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HOW UNIVERSAL IS THE PERIOD DOUBLING PHENOMENON IN EQUATIONS WITH QUADRATIC NONLINEARITY?

by

Coraci P. Malta and Cesar R. de Oliveira
Instituto de Física, Universidade de São Paulo

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HOW UNIVERSAL IS THE PERIOD DOUBLING PHENOMENON
IN EQUATIONS WITH QUADRATIC NONLINEARITY?

Coraci P. Malta and Cesar R. de Oliveira†
Departamento de Física Matemática, Instituto de Física da Universidade de São Paulo, CP 20516, 01000 São Paulo, SP, Brasil

ABSTRACT

Varying one parameter, the solution of nonlinear 1st order differential equation with time delay τ is Fourier analysed. After the Hopf bifurcation, period-doubling phenomenon always occurs when τ is one of the fixed parameters (both for small and large τ). Varying τ, there are values of the fixed parameters for which no period-doubling occurs. "Chaos" follows the period-doubling sequence and the rate at which "chaos" is approached is very close to the universal $\delta = 4.6692016...$ characterising the period-doubling sequence to chaos in nonlinear difference equations.

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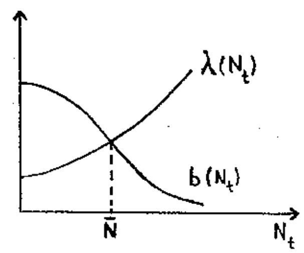
Rich dynamical behaviour is a common feature of nonlinear first order difference and differential equations depending on parameters(1-3). The simple first order nonlinear difference equations (noninvertible maps of the interval)(1-8), for instance, can give rise to cascades of periodic orbits by generic period doubling bifurcation process which goes on until a critical value of the parameter is reached beyond which periodicity ends ("period doubling route to chaos"(6)). The universality of this period doubling bifurcation sequence is not only qualitative(1-7), it also is quantitative(6-9). Feigenbaum(6) has shown that the period doubling sequence for non-area-preserving maps is characterized by two universal constants, $\alpha = 2.502907875...$ and $\delta = 4.6692016...$. The constant α is the asymptotic value of the scaling of the transformation while δ is the asymptotic value of the ratio between the ranges of parameter values in which successive cycles appear and then become unstable.

First order nonlinear differential equations with time delay may exhibit even richer dynamical behaviour. Sometime ago, Perez, Malta and Coutinho(10) proposed the following nonlinear equation with time delay τ

$$\frac{dN_t}{dt} = b(N_{t-\tau})\, N_{t-\tau} - \lambda(N_t)\, N_t \qquad (1)$$



to describe isolated population of Drosophila Sturtevantis flies (oscillations in isolated population of these flies were observed by Tadei and Mourão[11]). By linearizing around the equilibrium point $\bar{N}$, sufficient conditions for the stability of the equilibrium population were obtained. Violation of these sufficient conditions constitute necessary conditions for an oscillatory behaviour of the population. It was established that oscillations in the population result from a Hopf bifurcation occurring at a value τ_H of the time delay parameter. All these conditions were verified by numerically solving equation (1) assuming for $\underline{b(N_t)}$ and $\underline{\lambda(N_t)}$ the following simple form

$$b(N_t) = \begin{cases} a\left(1 - \frac{N_t}{N_0}\right) & \text{for} \quad N_t \leq N_0 \\ 0 & \text{for} \quad N_t > N_0 \end{cases} \tag{2}$$

and

$$\lambda(N_t) = \text{const} = \mu \quad .$$

For $\underline{b(N_t)}$ and $\underline{\lambda(N_t)}$ of form (2) τ_H is given by

$$\tau_H = \frac{\theta}{\sqrt{3\mu^2 - 4a\mu + a^2}} \tag{3}$$

$$\theta = \sin^{-1}\left(-\sqrt{1 - \left(\frac{\mu}{2\mu - a}\right)^2}\right) \quad , \quad \frac{\pi}{2} < \theta < \pi \quad .$$

Several sets of parameters (a, N_0, μ) were considered and τ varied over a wide range. For all parameters

sets, independently of the initial conditions, whenever τ passed through τ_H[3] the equilibrium point $\bar{N}$ became unstable and the population started to oscillate around $\bar{N}$. More detailed calculations were performed for the sets $\underline{a} = 4.0$, $\underline{N_0} = 400$ and $\underline{a} = 3.2$, $\underline{N_0} = 320$ with $\mu = 1.0$ (in both cases). In the former case, as τ was increased beyond τ_H (Hopf bifurcation), the oscillations seemed to have a single frequency for a certain range of τ but then a 2nd bump started to develop suggesting that a second frequency was being introduced, two frequencies remaining in a certain range and, finally, beyond a value τ_c the oscillations acquired a complex structure. For the second set of parameters, on the other hand, as τ was increased the oscillations seemed to have always a single frequency. The systematic introduction of new frequencies (new bumps in the graphs) in the solutions as τ was increased for some sets of parameters lead us to conjecture that the period doubling phenomena[1-8] could also be present in this equation (1). In fact, this very equation is also used to describe an optical cavity filled with nonlinear dielectric medium irradiated with a laser light of constant intensity and, indeed, Ikeda, Kondo and Akimoto[12] showed the existence of the period doubling cascade in this equation by comparing the graphs of the solutions as function of the parameters (no Fourier analysis was performed).

In this work we present the results of Fourier analysis of the solutions of equation (2) (i) fixing $\underline{a}$ and $\underline{N_0}$ and varying the time-delay τ and (ii) fixing τ and $\underline{a/N_0}$ and varying $\underline{a}$ (μ is always 1.0).

Results with τ varying are given in Table 1 ($\underline{a} = 3.200$, $\underline{N_0} = 320$) and in Table 2 ($\underline{a} = 4.000$, $\underline{N_0} = 400$). As the previous numerical studies[10] had indicated, for the first parameter set the solution contains always a single dominant frequency (see figure 1) the period T being approximately equal to the corresponding τ value (see Table 1). The second set of parameters, on the other hand, exhibits period doubling at the values $\tau = \tau_k$ given in the 3rd column of Table 2 (see figures 2 and 3 of the power spectrum for $\tau_1 < \tau = 2.850 < \tau_2$ and $\tau_2 < \tau = 3.160 < \tau_3$ respectively). The ratios of the ranges of τ values at which the period of the solution doubles,

$$\delta_k^{\tau} = \frac{\tau_{k+1} - \tau_k}{\tau_{k+2} - \tau_{k+1}} , \qquad (4)$$

(given in the 4th column of Table 2) are very close to the universal (asymptotic) value $\delta = 4.6692016...$ associated to nonlinear difference equations (non area-preserving maps)[6]. The value of τ at which "chaos" starts is $\tau_c \cong 3.260$.

Results with $\underline{a}$ varying are given in Table 3 ($\tau = 30.000$) and in Table 4 ($\tau = 3.260$, which is τ_c for $\underline{a} = 4.000$), $\underline{a/N_0} = .010$ in both cases. For both values of τ, after the Hopf bifurcation period-doubling is obtained at values $\underline{a = a_k}$ given in the 3rd column of Tables 3 and 4 (see figures 4 and 5 of the power spectrum for $\underline{a_H < a = 3.400 < a_1}$ and $\underline{a = 3.500 > a_1}$, respectively). The values of δ_1^a,

$$\delta_1^a = \frac{a_2 - a_1}{a_3 - a_2} ,$$

are given in the 4th column of Tables 3 and 4.

For $\tau = 3.260$ the determination of $\underline{a_k}$ is very difficult so that the corresponding δ_1^a is rather uncertain.

For $\tau = 30.000$ the "chaos" region starts at $\underline{a_c} \cong 3.57$ and for $\tau = 3.260$ it starts at $\underline{a_c} \cong 4.0$ (as it should as $\tau \cong 3.260$ is the value of τ at which chaos starts for $\underline{a} = 4.000$ fixed). These critical values were determined using the corresponding δ_1 value as we do not have the asymptotic δ value (they were verified numerically).

From this numerical analysis we conclude that fixing τ and varying $\underline{a}$, after the Hopf bifurcation period-doubling always occurs. Only if the time delay is very small no period-doubling is observed as in this case no Hopf bifurcation can occur (for $\tau \to 0$ the stationary solution of equation (1) is stable[10]). On the other hand, fixing $\underline{a}$ and varying τ, there seems to exist a minimum value of $\underline{a}$ ($a_{min} \cong 3.25$) for occurring period doubling, therefore the existence of Hopf bifurcation is necessary but not sufficient for period-doubling to occur.

These numerical results also indicate that the δ value for this time delayed nonlinear equation is very close to the δ value of non area-preserving maps suggesting that the time delay incorporates into the differential equation the behaviour of the difference equation associated to it $\left(\frac{dN_t}{dt} = 0\right)$.

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TABLE CAPTIONS

TABLE 1 – $a = 3.2$, $N_0 = 320$, $\mu = 1.0$; $\tau_H = 3.853$.

TABLE 2 – $a = 4.0$, $N_0 = 400$, $\mu = 1.0$; $\tau_H = 1.209$.

TABLE 3 – $\tau = 30.0$, $a/N_0 = .01$, $\mu = 1.0$; $a_H = 3.0051$.

TABLE 4 – $\tau = 3.260$, $a/N_0 = .01$, $\mu = 1.0$; $a_H = 3.2582$.

TABLE 1

τ	T
3.9	4.8
7.0	7.9
12.1	13.0
16.5	17.4
35.0	35.9
50.0	50.9
150.0	150.6

TABLE 2

τ	T	τ_k	δ_k^τ
1.2100	1.8		
2.4000	3.2		
2.5000	3.3	$\tau_1 = 2.5111$	
2.6000	6.8		
3.0900	7.9	$\tau_2 = 3.1000$	$\delta_1 \cong 4.730$
3.1060	15.9		
3.2240	16.3	$\tau_3 = 3.2245$	$\delta_2 \cong 4.560$
3.2280	32.3		
3.2510	32.5	$\tau_4 = 3.2518$	$\delta_3 \cong 4.550$
3.2525	66.6		
3.25783	121.3	$\tau_5 = 3.25782$	

TABLE 3

a	T	a_k	δ_1^a
3.2500	30.9		
3.5000	61.6	$a_1 = 3.4396$	
3.5500	122.9	$a_2 = 3.5449$	~ 4.659
3.5700	245.8	$a_3 = 3.5675$	

TABLE 4

a	T	a_k	δ_1^a
3.700	4.1		
3.900	8.2	$a_1 = 3.8117$	
3.980	16.5	$a_2 = 3.9625$	~ 5.09
3.995	32.6	$a_3 = 3.9921$	

FIGURE CAPTIONS

FIGURE 1 - Power spectrum for $\tau = 50.0$, $a = 3.200$, $N_0 = 320$.

FIGURE 2 - Power spectrum for $\tau_1 < \tau = 2.850 < \tau_2$, $a = 4.000$, $N_0 = 400$, $\mu = 1.000$.

FIGURE 3 - Power spectrum for $\tau_2 < \tau = 2.850 < \tau_3$, $a = 4.000$, $N_0 = 400$, $\mu = 1.000$.

FIGURE 4 - Power spectrum for $a_H < a = 3.400 < a_1$, $\tau = 30.000$, $a/N_0 = .010$.

FIGURE 5 - Power spectrum for $a_1 < a = 3.500 < a_2$, $\tau = 30.000$, $a/N_0 = .010$.

FIGURE 1

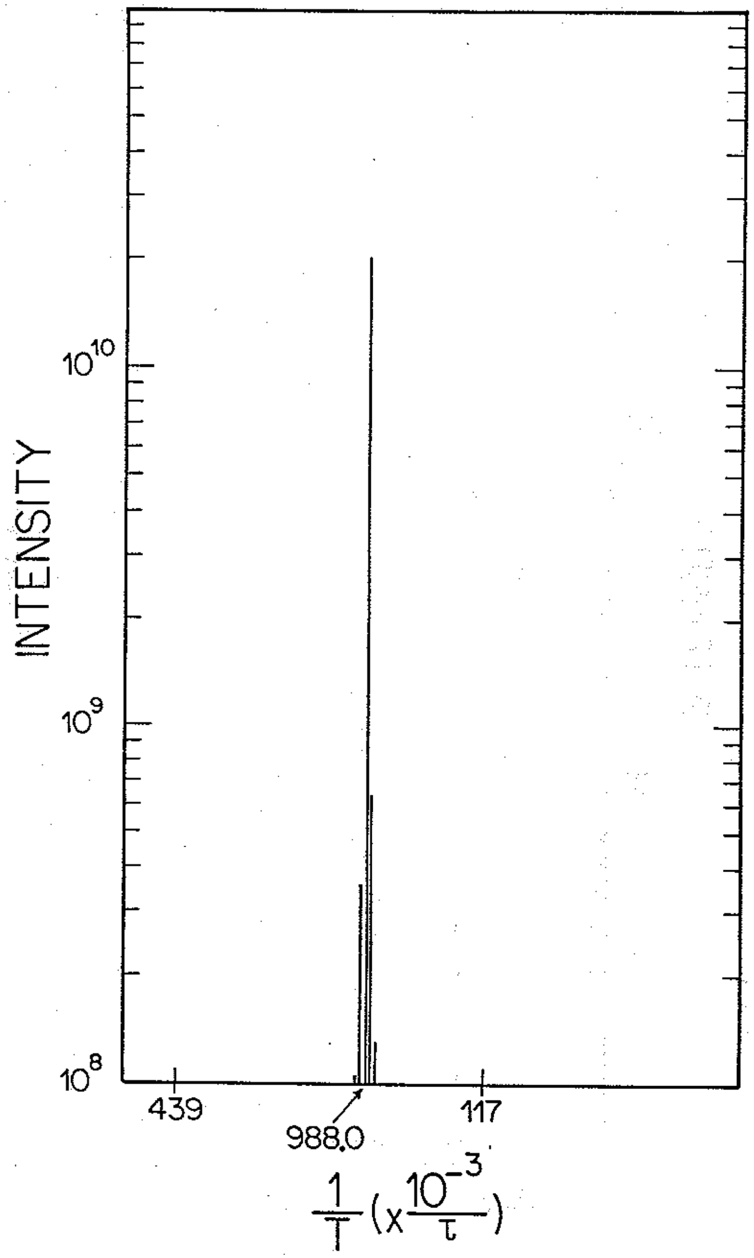

FIGURE 2

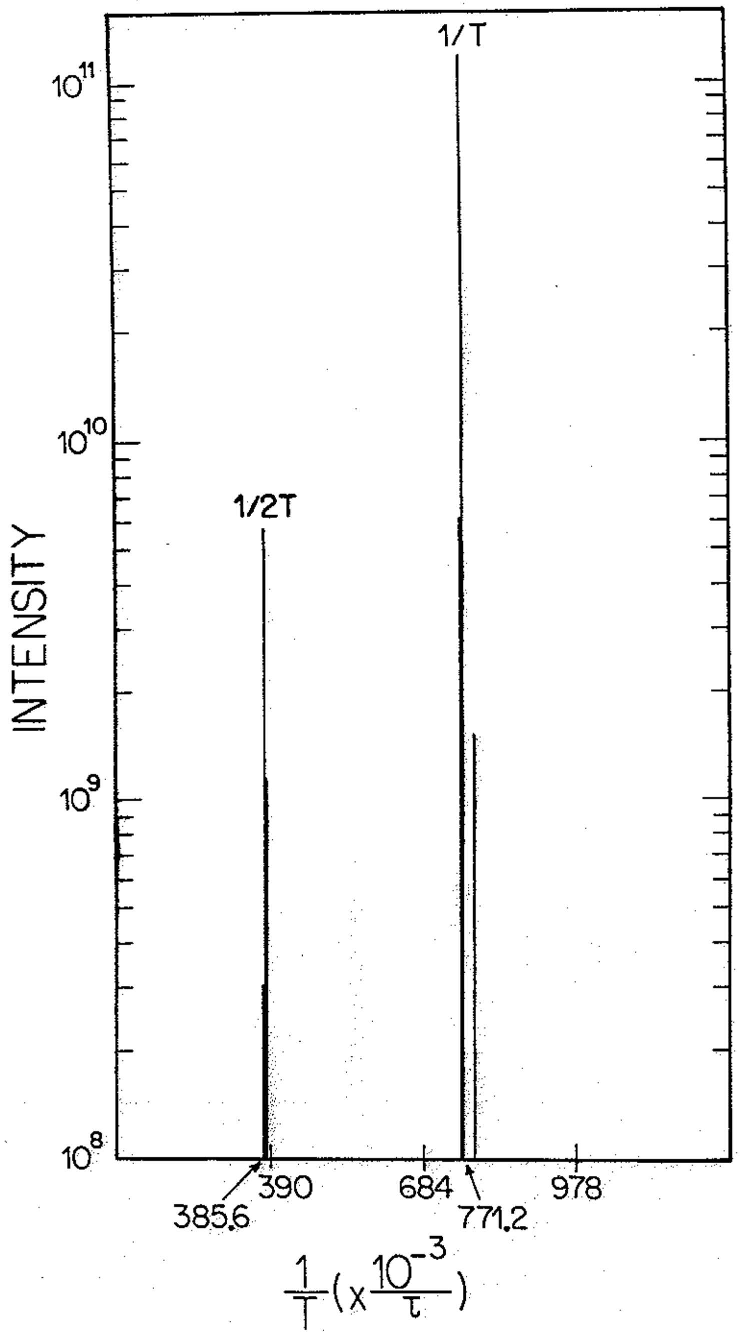

FIGURE 3

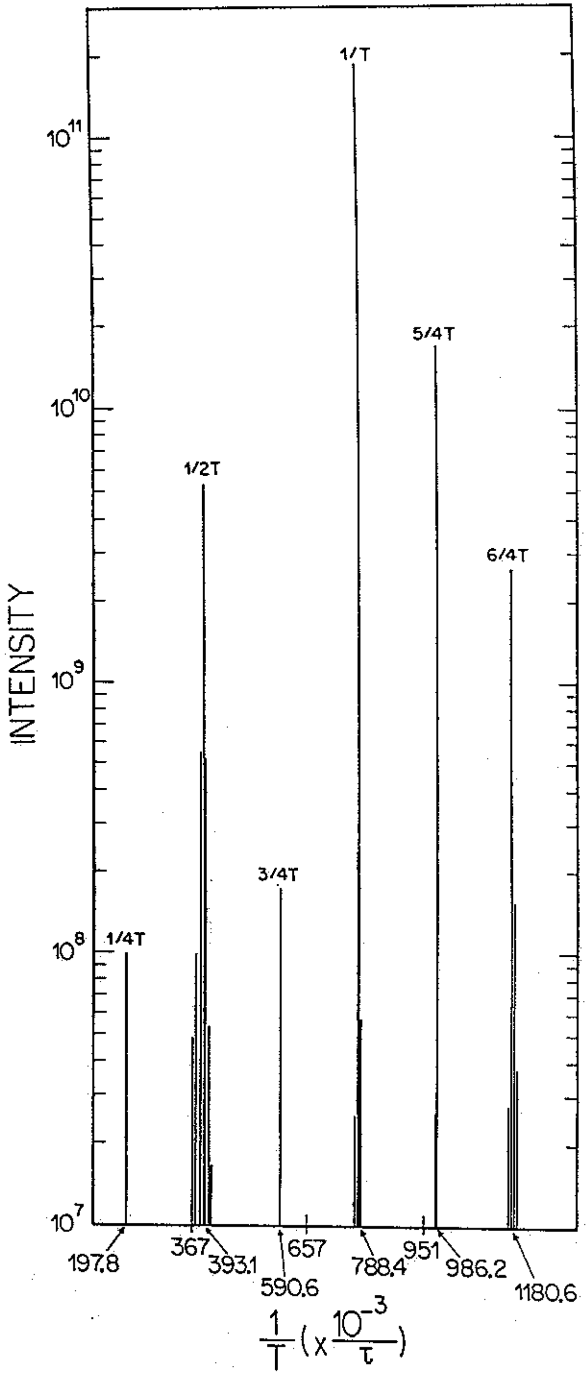

FIGURE 4

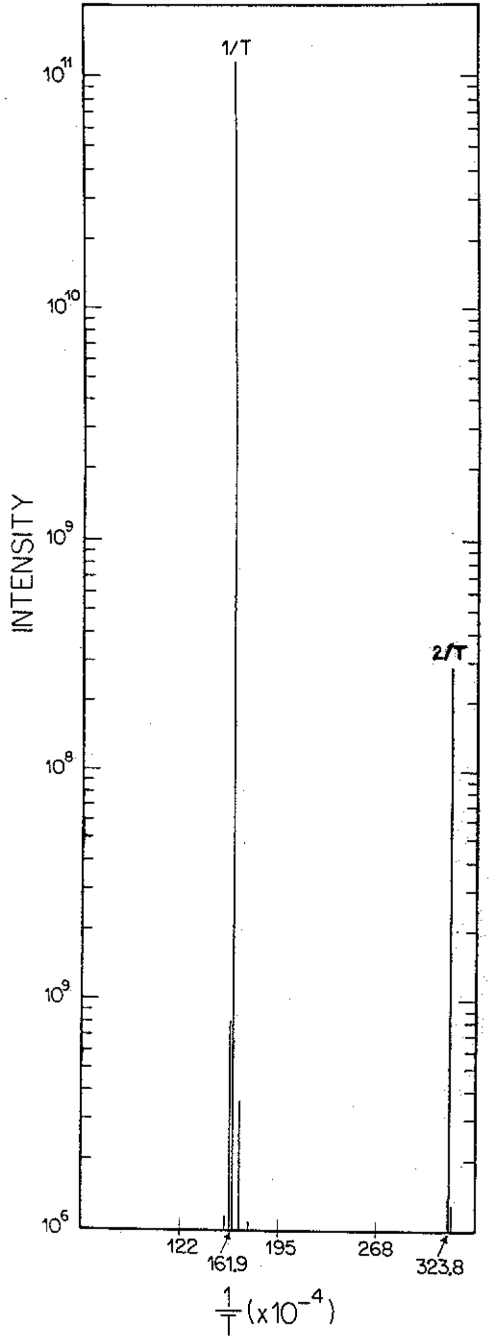

FIGURE 5