

UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01000 - SÃO PAULO - SP
BRASIL

IFUSP/P 399
B.I.F. - USP

1 publicações

2 IFUSP/P-399

08 AGO 1983



ON THE EXISTENCE OF BOUND STATES OF N-PARTICLE
SYSTEMS IN ONE- AND TWO-DIMENSIONS

by

F.A.B. Coutinho, C.P. Malta and J. Fernando Perez
Instituto de Física, Universidade de São Paulo

Maio/1983

ON THE EXISTENCE OF BOUND STATES OF
N-PARTICLE SYSTEMS IN ONE- AND TWO-DIMENSIONS

F.A.B. Coutinho^(*), C.P. Malta and J. Fernando Perez^(*)
Instituto de Física, Universidade de São Paulo
C.P. 20516 - São Paulo, Brasil

ABSTRACT

We give sufficient conditions for the existence, in one- and two-dimensions, of bound states of a system of N-particles interacting via two-body potentials.

It is well known that a quantum particle in ν dimensions in the presence of an attractive potential $V(x) \leq 0$ (with $\lim_{|\vec{x}| \rightarrow \infty} V(\vec{x}) = 0$) has at least one eigenstate of negative energy for $\nu = 1$ and 2 (see for instance refs. 1 and 2). If $\nu=1$ the statement remains true even under the weaker assumption that $\int V(x) dx < 0$ (2).

In this letter we present several extensions of these results. First we show that also for $\nu=2$ the assumption $\int V(\vec{x}) d^{\nu}x < 0$ is sufficient to ensure the existence of at least one bound state. Then we study the N-body problem with particles interacting via two body potentials $V_{ij}(\vec{x}_i - \vec{x}_j)$ with $\int V_{ij}(\vec{x}) d^{\nu}x < 0$ and prove for $\nu=1$ and 2 the existence of at least one bound state with energy below the continuum. Finally we show that if the system admits a partition into (exactly) two disjoint cluster C_1 and C_2 , which are internally bound with energies E_{C_1} and E_{C_2} respectively and such that the continuum spectrum of N-body system (after removal of the center of mass motion) starts at $E_{C_1} + E_{C_2}$ and the intercluster potential⁽³⁾ $V_{C_1 C_2}(\vec{x})$ satisfies $\int V_{C_1 C_2}(\vec{x}) d^{\nu}x < 0$, then there is at least one bound state with energy below $E_{C_1} + E_{C_2}$ (i.e. below the continuum).

Let us first consider the problem of one particle in the presence of a potential $V(\vec{x}) \in C_0(\mathbb{R}^{\nu})$ (continuous and of compact support) with $\int V(\vec{x}) d^{\nu}x < 0$, $\nu = 1$ and 2. The existence of at least one bound state with negative energy follows from the variational principle and Hunziker's theorem⁽¹⁾ (which locates the continuum spectrum there where common sense says it should be⁽³⁾) if we can exhibit an element $\phi \in L^2(\mathbb{R}^{\nu})$ such that $(\phi, (H_0 + V)\phi) < 0$ where $H_0 = -\frac{\Delta}{2m}$ (with the obvious domain restrictions).

^(*) Partially supported by CNPq.

(i) Case $v = 1$ (needed for later use)

Let $\text{supp } V \subset [-R, +R]$ and $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi(x) = 1$ for all $x \in [-R, +R]$. If we now set $\phi_\alpha(x) = \alpha^{\frac{1}{2}} \phi(\alpha x)$ ($\alpha > 0$) it is clear that for $\alpha \leq 1$ we have

$$(\phi_\alpha, (H_0 + V)\phi_\alpha) = \alpha^2 (\phi, H_0 \phi) + \alpha \int V(x) dx$$

Therefore $(\phi_\alpha, (H_0 + V)\phi_\alpha) < 0$ if $\alpha < \frac{|\int V(x) dx|}{(\phi, H_0 \phi)}$.

(ii) Case $v = 2$

The reader will have noticed that the proof for $v = 1$ essentially followed the intuition suggested by the uncertainty principle. In two dimensions this reasoning fails (as both the kinetic and potential energies scale in the same way) and a more refined trial wave function is required.

Let the support of V be contained in the circle of radius R with center in the origin. For $0 < \alpha \leq 1$ let

$$X_\alpha(\vec{x}) = \begin{cases} 1 & , \quad r \leq R \\ 2 \left(\frac{R}{r}\right)^{-\alpha} - 1 & , \quad R < r < 2^{\frac{1}{\alpha}} R \\ 0 & , \quad r > 2^{\frac{1}{\alpha}} R \end{cases}$$

and $\phi_\alpha(\vec{x}) = \frac{1}{N_\alpha^{\frac{1}{2}}} X_\alpha(\vec{x})$ where $N_\alpha = \int X_\alpha^2(\vec{x}) d^2x$.

A straightforward calculation (4) gives

$$2m(\phi_\alpha, H_0 \phi_\alpha) = \int (\vec{\nabla} \phi_\alpha)^2 d^2x \leq \frac{4\pi\alpha}{N_\alpha}$$

and $(\phi_\alpha, V \phi_\alpha) = \int V(x) d^2x$, thus if $\alpha < \frac{2m \int V(x) d^2x}{4\pi}$ then $(\phi_\alpha, (H_0 + V)\phi_\alpha) < 0$.

A simple limiting argument allows us to remove the assumption that $V \in C_0^\infty(\mathbb{R}^v)$. In fact the reader can quickly verify that the above results hold under the following assumption (denoted A from now on). There exist $R > 0$ and $I > 0$ such that

$$\int_{|\vec{x}| \leq R'} V(\vec{x}) d^v x \geq -I \quad \text{for all } R' \geq R \quad (A)$$

Of course we have to add an extra assumption on V in order to guarantee that the continuum starts at zero energy. The technical requirement is that $V \in L^2(\mathbb{R}^v) + L_e^\infty(\mathbb{R}^v)$.

Let us now consider the N -body problem. Denoting by \vec{x}_i and m_i , $i = 1, \dots, N$ the particles coordinates and masses and introducing Jacobi coordinates (5)

$$\vec{\xi}_i = \vec{x}_{i+1} - \left(\sum_{j \leq i} m_j\right)^{-1} \left(\sum_{j \geq i} m_j \vec{x}_j\right) \quad i = 1, \dots, N-1$$

the Hamiltonian (after removal of the center of mass motion) reads:

$$H_N = - \sum_{i=1}^{N-1} \frac{1}{2\mu_i} \Delta_{\xi_i} + \sum_{i < j} V_{ij}(\vec{x}_i - \vec{x}_j)$$

where $\mu_i^{-1} = m_{i+1}^{-1} + \left(\sum_{j \leq i} m_j\right)^{-1}$

that is

$$H_{N+1} = H_N - (2\mu_N)^{-1} \Delta_{\xi_N} + \sum_{i=1}^N V_{i,N+1}(\vec{x}_{N+1} - \vec{x}_i)$$

where the hamiltonian H_N involves only the coordinates $\vec{\xi}_1, \dots, \vec{\xi}_{N-1}$.

Let now all $V_{ij}(\vec{x})$ verify assumption (A). Proceeding by induction, let E_N be the energy of the bound state of N particles (with E_N below the continuum threshold) and $\phi_N(\vec{\xi}_1, \dots, \vec{\xi}_{N-1})$ its wave function. Then consider

$$\psi_\alpha(\vec{\xi}_1, \dots, \vec{\xi}_{N-1}, \vec{\xi}_N) = \phi_N(\vec{\xi}_1, \dots, \vec{\xi}_{N-1}) \phi_\alpha(\vec{\xi}_N)$$

where ϕ_α is the function given in (i) for $v=1$ and in (ii) for $v=2$.

It is clear that

$$(\psi_\alpha, H_{N+1} \psi_\alpha) = E_N + (\phi_\alpha, (H_0 + V) \phi_\alpha)$$

where $H_0 = -\frac{\Delta \xi_N}{2\mu_N}$ and

$$V(\vec{\xi}_N) = \sum_{i=1}^N \int |\phi_N(\vec{\xi}_1, \dots, \vec{\xi}_{N-1})|^2 V_{i,N+1}(\vec{x}_{N+1} - \vec{x}_i) d^v \xi_1 \dots d^v \xi_{N-1}$$

is the "effective" potential seen by the $(N+1)$ th particle in the presence of the bound state of the other N particles. Now

$$\int V(\vec{\xi}_N) d^v \xi_N = \sum_{i=1}^N \int V_{i,N+1}(\vec{x}) d^v x$$

since $\vec{x}_{i,N+1} = \vec{\xi}_N +$ linear combination of $(\vec{\xi}_1, \dots, \vec{\xi}_{N-1})$ and $\int |\phi_N|^2 d^v \xi_1 \dots d^v \xi_{N-1} = 1$. Therefore assumption (A) for the V_{ij} implies the validity of assumption (A) for V . Choosing then a sufficiently small (as in (i) and (ii)) we get

$(\phi_\alpha, (H_0 + V) \phi_\alpha) < 0$ and so $(\psi_\alpha, H_{N+1} \psi_\alpha) < E_N$, which concludes the proof.

Using the ideas and techniques described in ref. 3 we now prove the existence of bound states of N particle systems for $v=1$ and 2 , provided there exists a decomposition of the system into two disjoint clusters

$$C_1 = \{i_1, \dots, i_{N_1}\}, C_2 = \{j_1, \dots, j_{N_2}\}, N_1 + N_2 = N$$

both admitting bound states with energies E_{C_1} and E_{C_2} (below the respective continuum thresholds) such that the "intercluster" potential

$$V_{C_1 C_2}(\vec{x}) = \sum_{\substack{i \in C_1 \\ j \in C_2}} V_{ij}(\vec{x})$$

satisfies assumption (A), and such that the continuum spectrum of H_N starts at $E_{C_1} + E_{C_2}$.

In fact

$$H_N = H_{C_1} + H_{C_2} + \left[-\frac{\Delta \xi}{2\mu} + \sum_{\substack{i \in C_1 \\ j \in C_2}} V_{ij}(\vec{x}_i - \vec{x}_j) \right]$$

where $\mu^{-1} = (\sum_{i \in C_1} m_i)^{-1} + (\sum_{j \in C_2} m_j)^{-1}$ and $\vec{\xi}$ denotes the position of the C.M. of C_2 with respect to the C.M. of C_1 .

Taking

$$\psi_\alpha(\vec{x}_1, \dots, \vec{x}_{N_1-1}, \vec{y}_1, \dots, \vec{y}_{N_2-1}, \vec{\xi}) = \phi_{C_1}(\vec{x}_1, \dots, \vec{x}_{N_1-1}) \phi_{C_2}(\vec{y}_1, \dots, \vec{y}_{N_2-1}) \phi_\alpha(\vec{\xi})$$

where ϕ_{C_i} , $i=1,2$ are the wave functions of the bound states

of the cluster C_1 , we have as before

$$(\psi_\alpha, H_N \psi_\alpha) = E_{C_1} + E_2 + (\phi_\alpha, (H_0 + V) \phi_\alpha)$$

where $H_0 = -\frac{\Delta_\xi}{2\mu}$ and

$$V(\xi) = \sum_{\substack{i \in C_1 \\ j \in C_2}} \int |\phi_{C_1}(\vec{x}_1, \dots)| |\phi_{C_2}(\vec{y}_1, \dots)|^2 V_{ij}(\vec{x}_i - \vec{y}_j) \prod_{i=1}^{N_1-1} d^{\nu_{C_1}(C_1)} \xi_i \prod_{j=1}^{\nu_{C_2}(C_2)} d^{\nu_{C_2}(C_2)} \xi_j$$

(Here $\xi_i^{C_1}$, $i = 1, \dots, N_{i-1}$ are the Jacobi coordinates for cluster C_1). Again $\int V(\xi) d^{\nu} \xi = \sum_{\substack{i \in C_1 \\ i \in C_2}} \int V_{ij}(\xi) d^{\nu} \xi = \int V_{C_1 C_2}(\xi) d^{\nu} \xi$

and for α sufficiently small

$$(\psi_\alpha, H_N \psi_\alpha) < E_{C_1} + E_{C_2}$$

Since by hypothesis the continuum starts at

$E_{C_1} + E_{C_2}$ the proof is complete.

ACKNOWLEDGMENTS

We would like to thank profs. Y. Hama and W. Wreszinski for enlightening discussions.

REFERENCES

- 1) M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. IV, Academic Press (1978) N.Y.
- 2) M. Schechter, Operator Methods in Quantum Mechanics, North-Holland (1981) Amsterdam.
- 3) B. Simon, Helv. Phys. Acta 43, 607 (1970).
- 4) A. Klein, L.J. Landau and D. Schucker, J. Stat. Phys. 26, 505 (1981).
- 5) M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. III, Academic Press (1979) N.Y.