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ABSTRACT

We provide simple sufficient conditions for the existence of a bound state in the system of N particles interacting via a purely attractive two-body potential. Our method is based on a variational approach.

1. INTRODUCTION

The main purpose of this paper is to provide sufficient conditions for the existence of a bound state in the system of N particles interacting via an attractive two-body potential. For $N=2$ it is not difficult to obtain sufficient conditions for the existence of a bound state. However the case $N \geq 3$ poses the problem of locating the threshold. Thus, while for $N=2$ it is sufficient to find a trial wavefunction Φ such that $(\Phi, H\Phi) \leq 0$, for $N \geq 3$ the trial wavefunction must be such that $(\Phi, H\Phi) \leq -\alpha^2 (\Phi, \Phi)$, with $\alpha \neq 0$ in general.

The physical idea behind our method is the following: given two bound clusters of N_1 and N_2 particles it must be possible to bind them together provided there is at least one pair of particles in different clusters which can form a bound state.

The first thing we do is to derive variationally sufficient conditions for two particles to have a bound state with energy below a given quantity $-\alpha^2$. This is done in section 2 for one-, two- and three-dimensions. In one- and two-dimensions, by setting $\alpha^2 = 0$ we recover a previous result[1,2] that globally attractive potentials always possess at least one bound state. In three-dimension, setting $\alpha^2 = 0$ we obtain a condition that is simpler than that obtained by Chadan and Martin[3]; also, considering the potential to have spherical symmetry we recover Calogero's "best" sufficient condition[4]. Moreover we show that some of the other sufficient conditions provided by Calogero[4] can be improved by the variational approach.

In section 3 sufficient conditions are obtained for the existence of two-particle bound states of a given symmetry having energy less than $-\alpha^2$. In particular, we derive the sufficient condition given by Calogero[4] for the existence of a bound state with angular momentum ℓ in a spherically symmetric potential.

Finally, in section 4 we derive sufficient conditions for the existence of a bound state in the N-body problem in three-dimension. The method we use is very similar to that employed in reference 1 for one- and two-dimensions.

2. SUFFICIENT CONDITION FOR THE EXISTENCE OF TWO-PARTICLE BOUND STATE WITH ENERGY LESS THAN $-\alpha^2$

If a trial function ϕ_R exists such that $(\phi_R, H\phi_R) \leq -\alpha^2(\phi_R, \phi_R)$ then there will exist at least one bound state with energy less than $-\alpha^2$. So, in principle, it is not difficult to obtain sufficient conditions for the existence of a bound state of energy less than $-\alpha^2$. However, as pointed out by Chadan and Martin[3], it is preferable to obtain conditions in terms of simple integrals of the potential. The general recipe we propose for obtaining simple sufficient conditions for a given potential to possess a bound state with energy less than $-\alpha^2$ is to use the following trial function

$$\begin{aligned} \phi_R(r) &= e^{-\alpha r}\,\phi(r) \quad \text{for} \quad |r| < R\,, \\ \phi_R(r) &= e^{-\alpha R}\,\phi(r)\,\frac{H(\alpha r)}{H(\alpha r)} \quad \text{for} \quad |r| > R\,, \end{aligned} \tag{1}$$

where $H(\alpha r)$ is the solution of the modified Helmholtz equation (we are using units such that the two-particle reduced mass makes $\frac{\hbar^2}{2\mu} = 1$)

$$(-\Delta+\alpha^2)\,H(\alpha r) = 0 \tag{2}$$

and $\phi(r)$ is arbitrary.

Thus, in one-dimension we take

$$\phi_R(x) = e^{-\alpha|x-x_0|}\,\phi_R(x-x_0)$$

where $\phi_R(x-x_0) \in L^2(R^1)$ and is such that for $|x-x_0| < R$, $\phi_R(x-x_0) = 1$ and for $|x-x_0| > R$, $\phi_R(x-x_0)$ starts at 1 and $\phi_R(x-x_0) \to 0$ as $|x-x_0| \to \infty$. By making the scaling $\phi_R(x-x_0) \to \to \beta^{1/2}\,\phi_R(\beta(x-x_0))$ and letting $\beta \to 0$ we obtain the condition

$$2\alpha + \int_{-\infty}^{\infty} e^{-2\alpha|x-x_0|}\,V(x)\,dx \leq 0\,, \tag{3}$$

x_0 arbitrary. Setting $\alpha = 0$ we recover the known result[1,2] that a globally attractive potential in one-dimension always possesses at least one bound state.

In two-dimension we take

$$\begin{aligned} \phi_R(r) &= e^{-\alpha r} \quad \text{for} \quad r < R\,, \\ \phi_R(r) &= \frac{e^{-\alpha R}}{K_0(\alpha R)}\,K_0(\alpha r) \quad \text{for} \quad r > R\,, \end{aligned} \tag{4}$$

where $K_0(\alpha r)$ is the modified bessel function. And obtain the following sufficient condition for the existence of a bound state of energy less than $-\alpha^2$

$$-\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^R e^{-2\alpha r}\,V(r,\theta)\,r\,dr - \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_R^{\infty} e^{-2\alpha R}\,\frac{K_0(\alpha r)}{K_0(\alpha R)}\,V(r,\theta)\,r\,dr \geq$$

$$\geq \left(\frac{1-e^{-2\alpha R}}{2}\right) - \alpha R\,e^{-2\alpha R}\left\{1 + \frac{K_0'(\alpha R)}{K_0(\alpha R)}\right\}\,, \tag{5}$$

where $K_0'(\alpha R) = \left.\frac{dK_0(\alpha r)}{d(\alpha r)}\right|_{\alpha r=\alpha R}$.

Again setting $\alpha = 0$ we recover the result that a globally attractive potential in two-dimension[1,2] always possesses at least one bound state.

Finally, in three-dimension we take

$$\Phi_R(r) = \frac{1}{R^{1/2}} e^{-\alpha r} \quad \text{for} \quad r < R ,$$
$$\Phi_R(r) = R^{1/2} \frac{e^{-\alpha r}}{r} \quad \text{for} \quad r > R , \qquad (6)$$

obtaining the following sufficient condition for the existence of a bound state of energy less than $-\alpha^2$

$$-\frac{1}{4\pi}\int d\Omega \int_0^R \frac{e^{-2\alpha r}}{R} V(r,\Omega) r^2 dr - \frac{1}{4\pi}\int d\Omega \int_R^\infty R\frac{e^{-2\alpha r}}{r^2} V(r,\Omega) r^2 dr \geq$$
$$\geq \frac{1-e^{-2\alpha R}}{2\alpha R} . \qquad (7)$$

Setting $\alpha = 0$ obtains

$$-\frac{1}{4\pi}\int d\Omega \int_0^R \frac{1}{R} V(r,\Omega) r^2 dr - \frac{1}{4\pi}\int d\Omega \int_R^\infty R\, V(r,\Omega) dr \geq 1 , \qquad (8)$$

which is simpler than the condition obtained by Chadan and Martin[3]. In the particular case of spherically symmetric potential condition (8) reduces to Calogero's[4] "best" sufficient condition.

Concluding this section we present the variational version of the other sufficient conditions derived by Calogero in reference 4. Taking as trial function $\Phi_R(r) = R^{1/2}/(r+R)$, a sufficient condition for a spherically symmetric potential to hold a bound state is

$$\int_0^\infty \frac{R}{(r+R)^2} V(r) r^2 dr \leq -\frac{1}{3} . \qquad (9)$$

Taking $\Phi_R(r) = (R^{1/2}/r)(1-e^{-r/R})$ as trial function we get the condition

$$\int_0^\infty \frac{R}{r^2} (1-e^{-r/R})^2 V(r) r^2 dr \leq -\frac{1}{2} . \qquad (10)$$

Conditions (9) and (10) should be compared with Calogero's[4] conditions (3.15) and (3.17) respectively. In both cases the variational method produced improvement (the fact that the trial function is not square integrable does not matter).

3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A BOUND STATE OF A GIVEN SYMMETRY BELOW $-\alpha^2$

We shall consider only the case of obtaining sufficient conditions for the existence of a bound state of a given angular momentum ℓ having energy less than $-\alpha^2$. Our recipe in this case is to use as trial function the regular and irregular solutions of the modified Bessel equation,

$$-\frac{d^2\chi_\ell}{dr^2} - \frac{2}{r}\frac{d\chi_\ell}{dr} + \frac{\ell(\ell+1)}{r^2}\chi_\ell + \alpha^2\chi_\ell = 0 \quad , \tag{11}$$

matched at an arbitrary point $r = R$. Thus, the trial function is

$$\phi_R^\ell(r) = \frac{(\alpha R)^{1/2}}{K_{\ell+1/2}(\alpha R)} \frac{K_{\ell+1/2}(\alpha r)}{(\alpha r)^{1/2}} \quad \text{for} \quad r < R \quad ,$$

$$\phi_R^\ell(r) = \frac{(\alpha R)^{1/2}}{I_{\ell+1/2}(\alpha R)} \frac{I_{\ell+1/2}(\alpha r)}{(\alpha r)^{1/2}} \quad \text{for} \quad r > R \quad , \tag{12}$$

providing the following sufficient condition

$$-\frac{1}{4\pi}\int d\Omega \int_0^\infty |\phi_R^\ell|^2 \, V(r,\Omega) r^2 dr \geq \alpha R^2 \left\{\frac{(\ell+1)}{\alpha R} + \frac{K_{\ell-1/2}(\alpha R)}{K_{\ell+1/2}(\alpha R)}\right\} + \alpha R^2 \left\{\frac{\ell}{R} + \frac{I_{\ell+3/2}(\alpha R)}{I_{\ell+1/2}(\alpha R)}\right\} \quad . \tag{13}$$

For spherically symmetric potential, in the limit $\alpha \to 0$ the above condition reduces to Calogero's "best"[4] sufficient condition

$$-\frac{1}{R}\int_0^R \left(\frac{r}{R}\right)^{2\ell} V(r) r^2 dr - \frac{1}{R}\int_R^\infty \left(\frac{R}{r}\right)^{2\ell+2} V(r) r^2 dr \geq 2\ell+1 \quad . \tag{14}$$

4. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A N-BODY BOUND STATE IN THREE-DIMENSION

We shall prove that the N-particle hamiltonian,

$$H_N = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) \quad , \tag{15}$$

has at least one bound state if $V(\vec{r}_i - \vec{r}_j)$ is a purely attractive potential satisfying condition (8).

We shall first prove the result for $N = 3$. Following Simon[5] we use coordinates relative to particle 3,

$$\vec{x}_1 = \vec{r}_1 - \vec{r}_3 \quad ,$$

$$\vec{x}_2 = \vec{r}_2 - \vec{r}_3 \quad , \tag{16}$$

$$\vec{R} = \frac{1}{M}\sum_{i=1}^{3} m_i \vec{r}_i \quad ,$$

where $M = \sum_{i=1}^{3} m_i$ is the total mass of the system. In these coordinates, the hamiltonian H_3, with the centre of mass removed, becomes

$$H_3 = \frac{\vec{k}_1^{\,2}}{2\mu_{13}} + \frac{\vec{k}_2^{\,2}}{2\mu_{23}} + \frac{1}{m_3}\vec{k}_1\cdot\vec{k}_2 + V(\vec{x}_1) + V(\vec{x}_2) + V(\vec{x}_1 - \vec{x}_2) \quad , \tag{17}$$

where $\vec{k}_i$ is the momentum canonically conjugate to $\vec{x}_i$ and μ_{13} is the reduced mass of m_i and m_3.

Now, as trial function we take $\Psi = \Phi_0(\vec{x}_1)\Phi_R(x_2)$, where $\Phi_0(\vec{x}_1)$ is an eigenfunction of H_2 (particles 1 and 3) with energy E_2 and $\Phi_R(x_2)$ is given by (6) with $\alpha = 0$, that is,

$$\phi_R(x_2) = \frac{1}{R^{1/2}} \quad \text{for} \quad x_2 < R \quad ,$$
$$\phi_R(x_2) = \frac{R^{1/2}}{x_2} \quad \text{for} \quad x_2 > R \quad . \tag{18}$$

Now, as the two-body potential is purely attractive, throwing away $V(\vec{x}_1-\vec{x}_2)$ we have (in units of $\frac{\hbar^2}{2\mu_{23}}$)

$$\frac{(\Psi,H_3\Psi)}{(\Psi,\Psi)} \leq E_2 + 1 + \frac{1}{4\pi}\int d\Omega \int_0^R \frac{1}{R} V(x_2,\Omega)x_2^2 dx_2 + $$
$$+ \frac{1}{4\pi}\int d\Omega \int_R^\infty R\, V(x_2,\Omega)dx_2 \quad , \tag{19}$$

since the expectation value of the Hughes-Eckart term $\vec{k}_1.\vec{k}_2$ vanishes. Since by hypothesis the two-body potential satisfies condition (8) it follows that H_3 has at least one bound state.

Analogously if the two-body potential is purely attractive and satisfies condition (8) and if H_N has a bound state then H_{N+1} will have at least one bound state. The proof is immediate.

We now show that if the continuum of the N-body system starts at two-cluster[6] break up then, if the two-body potential satisfies condition (8) there exists at least one bound state of N particles. Let C_1 be the cluster comprising particles 1 to L and C_2 the cluster comprising particles L+1 to N. As coordinates we shall use the coordinates relative to the L^{th} particle for particles $\in C_1$, the coordinates relative to the N^{th} particle for particles $\in C_2$ and the coordinate of the L^{th} particle relative to the N^{th} particle,

$$\vec{x}_{iL} = \vec{r}_i - \vec{r}_L \quad , \quad i = 1,\dots,L-1 \quad ,$$
$$\vec{x}_{iN} = \vec{r}_i - \vec{r}_N \quad , \quad i = L+1,\dots,N-1 \quad , \tag{20}$$
$$\vec{x} = \vec{r}_L - \vec{r}_N \, .$$

With these coordinates the N-particle hamiltonian (after some kinematics[5]) can be written as (centre of mass removed)

$$H_N = \sum_{i=1}^{L-1} \frac{\vec{k}_{iL}^2}{2\mu_{iL}} + \sum_{i>j=1}^{L-1} \frac{\vec{k}_{iL}.\vec{k}_{jL}}{m_L} + \sum_{i=1}^{L-1} V(\vec{x}_{iL}) + $$
$$+ \sum_{i>j=1}^{L-1} V(\vec{x}_{iL} - \vec{x}_{jL}) + \sum_{i=L+1}^{N-1} \frac{\vec{k}_{iN}^2}{2\mu_{iN}} + \sum_{i>j=L+1}^{N-1} \frac{\vec{k}_{iN}.\vec{k}_{jN}}{m_N} + $$
$$+ \sum_{i=L+1}^{N-1} V(\vec{x}_{iN}) + \sum_{i>j=L+1}^{N-1} V(\vec{x}_{iN} - \vec{x}_{jN}) + \frac{\vec{k}^2}{2\mu_{LN}} + V(\vec{x}) + $$
$$+ \sum_{i=1}^{L-1} \frac{\vec{k}.\vec{k}_{iL}}{m_L} - \sum_{i=L+1}^{N-1} \frac{\vec{k}.\vec{k}_{iN}}{m_N} + \sum_{\substack{i,j \\ i\in C_1,\neq L \\ j\in C_2,\neq N}} V(\vec{x}_{iL}-\vec{x}_{jN}+\vec{x}) \quad , \tag{21}$$

where $\vec{k}_{iL}$, $\vec{k}_{iN}$ and $\vec{k}$ are the momenta canonically conjugate to $\vec{x}_{iL}$, $\vec{x}_{iN}$ and $\vec{x}$ respectively and μ_{iL}, μ_{iN} and μ_{LN} are the reduced masses of particles (i,L), (i,N) and (L,N) respectively. Expression (21) can be written as

$$H_N = H_{C_1} + H_{C_2} + \frac{\vec{k}^2}{2\mu_{LN}} + V(\vec{x}) + \sum_{C_1} \frac{\vec{k}.\vec{k}_{iL}}{m_L} - \sum_{C_2} \frac{\vec{k}.\vec{k}_{iN}}{m_N} + $$
$$+ \sum_{\substack{i\in C_1,\neq L \\ j\in C_2,\neq N}} V(\vec{x}_{iL}-\vec{x}_{jN}+\vec{x}) \quad . \tag{22}$$

Now, as trial function we take

$$\Psi = \Phi_{C_1} \Phi_{C_2} \Phi_R(x) \quad , \tag{23}$$

where Φ_{C_1} and Φ_{C_2} are eigenfunction of H_{C_1} and H_{C_2} with energies E_{C_1} and E_{C_2} respectively and $\Phi_R(x)$ is given by (18). Again, as the two-body interaction is purely attractive, throwing away from (22) the intercluster potential (last term), we get (in units of $\frac{\hbar^2}{2\mu_{LN}}$)

$$\frac{(\Psi, H_N \Psi)}{(\Psi,\Psi)} \leq E_{C_1} + E_{C_2} + \left\{1 + \frac{1}{4\pi} \int d\Omega \int_0^R \frac{1}{R} V(\vec{x}) x^2 dx + \right.$$

$$\left. + \frac{1}{4\pi} \int d\Omega \int_R^\infty R\, V(\vec{x}) dx \right\} \quad , \tag{23}$$

as the expectation value of all Hughes-Eckart terms vanishes. And again, if the two-body potential satisfies condition (8), it follows that H_N has at least one bound-state.

Finally, using for $\Phi_R(x)$ the function given by (6) it can easily be shown that if the two-body potential is purely attractive and satisfies condition (7) then the N-particle system will have at least one bound-state of energy less than $-\alpha^2$.

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