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A LOWER BOUND FOR THE ENERGY OF THE GROUND STATE
OF BOSONS MOVING IN ONE DIMENSION

by

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A LOWER BOUND FOR THE ENERGY OF THE GROUND STATE
OF BOSONS MOVING IN ONE DIMENSION

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SUMMARY

We show that the ground state energy of N bosons of mass m moving in one dimension is greater than $E = -\frac{m}{16\hbar^2} N^2(N-1) \left| \int_{-\infty}^{+\infty} V(x) dx \right|$ where V(x) is the two-body potential. We conjecture that $E = -\frac{m}{24\hbar^2} N(N^2-1) \left| \int_{-\infty}^{+\infty} V(x) dx \right|^2$ provides a lower bound.

The objective of this note is to combine three known results obtaining the following theorem: the ground state of N bosons in one-dimension, interacting via a two-body potential $V(x_i-x_j)$ such that $\int_{-\infty}^{+\infty} V dx = -I$ ($I > 0$), is greater than $E = -\frac{m}{16\hbar^2} I^2 N^2(N-1)$.

We start by recalling the three known theorems we shall use to prove the above result.

Theorem 1 - A system of N particles moving in one-dimension interacting via a globally attractive potential has always a bound state. This theorem⁽¹⁾ was proved by us.

Theorem 2 - Of all potentials V(x) with the same value of $I = -\int_{-\infty}^{+\infty} V(x) dx$ the δ -function potential has the lowest energy. This theorem is due to L. Spruch⁽²⁾. We repeat the argument here because the proof given by Spruch⁽²⁾ is difficult to follow due to many misprints. Alternatively a more general theorem⁽³⁾ was proved by J.B. Keller.

Proof of theorem 2 - Let ψ be the exact normalized ground state solution of $H_0 + V$.

So

$$E = \langle \psi | H_0 + V | \psi \rangle = \langle \psi | H_0 | \psi \rangle + \int V(x) |\psi(x)|^2 dx .$$

Let $|\psi(x_0)|$ be the maximum value of $|\psi(x)|$. Then

$$E \geq \langle \psi | H_0 | \psi \rangle - |\psi(x_0)|^2 I = \langle \psi | H_0 | \psi \rangle + \langle \psi | V_\delta | \psi \rangle$$

where $V_\delta(x) = -I \delta(x-x_0)$, thus concluding the proof.

Remark - V has been assumed to be purely attractive.

This assumption can be easily removed (see ref. 2).

We need now the following theorem due to Hall and Post⁽⁴⁾:

Theorem 3 (Hall and Post) - A lower bound for the ground state of a system of N particles is given by the ground state energy of the following two-body hamiltonian

$$H = - (N-1) \frac{\hbar^2}{2m} \Delta_{\rho_2}^2 + \frac{N}{2} (N-1) V(\sqrt{2} \rho_2) \quad (1)$$

where $\rho_2 = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2)$.

We are now ready to prove the theorem stated in the beginning of this note.

Hall-Post theorem is valid in any dimension and therefore we can say that a lower bound for the energy of N identical bosons moving in one dimension is given by

$$E_L = (N-1) E_b$$

where E_b is the ground state solution of

$$-\frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} \phi(x) + \frac{N}{2} V(x) \phi(x) = E_b \phi(x) \quad (2)$$

But by theorem 2 we have

$$E_b > - \frac{m}{16\hbar^2} N^2 I^2 \quad (3)$$

and therefore

$$E_L > - \frac{m}{16\hbar^2} (N-1) N^2 I^2 \quad (4)$$

thus proving the result.

We now make the following:

Conjecture: A lower bound for the ground state energy of N identical particles interacting via a globally attractive potential $\left[I = - \int_{-\infty}^{+\infty} V(x) dx \right]$ is given by the ground state of N particles interacting via $V_\delta = -I \delta(x_1 - x_j)$.

Quasi Proof - Let ψ be the exact symmetric wave function of $H_0 + V$. Then

$$E = \langle \psi | H_0 + V | \psi \rangle = \langle \psi | H_0 | \psi \rangle + \frac{N(N-1)}{2} \int d\rho \phi(\rho) V(\rho)$$

where $\phi(\rho) = \int |\psi|^2(x, \rho) dx$, $\rho = (x_1 - x_2)$ and x stands for all the other coordinates.

Let ρ_0 be the point where $\phi(\rho, x)$ is maximum.

Then

$$\begin{aligned}
E &\geq \langle \psi | H_0 | \psi \rangle - \frac{N(N-1)}{2} \phi(\rho_0) \int I = \\
&= \langle \psi | H_0 | \psi \rangle - \frac{N(N-1)}{2} \int \phi(\rho) \int I \delta(\rho - \rho_0) d\rho = \\
&= \langle \psi | H_0 | \psi \rangle - \frac{N(N-1)}{2} \langle \psi | \int I \delta(\rho - \rho_0) | \psi \rangle = \\
&= \langle \psi | H_0 + U_{\delta, \rho_0} | \psi \rangle
\end{aligned}$$

where

$$U_{\delta, \rho_0} = -\frac{1}{2} \sum_{ij} \int I \delta(x_i - x_j - \rho_0)$$

So we have proved that the ground-state energy of $H_0 + U_{\delta, \rho_0}$ provides a lower bound for the ground-state of $H_0 + V$.

We conjecture that a minimum occur for $H_0 + U_{\delta, \rho_0=0} = H_0 + U_{\delta}$ which is exactly solvable⁽⁵⁾.

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