UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
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by

F.A.B. Coutinho, C.P. Malta and J. Fernando Perez

Instituto de Física, Universidade de São Paulo

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ON THE EXISTENCE OF BOUND STATES OF
N-PARTICLE SYSTEMS IN ONE- AND TWO-DIMENSIONS

F.A.B. Coutinho(*), C.P. Malta and J. Fernando Perez(*)
Instituto de Física, Universidade de São Paulo
C.P. 20516 - São Paulo, Brasil

ABSTRACT

We give sufficient conditions for the existence, in one- and two-dimensions, of bound states of a system of N-particles interacting via two-body potentials.

(*) Partially supported by CNPq.

It is well known that a quantum particle in ν dimensions in the presence of an attractive potential $V(x) \leq 0$ (with $\lim_{|\vec{x}| \to \infty} V(\vec{x}) = 0$) has at least one eigenstate of negative energy for $\nu = 1$ and 2 (see for instance refs. 1 and 2). If $\nu=1$ the statement remains true even under the weaker assumption that $\int V(x)\,dx < 0$ [2].

In this letter we present several extensions of these results. First we show that also for $\nu=2$ the assumption $\int V(\vec{x})\,d^\nu x < 0$ is sufficient to ensure the existence of at least one bound state. Then we study the N-body problem with particles interacting via two body potentials $V_{ij}(\vec{x}_i - \vec{x}_j)$ with $\int V_{ij}(\vec{x})\,d^\nu x < 0$ and prove for $\nu = 1$ and 2 the existence of at least one bound state with energy below the continuum. Finally we show that if the system admits a partition into (exactly) two disjoint cluster C_1 and C_2, which are internally bound with energies E_{C_1} and E_{C_2} respectively and such that the continuum spectrum of N-body system (after removal of the center of mass motion) starts at $E_{C_1} + E_{C_2}$ and the intercluster potential[3] $V_{C_1C_2}(\vec{x})$ satisfies $\int V_{C_1C_2}(\vec{x})\,d^\nu x < 0$, then there is at least one bound state with energy below $E_{C_1} + E_{C_2}$ (i.e. below the continuum).

Let us first consider the problem of one particle in the presence of a potential $V(\vec{x}) \in C_0(\vec{R}^\nu)$ (continuous and of compact support) with $\int V(\vec{x})\,d^\nu x < 0$, $\nu = 1$ and 2. The existence of at least one bound state with negative energy follows from the variational principle and Hunziker's theorem[1] (which locates the continuum spectrum there where common sense says it should be[3]) if we can exhibit an element $\phi \in L^2(\vec{R}^\nu)$ such that $(\phi, (H_0+V)\phi) < 0$ where $H_0 = -\frac{\Delta}{2m}$ (with the obvious domain restrictions).

(i) Case $\nu = 1$ (needed for later use)

Let $\text{supp}\, V \subset [-R,+R]$ and $\phi \in C_0^\infty(\vec{R})$ be such that $\phi(x) = 1$ for all $x \in [-R,+R]$. If we now set $\phi_\alpha(x) = \alpha^{\frac{1}{2}}\phi(\alpha x)$ $(\alpha > 0)$ it is clear that for $\alpha \leq 1$ we have

$$(\phi_\alpha, (H_0+V)\phi_\alpha) = \alpha^2(\phi, H_0\phi) + \alpha\int V(x)\,dx \quad .$$

Therefore $(\phi_\alpha, (H_0+V)\phi_\alpha) < 0$ if $\alpha < \dfrac{|\int V(x)\,dx|}{(\phi, H_0\phi)}$.

(ii) Case $\nu = 2$

The reader will have noticed that the proof for $\nu = 1$ essentially followed the intuition suggested by the uncertainty principle. In two dimensions this reasoning fails (as both the kynetic and potential energies scale in the same way) and a more refined trial wave function is required.

Let the support of V be contained in the circle of radius R with center in the origin. For $0 < \alpha \leq 1$ let

$$\chi_\alpha(\vec{x}) = \begin{cases} 1 \quad , & r \leq R \\ 2\left(\frac{r}{R}\right)^{-\alpha} - 1 \quad , & R < r < 2^{\frac{1}{\alpha}} R \\ 0 \quad , & r > 2^{\frac{1}{\alpha}} R \end{cases}$$

and $\phi_\alpha(\vec{x}) = \dfrac{1}{N\alpha^{\frac{1}{2}}}\chi_\alpha(\vec{x})$ where $N_\alpha = \int \chi_\alpha^2(\vec{x})\,d^2x$.

A straightforward calculation (4) gives

$$2m(\phi_\alpha, H_0\,\phi_\alpha) = \int (\vec{\nabla}\phi_\alpha)^2 d^2x \leq \frac{4\pi\alpha}{N\alpha}$$

and $(\phi_\alpha, V\,\phi_\alpha) = \int V(x)\,d^2x$, thus if $\alpha < \dfrac{2m|\int V(x)\,d^2x|}{4\pi}$ then $(\phi_\alpha, (H_0+V)\phi_\alpha) < 0$.

A simple limiting argument allows us to remove the assumption that $V \in C_0(\vec{R}^\nu)$. In fact the reader can quickly verify that the above results hold under the following assumption (denoted A from now on). There exist $R > 0$ and $I > 0$ such that

$$\int_{|\vec{x}| \leq R'} V(\vec{x})\,d^\nu x \geq -I \quad \text{for all} \quad R' \geq R \quad . \qquad (A)$$

Of course we have to add an extra assumption on V in order to guarantee that the continuum starts at zero energy. The technical requirement is that $V \in L^2(\vec{R}^\nu) + L_\varepsilon^\infty(\vec{R}^\nu)$.

Let us now consider the N-body problem. Denoting by $\vec{x}_i$ and m_i , $i = 1,\ldots,N$ the particles coordinates and masses and introducing Jacobi coordinates (5)

$$\vec{\xi}_i = \vec{x}_{i+1} - (\sum_{j\leq i} m_j)^{-1}(\sum_{j\geq i} m_j\vec{x}_j) \qquad i = 1,\ldots,N-1 \quad .$$

the Hamiltonian (after removal of the center of mass motion) reads:

$$H_N = -\sum_{i=1}^{N-1} \frac{1}{2\mu_i}\Delta_{\xi_i} + \sum_{i<j} V_{ij}(\vec{x}_i - \vec{x}_j)$$

where $\mu_i^{-1} = m_{i+1}^{-1} + (\sum_{j\leq i} m_j)^{-1}$

that is

$$H_{N+1} = H_N - (2\mu_N)^{-1}\Delta_{\xi_n} + \sum_{i=1}^{N} V_{i,N+1}(\vec{x}_{N+1} - \vec{x}_i)$$

where the hamiltonian H_N involves only the coordinates $\vec{\xi}_1,\ldots,\vec{\xi}_{N-1}$.

Let now all $V_{ij}(\vec{x})$ verify assumption (A). Proceeding by induction, let E_N be the energy of the bound state of N particles (with E_N below the continuum threshold) and $\Phi_N(\vec{\xi}_1,\ldots,\vec{\xi}_{N-1})$ its wave function. Then consider

$$\psi_\alpha(\vec{\xi}_1,\ldots,\vec{\xi}_{N-1},\vec{\xi}_N) = \Phi_N(\vec{\xi}_1,\ldots,\vec{\xi}_{N-1})\ \phi_\alpha(\vec{\xi}_N)$$

where ϕ_α is the function given in (i) for $\nu=1$ and in (ii) for $\nu=2$.

It is clear that

$$(\psi_\alpha, H_{N+1}\ \psi_\alpha) = E_N + (\phi_\alpha, (H_o+V)\phi_\alpha)$$

where $H_o = -\dfrac{\Delta_{\xi_N}}{2\mu_N}$ and

$$V(\vec{\xi}_N) = \sum_{i=1}^{N} \int |\Phi_N(\vec{\xi}_1,\ldots,\vec{\xi}_{N-1})|^2\ V_{i,N+1}(\vec{x}_{N+1}-\vec{x}_i)\ d^\nu\xi_1\ldots d^\nu\xi_{N-1}$$

is the "effective" potential seen by the (N+1)th particle in the presence of the bound state of the other N particles. Now

$$\int V(\vec{\xi}_N)\ d^\nu\xi_N = \sum_{i=1}^{N} \int V_{i,N+1}(\vec{x})d^\nu x$$

since $\vec{x}_{i,n+1} = \vec{\xi}_N$ + linear combination of $(\vec{\xi}_1,\ldots,\vec{\xi}_{N-1})$ and $\int|\Phi_N|^2\ d^\nu\xi_1\ldots d^\nu\xi_{N-1} = 1$. Therefore assumption (A) for the V_{ij} implies the validity of assumption (A) for V. Choosing then α sufficiently small (as in (i) and (ii)) we get

$(\phi_\alpha, (H_o+V)\phi_\alpha) < 0$ and so $(\psi_\alpha, H_{N+1}\ \psi_\alpha) < E_N$, which concludes the proof.

Using the ideas and techniques described in ref. 3 we now prove the existence of bound states of N particle systems for $\nu = 1$ and 2, provided there exists a decomposition of the system into two disjoint clusters

$$C_1 = \{i_1,\ldots i_{N_1}\},\ C_2 = \{j_1,\ldots,j_{N_2}\},\ N_1+N_2 = N$$

both admitting bound states with energies E_{C_1} and E_{C_2} (below the respective continuum thresholds) such that the "intercluster" potential

$$V_{C_1C_2}(\vec{x}) = \sum_{\substack{i\varepsilon C_1\\ j\varepsilon C_2}} V_{ij}(\vec{x})$$

satisfies assumption (A), and such that the continuum spectrum of H_N starts at $E_{C_1}+E_{C_2}$.

In fact

$$H_N = H_{C_1} + H_{C_2} + \left[-\frac{\Delta_\xi}{2\mu} + \sum_{\substack{i\varepsilon C_1\\ i\varepsilon C_2}} V_{ij}(\vec{x}_i-\vec{x}_j)\right]$$

where $\mu^{-1} = (\sum_{i\varepsilon C_1} m_i)^{-1} + (\sum_{j\varepsilon C_2} m_j)^{-1}$ and $\vec{\xi}$ denotes the position of the C.M. of C_2 with respect to the C.M. of C_1.

Taking

$$\psi_\alpha(\vec{x}_1,\ldots,\vec{x}_{N_1-1}, \vec{y}_1,\ldots\vec{y}_{N_2-1}\ \vec{\xi}) = \Phi_{C_1}(\vec{x}_1,\ldots,\vec{x}_{N_1-1})\Phi_{C_2}(\vec{y}_1,\ldots,\vec{y}_{N_2-1})\ \phi_\alpha(\vec{\xi})$$

where Φ_{C_i}, $i = 1,2$ are the wave functions of the bound states of the cluster C_i , we have as before

$$(\psi_\alpha, H_N \psi_\alpha) = E_{c_1} + E_2 + (\phi_\alpha, (H_o+V)\phi_\alpha)$$

where $H_o = -\frac{\Delta_\xi}{2\mu}$ and

$$V(\bar{\xi}) = \sum_{\substack{i\varepsilon c_1 \\ j\varepsilon c_2}} \int |\phi_{c_1}(\bar{x}_1,\ldots)||\phi_{c_2}(\vec{y}_1,\ldots)|^2 V_{ij}(\vec{x}_i-\vec{y}_j) \prod_{i=1}^{N_1-1} d^\nu\xi_i^{(c_1)} \prod_{j=1}^{N_2-1} d^\nu\xi_j^{(c_2)}$$

(Here $\vec{\xi}_i^{c_i}$, $i = 1,\ldots,N_{i-1}$ are the Jacobi coordinates for cluster C_i). Again $\int V(\vec{\xi})d^\nu\xi = \sum_{\substack{i\varepsilon c_1 \\ i\varepsilon c_2}} \int V_{ij}(\vec{\xi})d^\nu\xi = \int V_{c_1c_2}(\vec{\xi})d^\nu\xi$

and for α suficciently small

$$(\psi_\alpha, H_N \psi_\alpha) < E_{c_1} + E_{c_2} .$$

Since by hypothesis the continuum starts at $E_{c_1} + E_{c_2}$ the proof is complete.

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REFERENCES

1) M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. IV, Academic Press (1978) N.Y.

2) M. Schechter, Operator Methods in Quantum Mechanics, North-Holland (1981) Amsterdam.

3) B. Simon, Helv. Phys. Acta 43, 607 (1970).

4) A. Klein, L.J. Landau and D. Schucker, J. Stat. Phys. 26, 505 (1981).

5) M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. III, Academic Press (1979) N.Y.