L}{1-Y^2}\left[1 + \frac{1}{2}\,\frac{X}{1-Y^2}\,(1 + \frac{3}{2}\,Y_T^2)\right] \tag{4.1}$$

Neglecting the poloidal field, it results from (3.6) and (2.1)

$$\frac{1}{1-Y^2} = \frac{1}{1-Y_o^2\cos^2\theta} \cong \frac{1}{1-Y_o^2} - Y_o^2 \text{sen}^2\theta$$

and the second term can be neglected. Also

$$\frac{3}{2}\,Y_T^2 << 1$$

can be neglected in the second order term in (4.1). Integrating (2.1) along the path of the wave we obtain

$$\alpha_F = \frac{1}{2}\,\frac{\omega}{c}\,\frac{Y_o n_o R_o}{(1-Y_o^2)}\,\frac{e^2}{m\varepsilon_o\omega}\left[I_1 + \frac{1}{2}\,\frac{n_o}{(1-Y_o^2)}\,\frac{e^2}{m\varepsilon_o\omega^2}\,I_2\right] \tag{4.2}$$

where

$$I_1 = \int_{-\theta_o}^{\theta_o} f(R_o,\theta)\,d\theta \tag{4.3}$$

and

$$I_2 = \int_{-\theta_o}^{\theta_o} f^2(R_o\theta)\,d\theta \tag{4.4}$$

Eq. (4.2) can be written in the form

$$\alpha_F = 2.632 \frac{B_o n_o R_o \lambda^2}{(1-Y_o^2)} \left[I_1 + 0.0045\, n_o \lambda^2 I_2 \right] \qquad (4.5)$$

where the units for λ , B_o , n_o , and R_o are respectively mm., Tesla , $10^{13}cm^{-3}$, and m .

The electron density is assumed of the form

$$n(R) = n_m \left[1 - \frac{(R-R_m)^2}{a^2} \right]^{\ell} \qquad (4.6)$$

where n_m is the maximum electron density in the tokamak. We obtain from Fig. 1 and Eq. (3.19)

$$f(R_o,\theta) = \left[1 - \frac{R_o^2}{a^2} \left(\sec\theta - \frac{R_m}{R_o} \right)^2 \right]^{\ell} \frac{n_m}{n_o} \qquad (4.7)$$

where

$$\frac{n_o}{n_m} = \left[1 - \frac{(R_o-R_m)^2}{a^2} \right]^{\ell} \qquad (4.8)$$

The values of I_1 and I_2 numerically calculated for $R_o=R_m$ are presented in Table 2. The Faraday rotation angles α_F are given in Table 3. We can see that the angle α_F has small dependence on the exact form of the density profile (for fixed R_o). It means that an acceptable value of n_o can be obtained from only a measurement of α_F (for $R_o=R_m$).

The density profile $n_o(R_o)$ can be obtained

from the values of α_F for different values of R_o . These measurements would be troublesome in the TBR Tokamak because of the reduced dimensions of the aligned vessel ports. But we will consider this case as a numerical illustration.

Neglecting the second term in (4.5), we obtain

$$\alpha_F(R_o) = 2.632 \frac{B_o \lambda^2 n_o(R_o)}{(1-Y_o^2)} R_o\, I_1(R_o) \qquad (4.9)$$

where the calculated values of $I_1(R_o)$ are shown in Table 4. $I_1(R_o)$ is weakly dependent on the density profile (or ℓ) for each value of R_o , while $n_o(R_o)$ is strongly dependent on ℓ (see Eq. (4.8)). It means that it would be simple to obtain the accurate values of $n_o(R_o)$ from $\alpha_F(R_o)$ by successive approximations.

5. DISCUSSION

The poloidal field in a tokamak can be obtained by measuring the Faraday rotation in a far infrared test beam[7,8]. The propagation direction must be perpendicular to the toroidal field. We show that if the propagation is tangential to the toroidal field lines, the effect on the polarization is a pure Faraday rotation unrelated to the poloidal field. Thus, the electron density in a tokamak can be measured this way. In particular, we show that the method is viable in the TBR Tokamak. Far-infrared lasers and techniques for measuring Faraday rotation angles have been developed at UNICAMP[9]. The wavelengths are available in the range from 0.04 to 1.2 mm.

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FIGURES

TABLES

Major Radius	R_m	0.30 m
Plasma Radius	a	$\sim$ 0.08 m
Plasma Current	I	$\leq$ 20 KA
Toroidal Field	B_t	$\lesssim$ 0.5 T
Poloidal Field	B_p	$\leq$ 0.05 T
Electron Density	n	$\sim 10^{13}\,cm^{-3}$
Maximum θ	θ_o	38^o

TABLE 1

ℓ	I_1	I_2
1	1.086	0.978
2	0.978	0.861
3	0.909	0.792
4	0.861	0.745
6	0.792	0.682
8	0.745	0.639

TABLE 2

α_F (degrees)					
n_o ($10^{13}cm^{-3}$)	ℓ=1	ℓ=2	ℓ=3	ℓ=4	ℓ=6
0.01	0.25	0.22	0.21	0.20	0.18
0.05	1.23	1.11	1.03	0.98	0.90
0.1	2.46	2.22	2.06	1.95	1.80
0.5	12.3	11.1	10.3	9.77	9.00
1.0	24.7	22.3	20.7	19.6	18.0
5.0	126	113	105	99.5	91.6
10.0	256	230	214	203	187

TABLE 3

R_o (cm)	ℓ=1	ℓ=2	ℓ=3	ℓ=4	ℓ=5
30	1.09	0.98	0.91	0.86	0.82
31	0.97	0.85	0.78	0.72	0.68
32	0.87	0.75	0.67	0.61	0.57
33	0.77	0.65	0.58	0.52	0.48
34	0.67	0.56	0.49	0.44	0.41
35	0.56	0.46	0.40	0.36	0.33
36	0.45	0.37	0.32	0.28	0.26
37	0.31	0.25	0.22	0.19	0.18

TABLE 4

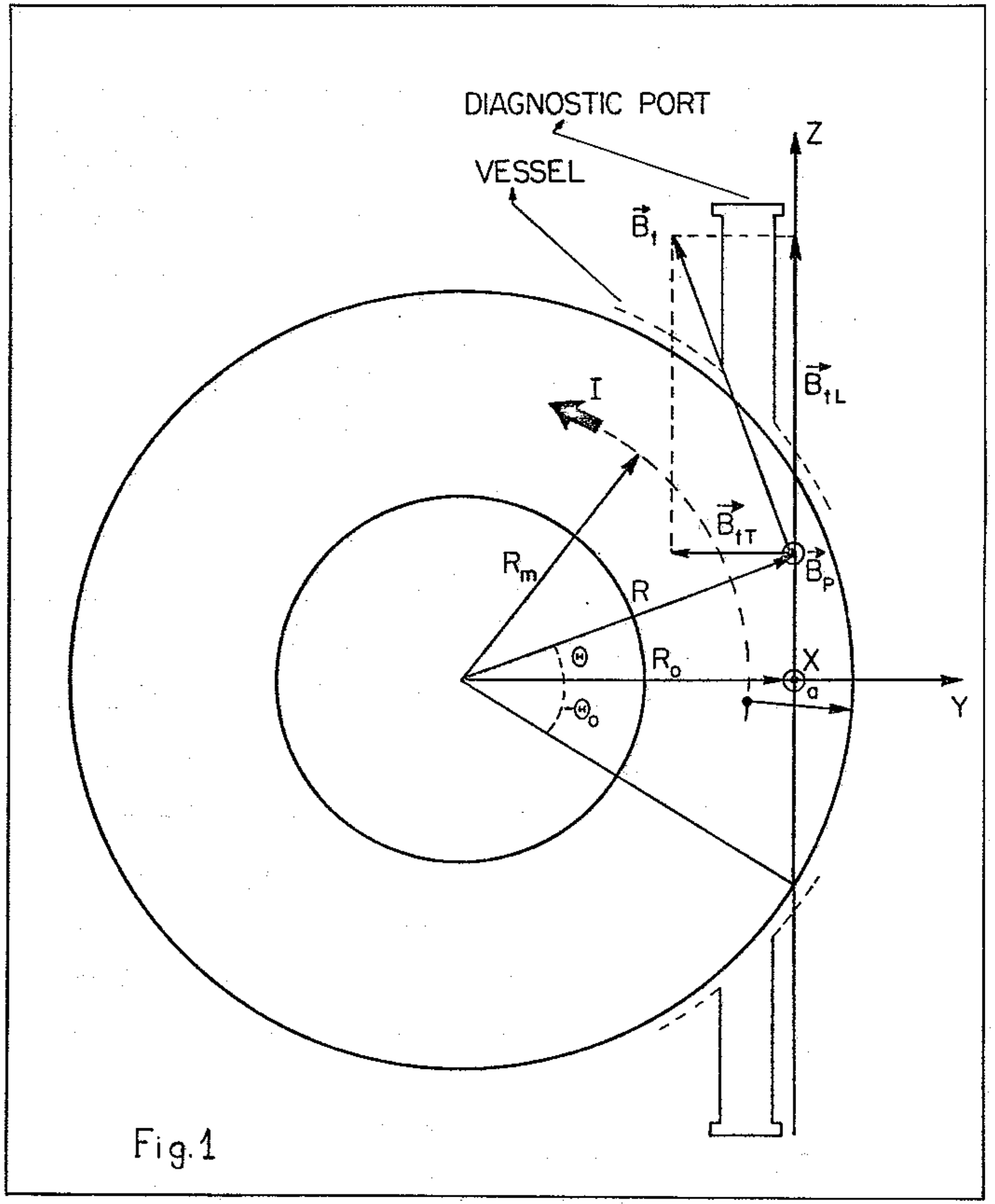

Fig.1

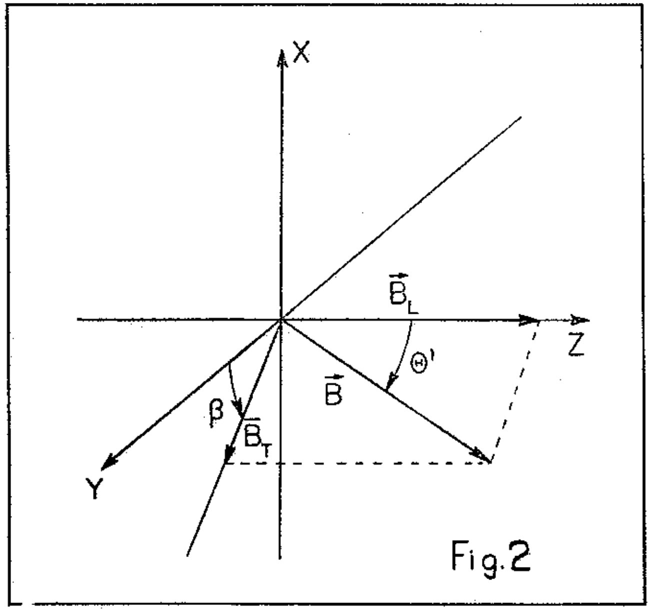

Fig.2

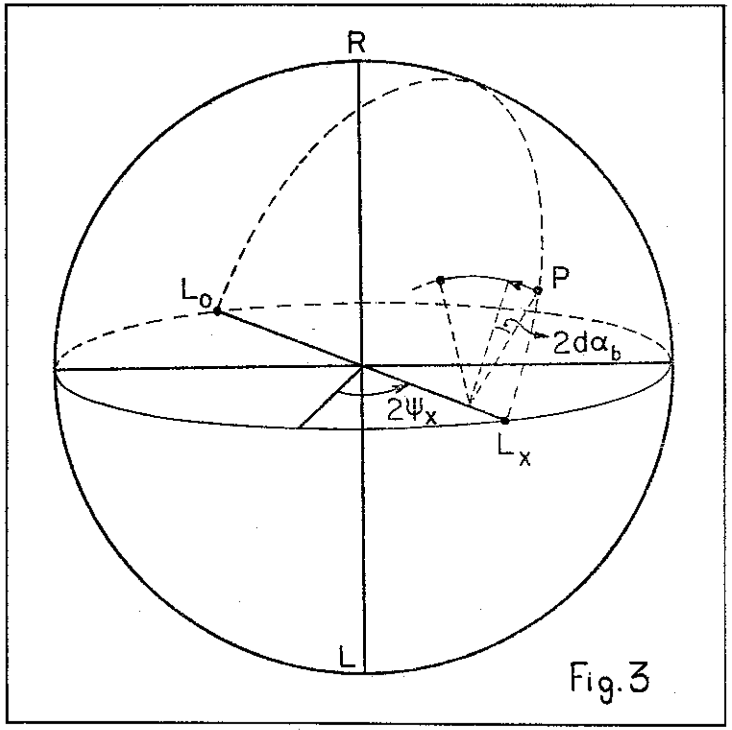

Fig.3

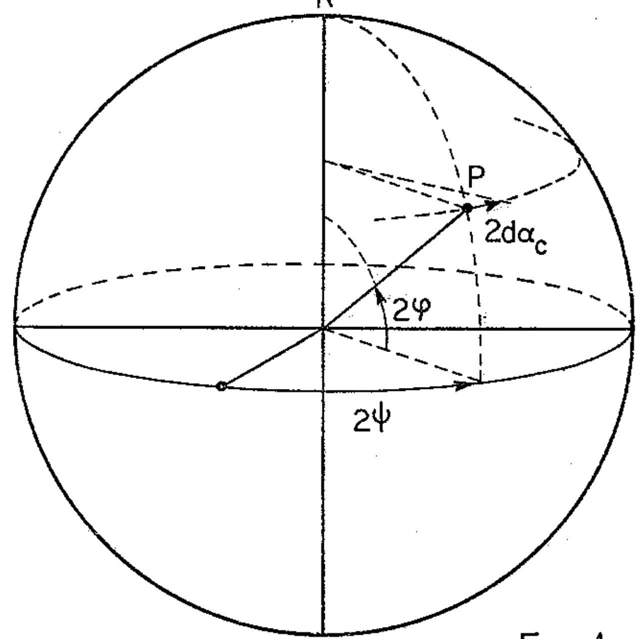

Fig. 4

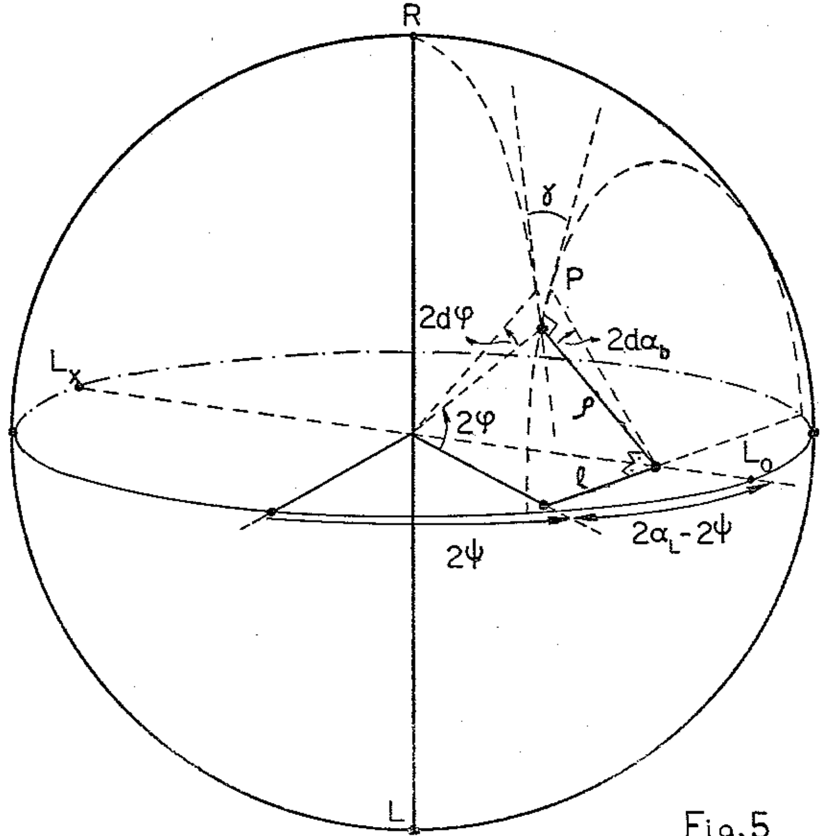

Fig. 5