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A LOWER BOUND FOR THE ENERGY OF THE GROUND STATE OF BOSONS MOVING IN ONE DIMENSION

by

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A LOWER BOUND FOR THE ENERGY OF THE GROUND STATE OF BOSONS MOVING IN ONE DIMENSION

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SUMMARY

We show that the gound state energy of N bosons of mass m moving in one dimension is greater than $E = -\frac{m}{16\hbar^2} N^2(N-1) \left|\int_{-\infty}^{+\infty} V(x)\,dx\right|$ where $V(x)$ is the two-body potential. We conjecture that $E = -\frac{m}{24\hbar^2} N(N^2-1) \left|\int_{-\infty}^{\infty} V(x)\,dx\right|^2$ provides a lower bound.

The objective of this note is to combine three known results obtaining the following theorem: the ground state of N bosons in one-dimension, interacting via a two-body potential $V(x_i - x_j)$ such that $\int_{-\infty}^{+\infty} V\,dx = -I$ $(I > 0)$, is greater than $E = -\frac{m}{16\hbar^2} I^2 N^2(N-1)$.

We start by recalling the three known theorems we shall use to prove the above result.

Theorem 1 - A system of N particles moving in one-dimension interacting via a globally attractive potential has always a bound state. This theorem[1] was proved by us.

Theorem 2 - Of all potentials $V(x)$ with the same value of $I = -\int_{-\infty}^{+\infty} V(x)\,dx$ the δ-function potential has the lowest energy. This theorem is due to L. Spruch[2]. We repeat the argument here because the proof given by Spruch[2] is difficult to follow due to many misprints. Alternatively a more general theorem[3] was proved by J.B. Keller.

Proof of theorem 2 - Let ψ be the exact normalized ground state solution of $H_0 + V$.

So

$$E = \langle\psi|H_0 + V|\psi\rangle = \langle\psi|H_0|\psi\rangle + \int V(x)\ |\psi(x)|^2\,dx\ .$$

Let $|\psi(x_0)|$ be the maximum value of $|\psi(x)|$. Then

$$E \geq \langle\psi|H_0|\psi\rangle - |\psi(x_0)|^2 I = \langle\psi|H_0|\psi\rangle + \langle\psi|V_\delta|\psi\rangle$$

where $V_\delta(x) = -I\,\delta(x-x_0)$, thus concluding the proof.

<u>Remark</u> - V has been assumed to be purely attractive. This assumption can be easily removed (see ref. 2).

We need now the following theorem due to Hall and Post[4]:

<u>Theorem 3</u> (Hall and Post) - A lower bound for the ground state of a system of N particles is given by the ground state energy of the following two-body hamiltonian

$$H = -(N-1)\frac{\hbar^2}{2m}\Delta^2_{\rho_2} + \frac{N}{2}(N-1)\,V(\sqrt{2}\,\rho_2) \quad , \tag{1}$$

where $\rho_2 = \frac{1}{\sqrt{2}}(\vec{r}_1-\vec{r}_2)$.

We are now ready to prove the theorem stated in the beginning of this note.

Hall-Post theorem is valid in any dimension and therefore we can say that a lower bound for the energy of N identical bosons moving in one dimension is given by

$$E_L = (N-1)\,E_b$$

where E_b is the ground state solution of

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\phi(x) + \frac{N}{2}V(x)\,\phi(x) = E_b\,\phi(x) \quad . \tag{2}$$

But by theorem 2 we have

$$E_b > -\frac{m}{16\hbar^2}N^2 I^2 \tag{3}$$

and therefore

$$E_L > -\frac{m}{16\hbar^2}(N-1)\,N^2 I^2 \tag{4}$$

thus proving the result.

We now make the following:

<u>Conjecture</u>: A lower bound for the ground state energy of N identical particles interacting via a globally attractive potential $\left(I = -\int_{-\infty}^{+\infty} V(x)\,dx\right)$ is given by the ground state of N particles interacting via $V_\delta = -I\,\delta(x_i-x_j)$.

<u>Quasi Proof</u> - Let ψ be the exact symmetric wave function of H_0+V. Then

$$E = \langle\psi|H_0+V|\psi\rangle = \langle\psi|H_0|\psi\rangle + \frac{N(N-1)}{2}\int d\rho\,\phi(\rho)\,V(\rho)$$

where $\phi(\rho) = \int |\psi|^2(x,\rho)\,dx$, $\rho = (x_1-x_2)$ and x stands for all the other coordinates.

Let ρ_0 be the point where $\phi(\rho,x)$ is maximum. Then

$$E \geq \langle\psi|H_0|\psi\rangle - \frac{N(N-1)}{2} \phi(\rho_0) I =$$

$$= \langle\psi|H_0|\psi\rangle - \frac{N(N-1)}{2} \int \phi(\rho) I \delta(\rho-\rho_0) d\rho =$$

$$= \langle\psi|H_0|\psi\rangle - \frac{N(N-1)}{2} \langle\psi|I \delta(\rho-\rho_0)|\psi\rangle =$$

$$= \langle\psi|H_0 + U_{\delta,\rho_0}|\psi\rangle$$

where

$$U_{\delta,\rho_0} = -\frac{1}{2} \sum_{ij} I \delta(x_i-x_j-\rho_0) \quad .$$

So we have proved that the ground-state energy of $H_0 + U_{\delta,\rho_0}$ provides a lower bound for the ground-state of $H_0 + V$.

We conjecture that a minimum occur for $H_0 + U_{\delta,\rho_0=0} = H_0 + U_\delta$ which is exactly solvable[5].

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