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GENERAL STATISTICS, SECOND QUANTIZATION AND QUARKS

by

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"... ce n'est qu'en adoptant des points de vue divers, quelquefois opposés, que les sciences progressent, et il ne faut pas mutiler l'esprit humain."

E. Picard

ABSTRACT

A somewhat new second quantization method is proposed in order to obtain commutation relations for intermediate states in quantum mechanics. These commutation rules are found to have a multilinear matricial form, whereas for bosons and fermions the usual bilinear relations are reobtained. It is shown that from a symmetric group point of view, it would be hard to accept the paraboson and parafermion concepts in quantum mechanics. Assuming that quarks can be represented by intermediate states, several properties of hadrons, such as quark confinement, baryonic number conservation, and 3-quarks saturation in baryons, seem to have a natural explanation.

1. INTRODUCTION

In a preceding paper[1] we have shown, using the irreducible representations of the symmetric group in Hilbert space, that boson (Y_S) and fermion (Y_A) states and also intermediate states (Y) are compatible with the postulates of Quantum Mechanics and with the Principle of Indistinguishability. Our analysis that gives support, within the framework of Quantum Mechanics, to the mathematical existence of intermediate states, justifies, in a certain sense, the general statistics proposed long ago by Gentile in a thermodynamical context[2-4]. We have improperly named para-bosons and para-fermions our intermediate states, only to be in agreement with Green's terminology[5]. Nevertheless, there are substantial differences between the two concepts as is seen in this paper.

In a few words, we have shown that an isolated system consisting of N identical particles (by particle we mean a particle or a quasiparticle) with total energy E has a $N!$ degenerate energy spectrum, due to the permutations P_i $(i = 1,2,\ldots,N!)$ of the labels $1,2,\ldots,N$ of the particles in their configuration space $\varepsilon^{(N)}$. Our analysis has been performed considering the eigenfunctions of the energy operator $\tilde{H}(1,2,\ldots,N)$, but it is easy to see that similar results could be obtained by taking into account any Hermitean operator $\tilde{f}(1,2,\ldots,N)$.

The energy eigenfunctions $\{e_i\}$ $(i = 1,2,\ldots,N!)$,

where $e_1 = u(1,2,\ldots,N)$ and $e_2, e_3, \ldots, e_{N!}$ are obtained from e_1 by permuting the labels $1,2,\ldots,N$, constitute a $N!$ dimensional basis of a Hilbert space that was indicated by $L_2(\varepsilon^{(N)})$. This $L_2(\varepsilon^{(N)})$ is decomposed into irreducible subspaces $h^{(\alpha)}$, that are the underlying subspaces of the representation of the symmetric group $S^{(N)}$ in $L_2(\varepsilon^{(N)})$, corresponding to the different partitions (α) of the number N. There are two one-dimensional subspaces that correspond to $(\alpha) = (N)$ and $(\alpha) = (1^N)$ and the wavefunctions associated with them are, respectively, Y_S, which is totally symmetric, and Y_A, which is totally antisymmetric under permutations. The remaining subspaces $h^{(\alpha)}$ have dimensions going from 2^2 up to $(N-1)^2$ with attached wavefunctions indicated by the column vectors

$$Y(\alpha) = \frac{1}{\sqrt{\tau}} \begin{pmatrix} Y_1(\alpha) \\ Y_2(\alpha) \\ \cdot \\ \cdot \\ \cdot \\ Y_\tau(\alpha) \end{pmatrix} \qquad (1.1)$$

where $\tau = (f^{(\alpha)})^2$ is the $h^{(\alpha)}$ dimension and $Y_i(\alpha)$ $(i = 1,2,\ldots,\tau)$ (which constitute the basis of $h^{(\alpha)}$), are given by a linear combination of the unitary vectors $\{e_j\}$ $(j = 1,2,\ldots N!)$.

By applying a permutation P to the particle labels in $\varepsilon^{(N)}$, the vector $Y(\alpha)$ becomes $PY(\alpha) = X(\alpha) = T_\alpha Y(\alpha)$, where T_α is a unitary matrix with τ^2 components. For the

one-dimensional subspaces we have $Y_S = Y_S$ and $Y_A = -Y_A$ so that the concepts of totally symmetric and totally antisymmetric wave-functions subsist. For the multidimensional $h^{(\alpha)}$ these concepts are meaningless because the permutation operation $P\,Y(\alpha)$ implies a rotation of $Y(\alpha)$, defined by a matrix T_α with τ^2 components (there are τ^2 numbers, put into a matrix form, associated to the permutation P, instead of only one number that can be $+1$ or -1).

Since T_α is a unitary matrix, it was also shown that the function

$$\phi(\alpha) = \sum_{i=1}^{\tau} |Y_i(\alpha)|^2 \qquad (1.2)$$

is permutation invariant and has been defined as the probability density function.

The first to note the need of extending the definition of the one-dimensional wavefunctions was Okayama[6]. However, he has obtained multidimensional wavefunctions where the over-all implied symmetries are not clearly displayed. Improving their results, we have written, in a τ-dimensional subspace, the state vector Y, as an orthonormalized τ-vector. The number of columns of the first row of the Young shape associated to each irreducible subspace $h^{(\alpha)}$ will determine the possible maximal occupation number $\underline{d}$ of the intermediate state. This maximal order $\underline{d}$ will be named statistical order of the intermediate state.

The present paper, is arranged as follows: in Section 2 we study some geometric properties of the $Y(\alpha)$ states, by carefully analysing a system formed by 3 particles and by extending to the N-particles systems several conclusions on symmetry properties. In Section 3 we develop a somewhat different process of second quantization and multi-linear commutation relations for the creation and annihilation operators, and their Hermitean conjugates are obtained. The usual bilinear commutation and anticommutation relations for bosons and fermions are particular cases of our general expressions. In Section 4 our results are compared with the parastatistics theory proposed by Green[5]. In Section 5 we try to interpret the Y states as representing a new kind of particles. In Section 6, as an application of our intermediate states, we consider the standard $SU(3)$ model of strongly interacting particles in a non-relativistic approximation for the internal dynamics.

2. ROTATIONS IN HILBERT SPACE

In this section we analyse some transformation properties of the wavefunctions of a system of $\underline{N}$ non-interacting particles. A geometrical interpretation of the transformations, based on the representation of the symmetric group, is given in terms of the basis vectors of a Hilbert space. We will restrict

ourselves to the detailed study of the simplest non trivial 3 particle case. The generalizations of the essential results which can be extracted from this simple case appear at the end of the section.

If the system is composed of 3 particles, the Hilbert space has dimension 6, comprising two one-dimensional sub-spaces and one four-dimensional sub-space. It was shown[1] that, if we indicate by

$$e = \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_6 \end{pmatrix}$$

the vectors of the basis of $L_2(\epsilon^{(3)})$, the one-dimensional wavefunctions Y_S and Y_A are given by

$$Y_S = (1/\sqrt{6}) \sum_{i=1}^{6} e_i$$

and $Y_A = (1/\sqrt{6})\ (e_1 - e_2 - e_3 + e_4 + e_5 - e_6)$ and the 4-dimensional Y is given by

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = E\,e \qquad (2.1)$$

where E is a 4×6 matrix seen in Appendix 1.

These wavefunctions can be put into a compact form:

$$\psi = \begin{pmatrix} Y_S \\ Y \\ Y_A \end{pmatrix} = U\,e \qquad (2.2)$$

where U is a 6×6 unitary matrix shown in Appendix 1 which determines the structure of the functions Y_S, Y and Y_A.

Now, if we assume that the particles do not interact and we indicate by α, β and γ the states allowed for them, we see that the basis vectors $\underline{e}$ can be written as

$$e(\alpha\gamma\beta) = \begin{pmatrix} e_1(\alpha\beta\gamma) \\ e_2(\alpha\beta\gamma) \\ e_3(\alpha\beta\gamma) \\ e_4(\alpha\beta\gamma) \\ e_5(\alpha\beta\gamma) \\ e_6(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} \alpha(1)\ \beta(2)\ \gamma(3) \\ \alpha(1)\ \beta(3)\ \gamma(2) \\ \alpha(2)\ \beta(1)\ \gamma(3) \\ \alpha(2)\ \beta(3)\ \gamma(1) \\ \alpha(3)\ \beta(1)\ \gamma(2) \\ \alpha(3)\ \beta(2)\ \gamma(1) \end{pmatrix} .$$

If, instead of the order $\alpha\beta\gamma$ we have, for instance, $\alpha\gamma\beta$ the basis vector is given by

$$e(\alpha\gamma\beta) = \begin{pmatrix} e_1(\alpha\gamma\beta) \\ e_2(\alpha\gamma\beta) \\ e_3(\alpha\gamma\beta) \\ e_4(\alpha\gamma\beta) \\ e_5(\alpha\gamma\beta) \\ e_6(\alpha\gamma\beta) \end{pmatrix} = \begin{pmatrix} \alpha(1)\ \gamma(2)\ \beta(3) \\ \alpha(1)\ \gamma(3)\ \beta(2) \\ \alpha(2)\ \gamma(1)\ \beta(3) \\ \alpha(2)\ \gamma(3)\ \beta(1) \\ \alpha(3)\ \gamma(1)\ \beta(2) \\ \alpha(3)\ \gamma(2)\ \beta(1) \end{pmatrix} .$$

Since $e_1(\alpha\gamma\beta) = e_2(\alpha\beta\gamma)$, $e_2(\alpha\gamma\beta) = e_1(\alpha\beta\gamma)$, $e_3(\alpha\gamma\beta) = e_4(\alpha\beta\gamma)$, $e_4(\alpha\gamma\beta) = e_3(\alpha\beta\gamma)$, $e_5(\alpha\gamma\beta) = e_6(\alpha\beta\gamma)$ and $e_6(\alpha\gamma\beta) = e_5(\alpha\beta\gamma)$, the basis transformation $e(\alpha\beta\gamma) \to e(\alpha\gamma\beta)$ can be described by the matricial relation

$$e(\alpha\gamma\beta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1(\alpha\beta\gamma) \\ e_2(\alpha\beta\gamma) \\ e_3(\alpha\beta\gamma) \\ e_4(\alpha\beta\gamma) \\ e_5(\alpha\beta\gamma) \\ e_6(\alpha\beta\gamma) \end{pmatrix} = P\begin{pmatrix} \alpha\beta\gamma \\ \alpha\gamma\beta \end{pmatrix} e(\alpha\beta\gamma)$$

where $P\begin{pmatrix} \alpha\beta\gamma \\ \alpha\gamma\beta \end{pmatrix}$ is a unitary matrix.

Thus, for a generic transformation $(\alpha\beta\gamma) \to (ijk)$ where the indices i, j and k can assume the values α, β and γ in an arbitrary order, we write:

$$e(ijk) = P\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} e(\alpha\beta\gamma) \qquad (2.3)$$

Equation (2.3) means the following: when the indices $(\alpha\beta\gamma)$ are permuted in an arbitrary way, the basis $\underline{e}$ is rotated in the Hilbert space $L_2(\varepsilon^{(3)})$.

If we choose the order $(\alpha\beta\gamma)$, the wavefunction $\Psi(\alpha\beta\gamma)$ is, according to equation (2.2), given by:

$$\Psi(\alpha\beta\gamma) = \begin{pmatrix} Y_S(\alpha\beta\gamma) \\ Y(\alpha\beta\gamma) \\ Y_A(\alpha\beta\gamma) \end{pmatrix} = U\, e(\alpha\beta\gamma)$$

and, in a generic order (ijk) we have, using equations (2.2) and (2.3):

$$\Psi(ijk) = U e(ijk) = U P\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} e(\alpha\beta\gamma) =$$

$$= U P\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} U^{\dagger} U\, e(\alpha\beta\gamma) = M\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} \Psi(\alpha\beta\gamma) .$$

That is

$$\Psi(ijk) = M\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} \Psi(\alpha\beta\gamma) \qquad (2.4)$$

where the matrix $M\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix}$, defined as $M\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} = U P\begin{pmatrix} \alpha\beta\gamma \\ ijk \end{pmatrix} U^{\dagger}$ depends on the structure matrix U.

In appendix 1 we show explicitly the $\underline{M}$ matrices for all possible values of i, j and k. These 6×6 matrices have the general form

$$M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} = \left(\begin{array}{c|ccc|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & 0 \\ 0 & & G\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} & & 0 \\ 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & \pm 1 \end{array}\right) \qquad (2.5)$$

where $G\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix}$ is a unitary 4×4 matrix associated with the intermediate states.

So, when the indices $(\alpha\beta\gamma)$ are permuted, assuming the (ijk) values, the wavefunctions Y_S, Y_A and Y undergo rotations according to the following relations:

$$Y_S(ijk) = Y_S(\alpha\beta\gamma) \ , \ Y_A(ijk) = \pm Y_A(\alpha\beta\gamma)$$

and $Y(ijk) = G\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} Y(\alpha\beta\gamma)$. Of course, due to the unitarity of the matrices $\underline{M}$, the functions $|Y_S|^2$, $|Y_A|^2$ and $|Y|^2 = \sum_{i=1}^{4} |Y_i|^2$ are invariants under permutations. This assures that the physical interpretations of the wavefunctions are unaltered by the unobservables transformations $(\alpha\beta\gamma) \rightarrow (ijk)$.

There are no restrictions on the occupation numbers of the states $(\alpha\beta\gamma)$ in the basis vectors $e(\alpha\beta\gamma)$, i.e., one, two or three particles can occupy the same state in $e(\alpha\beta\gamma)$. However, if there are three particles in the same state, we have $Y_A = Y = 0$ and $Y_S \neq 0$. For two particles in the same state, $Y_S \neq 0$, $Y_A = Y_2 = Y_4 = 0$ and $Y_1 \neq 0$ and $Y_3 \neq 0$. Here

we see that Y_2 and Y_4 have a fermionic behavior and Y_1 and Y_3 a bosonic behavior at least when the numbers of particles that occupy the same state is smaller than $d=2$.

When two particles occupy the same state the basis vector $\underline{e}$ has only three independent components. Thus, if we place two particles in the same state α, for instance, $\alpha=\beta$ and one in the state γ, $\Psi(\alpha\alpha\gamma)$ can be written in a compact form:

$$\Psi(\alpha\alpha\gamma) = \begin{pmatrix} Y_S \\ Y_1 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} e_1(\alpha\alpha\gamma) \\ e_2(\alpha\alpha\gamma) \\ e_4(\alpha\alpha\gamma) \end{pmatrix} = u\ e(\alpha\alpha\gamma) \qquad (2.6)$$

remembering that $Y_2 = Y_4 = Y_A = 0$.

If, instead of $(\alpha\alpha\gamma)$ we have $(\alpha\gamma\alpha)$ it is easy to see that $\Psi(\alpha\gamma\alpha) = u\,e(\alpha\gamma\alpha) = u\ P\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} e(\alpha\alpha\gamma) = u\ P\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} u^{\dagger}\Psi(\alpha\alpha\gamma) = m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} \Psi(\alpha\alpha\gamma)$, since $\underline{u}$ is a unitary matrix. That is, $\Psi(\alpha\gamma\alpha) = m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} \Psi(\alpha\alpha\gamma)$, where $m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} = u\ P\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} u^{\dagger}$. Similarly, for $(\gamma\alpha\alpha)$, $\Psi(\gamma\alpha\alpha) = m\begin{pmatrix}\alpha\alpha\gamma\\ \gamma\alpha\alpha\end{pmatrix} \Psi(\alpha\alpha\gamma)$.

Since

$$e(\alpha\gamma\alpha) = \begin{pmatrix} e_2(\alpha\gamma\alpha) \\ e_1(\alpha\gamma\alpha) \\ e_3(\alpha\gamma\alpha) \end{pmatrix} \quad \text{and} \quad e(\gamma\alpha\alpha) = \begin{pmatrix} e_5(\gamma\alpha\alpha) \\ e_3(\gamma\alpha\alpha) \\ e_1(\gamma\alpha\alpha) \end{pmatrix}$$

one can verify that $m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} = m\begin{pmatrix}\alpha\alpha\gamma\\ \gamma\alpha\alpha\end{pmatrix} = I$ is the identity matrix. Consequently, $Y_S(\alpha\alpha\gamma) = Y_S(\alpha\gamma\alpha) = Y_S(\gamma\alpha\alpha)$ and $Y(\alpha\alpha\gamma) = Y(\gamma\alpha\alpha) = Y(\alpha\gamma\alpha)$.

As was pointed out before, the basis vectors $\underline{e}$ do not present any restrictions on the occupation numbers. The restrictions only appear in the structure of Y_S, Y and Y_A.

The totally symmetric function Y_S and the totally antisymmetric Y_A represent, as is well known, bosons and fermions. The multidimensional hybrid function Y could represent, as will be seen in Section 5, a new kind of particle.

For N non-interacting particles, we can also show that the basis vector $\underline{e}$ and the wavefunctions transformations are generically given by the matrix relations:

$$e(\ldots ijk \ldots) = P\begin{pmatrix}\ldots \alpha\beta\gamma \ldots\\ \ldots ijk \ldots\end{pmatrix} e(\ldots \alpha\beta\gamma \ldots)$$

and

$$\Psi(\ldots ijk \ldots) = M\begin{pmatrix}\ldots \alpha\beta\gamma \ldots\\ \ldots ijk \ldots\end{pmatrix} \Psi(\ldots \alpha\beta\gamma \ldots)$$

where the matrix $\underline{P}$ can easily be computed, but the matrix $\underline{M}$ requires an extremely laborious calculation, since it depends on the structure of the multidimensional manifolds.

3. SECOND QUANTIZATION

In the second quantization that we propose, the creation a_k^* and annihilation a_k operators act, row by row, on the column vector $\underline{e}$.

Indicating by $e_0 = e(000)$ the "basis-vector vacuum state", the creation operators are defined by the following relations:

$$a_\alpha^* a_\beta^* a_\gamma^* e(000) = a_\alpha^* a_\beta^* e(00\gamma) = a_\alpha^* e(0\beta\gamma) = e(\alpha\beta\gamma)$$

$$a_\alpha^* e(0\alpha\beta) = \sqrt{2}\, e(\alpha\alpha\beta)$$

$$a_\alpha^* a_\alpha^* a_\alpha^* e(000) = a_\alpha^* a_\alpha^* e(00\alpha) = \sqrt{2}\; a_\alpha^* e(0\alpha\alpha) = \sqrt{2}\,\sqrt{3}\, e(\alpha\alpha\alpha) \qquad (3.1)$$

$$a_\alpha^* e(\beta 0\gamma) = P\begin{pmatrix}0\beta\gamma\\ \beta 0\gamma\end{pmatrix} a_\alpha^* e(0\beta\gamma) = P\begin{pmatrix}0\beta\gamma\\ \beta 0\gamma\end{pmatrix} e(\alpha\beta\gamma)$$

$$a_\alpha^* e(\beta\alpha 0) = P\begin{pmatrix}0\alpha\beta\\ \beta\alpha 0\end{pmatrix} a_\alpha^* e(0\alpha\beta) = P\begin{pmatrix}0\alpha\beta\\ \beta\alpha 0\end{pmatrix} \sqrt{2}\, e(\alpha\alpha\beta) \quad .$$

Similarly, the annihilation operators are defined by:

$$a_\alpha e(\alpha\beta\gamma) = e(0\beta\gamma)$$

$$a_\alpha e(0\alpha\gamma) = e(00\gamma)$$

$$a_\alpha e(\alpha\alpha\beta) = \sqrt{2}\; e(0\alpha\beta)$$

$$a_\alpha e(\alpha 0\gamma) = P\begin{pmatrix}0\alpha\gamma\\ \alpha 0\gamma\end{pmatrix} a_\alpha e(0\alpha\gamma) = P\begin{pmatrix}0\alpha\gamma\\ \alpha 0\gamma\end{pmatrix} e(00\gamma)$$

$$a_\alpha e(\gamma\alpha\alpha) = P\begin{pmatrix}\alpha\gamma\alpha\\ \gamma\alpha\alpha\end{pmatrix} a_\alpha e(\alpha\gamma\alpha) = \sqrt{2}\; P\begin{pmatrix}\alpha\gamma\alpha\\ \gamma\alpha\alpha\end{pmatrix} e(0\gamma\alpha) \quad .$$

Since, by equation (2.3) we have, $e(ijk) = P\binom{\alpha\beta\gamma}{ijk}\, e(\alpha\beta\gamma)$, it is straightforward to show that

$$
\begin{aligned}
a_i^* a_j^* a_k^* e &= P\binom{\alpha\beta\gamma}{ijk}\, a_\alpha^* a_\beta^* a_\gamma^* e \\
a_i a_j a_k e &= P\binom{kji}{\gamma\beta\alpha}\, a_\alpha a_\beta a_\gamma e \\
a_\beta^* a_\gamma a_\alpha e &= P\binom{\alpha\gamma\beta}{\gamma\alpha\beta}\, a_\beta^* a_\alpha a_\gamma e \\
a_\alpha a_\beta^* a_\gamma e &= P\binom{\gamma\alpha\beta}{\gamma\beta\alpha}\, a_\beta^* a_\alpha a_\gamma e \\
a_\alpha^* a_\beta a_\gamma^* e &= P\binom{\alpha\beta\gamma}{\alpha\gamma\beta}\, a_\gamma^* a_\beta a_\alpha^* e
\end{aligned}
\qquad (3.3)
$$

and so on.

The creation and annihilation operators have been defined in such a way that $a_\alpha^* a_\alpha e = \tilde{N}_\alpha e = N_\alpha e$, where N_α is the occupation number of the α state in the basis vector $\underline{e}$. It can also be verified that $a_\alpha a_\alpha^* = 1 + a_\alpha^* a_\alpha$, or, equivalently $[a_\alpha^*, a_\alpha]_- = 1$.

The equations (3.1), (3.2) and (3.3) define a composition law for the creation and annihilation operators when they act on the basis vector $\underline{e}$. We see that the permutation of the indices α and β, for instance, depends on the third element γ. This generalized result belongs to more complicated structures defined in a third order algebraic system[7] and comprises several equivalent relations deduced in Grassmann algebras.

Let us now obtain the algebraic relations for a_k^* and a_k when they act on the wavefunctions Y_S, Y and Y_A. Thus, using equations (2.2), (2.4) and (3.1) we get for $\alpha \neq \beta \neq \gamma \neq \alpha$:

$$\Psi(ijk) = U\, e(ijk) = a_i^* a_j^* a_k^* U\, e(000) = a_i^* a_j^* a_k^* \Psi(000)$$

and

$$\Psi(ijk) = M\binom{\alpha\beta\gamma}{ijk}\, \Psi(\alpha\beta\gamma) = M\binom{\alpha\beta\gamma}{ijk}\, a_\alpha^* a_\beta^* a_\gamma^* \Psi(000)$$

which allow us to conclude that:

$$a_i^* a_j^* a_k^* = M\binom{\alpha\beta\gamma}{ijk}\, a_\alpha^* a_\beta^* a_\gamma^*$$

Similarly, it can be shown that

$$
\begin{aligned}
a_i a_j a_k \Psi &= M\binom{kji}{\gamma\beta\alpha}\, a_\alpha a_\beta a_\gamma \Psi \\
a_\beta^* a_\gamma a_\alpha \Psi &= M\binom{\alpha\gamma\beta}{\gamma\alpha\beta}\, a_\beta^* a_\alpha a_\gamma \Psi \\
a_\alpha a_\beta^* a_\gamma \Psi &= M\binom{\gamma\alpha\beta}{\gamma\beta\alpha}\, a_\beta^* a_\alpha a_\gamma \Psi \\
a_\alpha^* a_\beta a_\gamma^* \Psi &= M\binom{\alpha\beta\gamma}{\alpha\gamma\beta}\, a_\gamma^* a_\beta a_\alpha^* \Psi
\end{aligned}
\qquad (3.4)
$$

and so on.

We must note that the operator $a_\alpha^* a_\alpha = \tilde{N}_\alpha$ gives the

occupation number of the α state, that is, $a^*_\alpha a_\alpha \Psi = N_\alpha \Psi$.

When $\alpha = \beta \neq \gamma$, for instance, using the results of section 2 and following the reasoning delineated above, we get

$$
\begin{aligned}
a^*_\gamma a^*_\alpha a^*_\alpha \Psi &= m\begin{pmatrix}\alpha\alpha\gamma\\ \gamma\alpha\alpha\end{pmatrix} a^*_\alpha a^*_\alpha a^*_\gamma \Psi = m\begin{pmatrix}\alpha\gamma\alpha\\ \gamma\alpha\alpha\end{pmatrix} a^*_\alpha a^*_\gamma a^*_\alpha \Psi \\
a_\gamma a_\alpha a_\alpha \Psi &= m\begin{pmatrix}\alpha\alpha\gamma\\ \gamma\alpha\alpha\end{pmatrix} a_\alpha a_\alpha a_\gamma \Psi = m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} a_\alpha a_\gamma a_\alpha \Psi \\
a_\alpha a^*_\gamma a_\alpha \Psi &= m\begin{pmatrix}\alpha\alpha\gamma\\ \alpha\gamma\alpha\end{pmatrix} a^*_\gamma a_\alpha a_\alpha \Psi = m\begin{pmatrix}\gamma\alpha\alpha\\ \alpha\gamma\alpha\end{pmatrix} a_\alpha a_\alpha a^*_\gamma \Psi \\
a^*_\alpha a_\gamma a^*_\alpha \Psi &= m\begin{pmatrix}\alpha\gamma\alpha\\ \gamma\alpha\alpha\end{pmatrix} a_\gamma a^*_\alpha a^*_\alpha \Psi = m\begin{pmatrix}\alpha\gamma\alpha\\ \alpha\alpha\gamma\end{pmatrix} a^*_\alpha a^*_\alpha a_\gamma \Psi
\end{aligned}
\qquad (3.5)
$$

When the operator $a_\alpha a^*_\alpha$ acts row by row on Y_S, Y and Y_A, results,

$$
\begin{aligned}
a_\alpha a^*_\alpha &= 1 + a^*_\alpha a_\alpha \quad \text{for} \quad Y_S \\
a_\alpha a^*_\alpha &= 1 + a^*{}_\alpha a_\alpha \quad \text{for} \quad Y_1 \text{ and } Y_3 \text{ if } N_\alpha = 0,1 \\
a_\alpha a^*_\alpha &= d - a^*_\alpha a_\alpha \quad \text{for} \quad Y_1 \text{ and } Y_3 \text{ if } N_\alpha \leqslant 2 \text{ , where } d=2 \text{ is the order of the statistics} \\
a_\alpha a^*_\alpha &= 1 - a^*_\alpha a_\alpha \quad \text{for} \quad Y_2 \text{ and } Y_4 \\
a_\alpha a^*_\alpha &= d_F - a^*_\alpha a_\alpha \quad \text{for} \quad Y_A \text{ , where } d_F=1 \text{ is the order of the Fermi statistics .}
\end{aligned}
\qquad (3.6)
$$

By taking into account the matrices $\underline{M}$ and $\underline{m}$ (given in appendix 1), we analyse now the action, row by row,

of the creation and annihilation operators on Ψ:

a) Bosons and Fermions

When a_k and a^*_k act on Y_S and Y_A, we see that the following usual commutation relations are satisfied:

$$
\begin{aligned}
\left[a^*_\alpha , a^*_\gamma\right]_\pm &= \left[a_\alpha , a_\gamma\right]_\pm = 0 \\
\left[a_\alpha , a^*_\gamma\right] &= \delta_{\alpha\gamma} \quad \text{and, consequently,} \\
\left[\hat{N}_\alpha , a^*_\gamma\right]_\pm &= \delta_{\alpha\gamma}\, a^*_\gamma \\
\left[\hat{N}_\alpha , a_\gamma\right]_\pm &= -\delta_{\alpha\gamma}\, a_\gamma
\end{aligned}
\qquad (3.7)
$$

where the (-) sign corresponds to Y_S (bosons) and the (+) sign to Y_A (fermions)

b) Intermediate States

On the other hand, when a_k and a^*_k act on the state Y, we get, for $\alpha \neq \beta \neq \gamma \neq \alpha$:

$$
\begin{aligned}
a_i^* a_j^* a_k^* Y &= G\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} a_\alpha^* a_\beta^* a_\gamma^* Y \\
a_i a_j a_k Y &= G\begin{pmatrix}kji\\ \gamma\beta\alpha\end{pmatrix} a_\alpha a_\beta a_\gamma Y \\
a_\beta^* a_\gamma a_\alpha Y &= G\begin{pmatrix}\alpha\gamma\beta\\ \gamma\alpha\beta\end{pmatrix} a_\beta^* a_\alpha a_\gamma Y \qquad (3.8)\\
a_\alpha a_\beta^* a_\gamma Y &= G\begin{pmatrix}\gamma\alpha\beta\\ \gamma\beta\alpha\end{pmatrix} a_\beta^* a_\alpha a_\gamma Y \\
a_\alpha^* a_\beta a_\gamma^* Y &= G\begin{pmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\end{pmatrix} a_\gamma^* a_\beta a_\alpha^* Y \quad \text{, and so on...}
\end{aligned}
$$

When $\alpha = \beta \neq \gamma$, we have, for $i = 2$ and 4 :

$$a_\alpha^* a_\alpha^* Y_i = a_\alpha a_\alpha Y_i = 0$$

and for $i = 1$ and 3 :

$$
\begin{aligned}
a_\gamma^* a_\alpha^* a_\alpha^* Y_i &= a_\alpha^* a_\alpha^* a_\gamma^* Y_i = a_\alpha^* a_\gamma^* a_\alpha^* Y_i \\
a_\gamma a_\alpha a_\alpha Y_i &= a_\alpha a_\alpha a_\gamma Y_i = a_\alpha a_\gamma a_\alpha Y_i \qquad (3.9)\\
a_\alpha a_\gamma^* a_\alpha Y_i &= a_\gamma^* a_\alpha a_\alpha Y_i = a_\alpha a_\alpha a_\gamma^* Y_i \\
a_\alpha^* a_\gamma a_\alpha^* Y_i &= a_\gamma a_\alpha^* a_\alpha^* Y_i = a_\alpha^* a_\alpha^* a_\gamma Y_i
\end{aligned}
$$

Finally, when $\alpha = \beta = \gamma$, we have

$$a_\alpha^* a_\alpha^* a_\alpha^* Y_i = a_\alpha a_\alpha a_\alpha Y_i = 0 \quad \text{for all} \quad Y_i .$$

It is very important to remark that for the

components Y_2 and Y_4 we have $\left[\hat{N}_\alpha , a_\gamma^*\right]_+ = \delta_{\alpha\gamma} a_\gamma^*$ and $\left[\hat{N}_\alpha , a_\gamma\right]_+ = -\delta_{\alpha\gamma} a_\gamma$ and that for the components Y_1 and Y_3 , $\left[\hat{N}_\alpha , a_\gamma^*\right]_- = \delta_{\alpha\gamma} a_\gamma^*$ and $\left[\hat{N}_\alpha , a_\gamma\right]_- = -\delta_{\alpha\gamma} a_\gamma$, when $N_\alpha \leq 2$. This means that two components of Y , namely Y_2 and Y_4 have a fermionic behavior when $N_\alpha \leq d = 2$. It is commonly accepted[8] as a natural requirement, that these relations remain valid, in general, in the theory of free para-particles. Here we show that this assumption is actually correct and consistent.

For the N-particles case, following the above reasoning and taking into account the general results of Section 2, we obtain, for bosons and fermions, the usual commutation relations. However, for the intermediate states, the commutation relations have multilinear matricial forms governed by the $G\begin{pmatrix}\dots\alpha\beta\gamma\dots\\ \dots ijk\dots\end{pmatrix}$ matrices. We are not aware of the exact form of these matrices, since they depend on the structure of the irreducible manifolds. Nevertheless, in principle, they can be calculated.

Summarizing, we can say that for the one-dimensional sub-spaces, where the permutation of particles changes the state-vectors only by a numerical factor (± 1), the commutation relations are very simple. They are bi-linear single valued relations with the properties:

(1) The commutation relation between any two operators does not depend on the position of the remaining ones.

(2) For Y_A there are only commutation relations whereas for Y_S we have only anti-commutation relations.

For the multi-dimensional sub-spaces, where the symmetry properties of Y under permutations of particles are defined by a matrix, the commutation relations have multi-linear matricial forms obeying:

(1') The commutation relation between two operators depends on the position of the remaining ones.

(2') It depends on the particular row Y_i where it is applied.

4. PARASTATISTICS AND QUANTUM MECHANICS

As is easy to see, our second quantization procedure is a natural extension of the usual second quantization method adopted in the literature[9]. The well known boson and fermion commutation relations are obtained as particular cases of our general expressions when we restrict ourselves to the one-dimensional subspaces. In our approach, the symmetry properties of the multi-dimensional subspaces, induced by the group of permutations, are preserved and the occupation numbers $N_\alpha = 2, 3, 4, \ldots$ arise as a natural consequence of the symmetries of the wavefunctions ψ.

As one can verify from equations (3.8) and (3.9),

for $N=3$ the commutation relations for the components Y_i are tri-linear. It is worthy to note that for Y_2 and Y_4 we have the commutation relations $[\hat{N}_\alpha , a^*_\gamma]_+ = \delta_{\alpha\gamma} a^*_\gamma$ and $[\hat{N}_\alpha , a_\gamma]_+ = -\delta_{\alpha\gamma} a_\gamma$, which are peculiar to fermions. On the other hand, for Y_1 and Y_3, when $N_\alpha \leq \underline{d} = 2$, the relations $[\hat{N}_\alpha , a^*_\gamma]_- = \delta_{\alpha\gamma} a^*_\gamma$ and $[\hat{N}_\alpha , a_\gamma]_- = -\delta_{\alpha\gamma} a_\gamma$ are satisfied. This means that Y_1 and Y_3, when $N_\alpha \leq 2$, show a bosonic character. However, when $N_\alpha = 3$, $Y_1 = Y_3 = 0$, i.e., it is impossible to accomodate more than 2 particles in the same state. Thus, the components Y_1 and Y_3 do not have a genuine bosonic behavior. To sum up we can say that, at least in the non relativistic quantum mechanical limit, the state Y does not have a pure fermionic or bosonic behavior. It is a fermion-boson hybrid. The same considerations remain also valid when the system is composed of N particles. If a convenient basis is chosen for the representation space, as we have done above, the state vectors for the intermediate states will be constituted by τ-vectors, where τ is the dimension of the $h^{(\alpha)}$, showing always the hybrid character, analogous to the case $N=3$.

This work and the preceding one[1] about para-bosons and para-fermions in quantum mechanics, are based on several concepts and results, as those derived from the classical spectral theory of partial differential equations, that presuppose a classical heritage[10,11] which, evidently, is present in our wavefunctions and commutation relations. We are not aware, at the moment, of a method for generalizing our multilinear

matricial commutation relations in order to apply them to the study of relativistic phenomena, which is the main purpose of quantum field theory. In elaborating a quantum field theory we try to define, in a consistent way, a set of field operators which are completely characterized by defining all possible algebraic relations between them. A more rigorous formulation can be given in terms of bounded operators and their algebras, but this is beyond the scope of the present work[12]. In our concern here, we must expect that the formalism of quantum theory could give the exact occupation numbers of the particles, which are the crucial observables and that, in the non relativistic limit, our wavefunctions and commutation relations should be reproduced.

Let us consider now the generalized method of field quantization developed by Green[5], that is analysed with great detail in the excellent book by Ohnuki and Kamefuchi[13]. Founded on the idea that the hypothesis of complete symmetry or complete anti-symmetry of the state vector of a system of particles is stronger than the assumption of the physical identity of the particles, Green suggested the tri-linear commutation relations

$$\left[a^*_\lambda\, a_\mu - \sigma\, a_\mu\, a^*_\lambda\ ,\ a_\nu\right]_- = -2\,\delta_{\lambda\nu}\, a_\mu \qquad (4.1)$$

$$\left[a_\lambda\, a_\mu - \sigma\, a_\mu\, a_\lambda\ ,\ a_\nu\right]_- = 0$$

for the para-fields. The parameter σ ($\sigma=1$ for fermions and $\sigma=-1$ for bosons) will characterize the two possible para-statistics.

To solve the system (4.1) he has also suggested a decomposition of the para-fields by the now well established "Green's anzatz"

$$a_\nu = \sum_{\alpha=1}^{p} b_\nu^{(\alpha)} \qquad (4.2)$$

where p is called the order of the para-statistics and the $b_\nu^{(\alpha)}$ are fermion fields for $\sigma=1$ and boson fields for $\sigma=-1$. Obviously, when in equation (4.2) we put $p=1$ and substitute into (4.1), we get the usual commutation relations for fermions and bosons.

Despite the fact that Green's formalism has solely used the quantum field theoretic framework, there is no reason why para-statistics cannot be applied to the corresponding quantum mechanical description, at least in the non relativistic limit. There are some differences between the quantum mechanical approach and that of the associated field theory. However, when the number N of particles is constant, some resemblances could be expected. Now, if we try to translate Green's field theoretic results into our quantum mechanical language, we fall in with several difficulties from the onset. The first one is the decoupling into two kinds of para-statistics implied by the

two-valuedness of σ. The second is the decomposition of the para-field into usual fields which belong to one-dimensional representations of the group of permutations. The third is the problematic interpretation of the order parameter p. The difficulties pointed out above seem to disguise the true character of the symmetries involved in the multi-dimensional state vector. Thus, it could be inferred that, from a symmetric group point of view, it is hard to accept the para-bosons and para-fermions concepts in Quantum Mechanics.

5. AN INTERPRETATION OF Y STATES

In Sections 1, 2 and 3 we deduced several important properties of general statistics and its corresponding second quantization. Some of them can be summarized as follows:

1) Boson and fermion creation and annihilation operators obey the usual bilinear commutation relations.

2) For the general states, the commutation relations have a multi-linear matricial form depending on the structure of the irreducible manifolds. These relations impose severe restrictions on the quantum properties associated to the irreducible subspaces of $L_2(\varepsilon^N)$. The particles must behave like a single cluster within the irreducible multi-dimensional subspaces of $L_2(\varepsilon^N)$. Of course, this indicates that there is a strong correlation among the particles, even in the absence of interaction.

3) The state vector Y does not have a pure fermionic or bosonic behavior, but is a fermion-boson hybrid. The occupation number $\underline{d}$ for Y states runs from 2 up to $N-1$.

Furthermore, as a consequence of establishing a one-to-one correspondence between the Young shapes and the state vectors Y, we get a quantization of the system for each shape. Also, for the subspaces defined by the symmetric group, there is a Geometric Superselection Rule[14]: "transitions between different irreducible subspaces are forbidden".

In this Section we propose to interpret the hybrid states Y as corresponding to a new kind of particle which in the ensuing will be named gentileons. Let us characterize now the novel features of these particles. Although there is a wide collection of possible intermediate states, many internal quantum numbers such as spin, isospin and others arising from internal symmetries or dynamical arguments can be used to drastically reduce the available number of states[15]. These selection rules would depend on the specific gentileons constituting the system. It is important to note that if we have only $N=3$ gentileons, there is only one intermediate four-dimensional state. For $N=2$, there is no intermediate state and the system is represented by Y_A or Y_S.

On considering the collision problem of two systems with gentilionic internal structures, we can get some insight on the geometric properties of gentileons. System 1 is composed by N_1 gentileons with internal symmetries defined by the Young

shape $S_p(N_1)$ whereas the system 2, composed by N_2 gentileons, is characterized by the Young shape $S_q(N_2)$. The gentileons are assumed to be identical and their total number $N = N_1+N_2$ is conserved during the collision. By taking into account the Geometric Superselection Rule (GSR), we verify that the symmetries of the internal states are unaltered:

$$S_p(N_1) + S_q(N_2) \nsim S_p(N_1) + S_q(N_2) \quad .$$

The ensuing consequences follow from this symmetry conservation law: two systems cannot coalesce and a free gentileon cannot be absorbed or emitted by a system. This suggests that, at least, when the number is conserved, gentileons cannot escape from a system.

Sticking to this geometric standpoint we can conclude that the gentileons could be understood as "confined entities" with saturation properties. These geometric features are intrinsic to the gentileons as the total symmetrisation (antisymmetrisation) is inherent to the bosons (fermions), not depending on their physical interpretation. Thus, they could be assimilated to individual real particles or to dynamical entities as quantum collective states. Nevertheless, one thing is certain: the gentileons, although being a consequence of the general results of quantum mechanics, seem to be quite different from the usual physical particles, and thus, we have no illusion as to whether these interpretations have any completeness or

broad validity. Anyway, the geometric confinement would explain why only bosons and fermions have been observed in laboratories and why gentileons have never been detected as free elementary particles in the physical world. It is also implied that, if we have a set of identical systems, each one consisting of N gentileons, and if we identify the evolution space with the group itself with respect to which the systems are elementary[16], only two descriptions are possible: bosonic or fermionic.

6. QUARKS AND GENERAL STATISTICS

As an application of the quantum geometric reasoning developed until now, let us consider the standard SU(3) model of strongly interacting particles in a non relativistic approximation for the internal dynamics[17]. If we assume that the fundamental triplet $(np\lambda)$ associated with a baryon is constituted by spin-half gentileons described by a four-dimensional hybrid Y defined on SU(3) space, several interesting possibilities are suggested. Naturally, since Y is not necessarily symmetric or antisymmetric under permutations, no specific symmetrisation is required for its radial part. Also, by adopting a Y state for the description of $(np\lambda)$ in SU(3) space, it is clear that we get the possibility of accomodating two identical particles in the same quantum state, without assuming parastatistics[18] or the existence of three triplets of quarks[19].

As shown in Section 5, the symmetries of the internal states of systems formed by gentileons are unaltered in collisions. In the case of baryons, described by Y states, this symmetry law accounts for several fundamental properties:

a) baryonic number conservation

b) quark confinement and

c) 3-quark saturation in baryons.

Next, we want to specialize the preceding discussion to mesons. To this effect, we point out that the set of accessible states of a system composed of three gentileons is completely inequivalent to the set which corresponds to a system composed of two gentileons. This extremely strong condition is the basis of the entire discussion on meson states. Structural differences between baryon and meson quark contents are expected to occur. The mesons could not be constructed with two flavours coming from the baryonic set $(np\lambda)$. Thus we would be compelled to construct a meson by introducing a new set of states. This new set is naturally generated by the $\overline{3}$ representation of $SU(3)$. It is worthy to observe that quark confinement in mesons should also be a consequence of GSR.

The question of quark confinement is at the centre of much current research. In the context of the one-dimensional representations of the symmetric group, the hypothesis that quarks are some kind of quasiparticles that, like phonons inside a crystal lattice, have meaning only as dynamic entities inside the hadrons, is a very reasonable one.

On the other hand, experiments performed with electrons at very high energies, have revealed that protons and neutrons have their charges concentrated on pointlike particles named partons. On considering partons and quarks as being the same thing, we could get a corpuscular interpretation for the quarks.

Of course, both interpretations are based upon familiar concepts of quantum mechanics and probably, the quarks, being gentileons, require a more complex description than those adopted for the usual particles.

APPENDIX 1 - THE MATRICES E, U and $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix}$

The matrices E and U, seen in equations (2.1) and (2.2), respectively, are given by:

$$E = \begin{pmatrix} 1/2 & 0 & 1/2 & -1/2 & 0 & -1/2 \\ 1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & 1/2\sqrt{3} & -1/\sqrt{3} & -1/2\sqrt{3} \\ -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} \\ 1/2 & 0 & -1/2 & -1/2 & 0 & 1/2 \end{pmatrix}$$

$$U = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/2 & 0 & 1/2 & -1/2 & 0 & -1/2 \\ 1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & 1/2\sqrt{3} & -1/\sqrt{3} & -1/2\sqrt{3} \\ -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} & -1/2\sqrt{3} & 1/\sqrt{3} & -1/2\sqrt{3} \\ 1/2 & 0 & -1/2 & -1/2 & 0 & 1/2 \\ 1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

The matrices $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix}$, that are defined by $M\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} = U\, P\begin{pmatrix}\alpha\beta\gamma\\ ijk\end{pmatrix} U^{\dagger}$, according to equation (2.4), are the following:

1) $\begin{pmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

2) $\begin{pmatrix}\alpha\beta\gamma\\ \gamma\beta\alpha\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \gamma\beta\alpha\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 0 & -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

3) $\begin{pmatrix}\alpha\beta\gamma\\ \beta\gamma\alpha\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \beta\gamma\alpha\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 0 & \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4) $\begin{pmatrix}\alpha\beta\gamma\\ \beta\alpha\gamma\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \beta\alpha\gamma\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

5) $\begin{pmatrix}\alpha\beta\gamma\\ \gamma\alpha\beta\end{pmatrix}$

$$M\begin{pmatrix}\alpha\beta\gamma\\ \gamma\alpha\beta\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 & 0 & 0 & 0 \\ 0 & -\sqrt{3}/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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