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GEOMETRY AS AN ASPECT OF DYNAMICS

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ABSTRACT. Contrary to the predominant way of doing physics, we show that the geometric structure of a general differentiable space-time manifold can be determined by means of the introduction in that manifold of a minimal set of fundamental dynamical quantities associated to a particle endowed with the fundamental property of covariant momentum. Thus, general relativistic physics implies a general pseudo-Riemannian geometry, whereas the physics of the special theory of relativity is tied up with Minkowski space-time, and Newtonian dynamics is bound to Newtonian space-time. While in the relativistic instance, the Riemannian character of the manifold is basically fixed by means only of the Hamiltonian state function of the particle (its energy), in the latter case, we have to resort, perhaps not unexpectedly, to the two dynamical entities mass and energy, separately.

1. INTRODUCTION

It has been claimed[1] that maybe the most important consequence of relativity theory is that space and time are not concepts which can be considered independently of each other, but that they must be combined in such a fashion as to give a four-dimensional description of physical phenomena. In this context, it has been stated, then, that *dynamics becomes an aspect of geometry*. This establishes an intimate relation between dynamics and geometry which can be considered even in prerelativistic theories .

In the usual manner of describing the natural world, this association between dynamics and geometry is customarily considered as follows. When building a dynamical model of Nature one always begins by postulating a certain space-time and from there proceeds to develop a certain physics in that arena, which is then considered as a substratum to the physical world. That is, one usually starts from a given, preestablished geometry, upon which a consequential dynamics is established, and it is well known that the choice of the geometry (of the postulated space-time) uniquely determines the physics that can be constructed in that postulated space-time. Thus, just as the only dynamics compatible with the absolute space-time of Newton is precisely Newtonian dynamics, correspondingly, in Minkowski space-time, only the dynamics of special relativity can be naturally built.

In the present work, we intend to show how the introduction from the onset, in a general differentiable space-time manifold, of a certain well defined minimal set of fundamental dynamical quantities allows the specific geometric structure of that manifold to be fixed. This view is basically contrary to the usual one and will be detailed below.

Before turning over to the point of view to be developed here, let us present the special relativistic and Newtonian cases, as they are usually stated. The four-dimensional space-time manifold of Minkowski consists of a three-dimensional spatial hypercone with time pointing along its symmetry axis. The geometry of this manifold has as its invariance group the full Lorentz group (or group of Poincaré):

$$x'^{\mu} = L^{\mu}_{\nu}\, x^{\nu} + a^{\mu} \tag{1.1}$$

with greek indices running from 1 to 4. Here, (L^{μ}_{ν}) is a (4×4) orthogonal matrix and a^{μ} is an arbitrary (constant) 4-vector.

Since it is perhaps somewhat less familiar than its Minkowski counterpart, let us dwell - although still in a cursory fashion- with the Newtonian case in a little more detail. In the Newtonian case, the 4-dimensional space-time manifold was first introduced by E. Cartan[2] as an affine manifold E_4, consisting of a 3-dimensional space-like hypersurface, orthogonal to the absolute time axis. This geometry fixes the group of symmetry

$$x'^{\alpha} = G^{\alpha}_{\beta}\, x^{\beta} + k^{\alpha} \tag{1.2}$$

Here, the matrix (G^{α}_{β}) has the $(3+1)\times(3+1)$ block form:

$$(G^{\alpha}_{\beta}) = \begin{pmatrix} G & \vec{v} \\ 0 & 1 \end{pmatrix} \tag{1.3}$$

where G is a (3×3) orthogonal matrix and the (3×1) column vector $\vec{v}$ is arbitrary. This geometry (and its related symmetry group) determines both the absolute kynematical and dynamical entities, that is, those entities which are left invariant by the

transformations (1.2).

The matrix (G^{α}_{β}) can be diagonalized and put in the form

$$\begin{pmatrix} GG^{T} & \vec{v} \\ 0 & 0 \end{pmatrix}$$

From this, it is seen that the metric (or fundamental) tensor ${}^{(n)}g_{\alpha\beta} = {}^{(n)}\eta_{\alpha\beta}$ of the affine Newtonian space-time E_4 is *singular*[3,4]. This fact immediately distinguishes Newtonian space-time from its special-relativistic counterpart. In fact, while in this latter case one can introduce dual metric tensors ${}^{(r)}g_{\alpha\beta}$ and ${}^{(r)}g^{\alpha\beta}$, one being the inverse of the other, this cannot be done in E_4, since there the inverse does not exist. Therefore, it is precisely in E_4, the Newtonian space-time, where the distinction between covariant and contravariant 4-vectors will be expected to be more fundamental than in the special relativistic case, where there exists a complete transposition between contravariant and covariant quantities. This, of course, should not be taken as meaning that in the 3-dimensional space-like hipersurface E_3 of E_4 this raising or lowering of indices is not fully justified, since that submanifold E_3 is Euclidean. This last fact leads to the consideration made a long time ago by E. Cartan[3] that E_4 is *not* an Euclidean manifold, but its affine connection, ${}^{(n)}V_4$ is Euclidean, which is just another way of seeing that the metric tensor of E_4 is singular.

2. CONTRAVARIANT AND COVARIANT VECTORS

When examining the interconnection between physics and geometry it is of paramount importance to establish the essential distinction that exists between contravariant and covariant entities. A very striking aspect of this distinction was pointed out by M. Schönberg[5] who observed that while the contravariant vectors are the ones which are more intimately related with geometry, the covariant vectors are the ones which are more closely connected with physics. In this regard, two instances come up immediately to mind: the position vector $\vec{x}$, which is essentially contravariant, and the momentum $\vec{p}$, which is essentially covariant. In this section, we discuss some aspects which manifest this distinction.

Given the vector affine space E_n, the linear mapping $\omega: E_n \to R$ of E_n over R defines a linear form over E_n. The vectors of E_n are the *contravariant* vectors $\vec{x}$, which, in a given basis $\{\vec{e}_i\}$ are written as:

$$\vec{x} = x^i \vec{e}_i \tag{2.1}$$

The linear forms over E_n belong to another vector affine space E_n^*, dual of E_n. The vectors $\vec{x}^* \in E_n^*$ are the *covariant* vectors $\omega(\vec{x})$:

$$\vec{x}^* = \omega(\vec{x}) = \omega(\vec{e}_i)\vec{x}^{\,i} = a_i \vec{x}^i \tag{2.2}$$

where we can consider the $a_i \equiv \omega(\vec{e}_i)$ as the components of the covariant vector ω in the dual basis $\{\vec{x}^i\} \equiv \{\vec{e}^i\}$, i.e., we may write a covariant vector $\vec{x}^* \epsilon E_n^*$ as:

$$\vec{x}^* = x_i \vec{e}^i \tag{2.3}$$

with $x_i \equiv a_i$. (While the x^i are considered as vectors components in E_n, in the dual space E_n^* they are linearly independent one-forms.)

The geometrical meaning of the contravariant and covariant vectors is obtained through the introduction of an affine space $(0, E_n) \equiv \mathcal{E}_n$, which is a space of points having a structure of a vector space depending on the point 0, taken as the origin[6]. It should be noticed that neither a metric was defined in E_n, nor a distance in $\mathcal{E}_n$.

The contravariant vector $\vec{x} = x^i \vec{e}_i \epsilon E_n$ is represented geometrically by an oriented line, whereas the covariant vector $\vec{x}^* = x_i \vec{e}^i \epsilon E_n^*$ is represented by two parallel hyperplanes, since we have a family $\vec{x}^* = x_i \vec{e}^i = \omega(\vec{x}) = a_i x^i = k$ of parallel hyperplanes, depending on the parameter k. Since the coordinate axes are intercepted at $x^i = k/a_i$, the components of a contravariant vector have dimensions of length - an *extensive* quantity - while the covariant vector components have dimensions of the inverse of a length - an *intensive* quantity.

As appropriate examples, we notice that the *position* vector $\vec{x}$ is essentially contravariant, while the gradient $\partial\phi/\partial\vec{x}$ of a scalar function $\phi(\vec{x})$ of position is essentially covariant. Recalling that in physics the dynamical quantity *momentum* $\vec{p}$ is defined as $\propto \partial\phi/\partial\vec{x}$, this definition makes momentum a covariant vector, and hence it is much more appropriate to write down the fundamental equation of Newtonian dynamics as $\vec{f} = -dp/dt$, then in the form $\vec{f} = m\, d^2\vec{x}/dt^2$.

With contravariant and covariant vectors, many different kinds of algebras can be built[7]. Thus, let the contravariant vector $\vec{V} = V^j \vec{I}_j$ and the covariant vector $\vec{U} = U_j \vec{I}^j$ be written in the reciprocal basis $\vec{I}_j$ and $\vec{I}^j$ of a certain n-dimensional affine space. The invariant $U_j V^j$ is denoted here by $\langle\vec{U},\vec{V}\rangle$. Introducing the symbols $(\vec{V})$ and $(\vec{U})$ associated to the vectors $\vec{V}$ and $\vec{U}$ by the anticommutation rules

$$[(\vec{V}),(\vec{V}')]_+ = 0$$
$$[(\vec{U}),(\vec{U}')]_+ = 0 \qquad (2.4)$$
$$[(\vec{V}),(\vec{U})]_+ = \langle\vec{V},\vec{U}\rangle 1_{G_n}$$

we obtain the *Grassmann algebra* G_n (1_{G_n} is the unit of G_n). This algebra is generated by the elements $(\vec{I}_j)$ and $(\vec{I}^j)$ through the anticommutation rules:

$$[(\vec{I}_j),(\vec{I}_k)]_+ = 0$$
$$[(\vec{I}^j),(\vec{I}^k)]_+ = 0 \qquad (2.5)$$
$$[(\vec{I}_j),(\vec{I}^k)]_+ = \delta_j^k \, 1_{G_n}$$
$$\langle\vec{I}_j,\vec{I}^k\rangle = \delta_j^k$$

Equations (2.5) show that, although G_n is an algebra of a n-dimensional space, it has the structure of a *Clifford algebra* C_{2n} of a 2n-dimensional space. The theory of G_n is, essentially, that of the spinors of E_{2n}. The Grassmann algebra G_n, taken over the complex numbers, is equivalent to a n- dimensional Jordan-Wigner algebra. Taking the adjoint $(\vec{I}^j) = (\vec{I}_j)^\dagger$, the anticommutation rules (2.5) become the n-dimensional equivalent to emission and absorption operators of the second quantization for fermions[8].

Similarly, one can define an associative algebra L_n, with elements denoted by $\{\vec{V}\}$ and $\{\vec{U}\}$, satisfying the commutation rules:

$$[\{\vec{V}\},\{\vec{V}'\}] = 0$$
$$[\{\vec{U}\},\{\vec{U}'\}] = 0 \qquad (2.6)$$
$$[\{\vec{V}\},\{\vec{U}\}] = \langle\vec{V},\vec{U}\rangle 1_{L_n}$$

(1_{L_n} being the unit element of L_n, and the generators of L_n satisfying the commutation rules:

$$[\{\vec{I}_j\},\{\vec{I}_k\}] = 0$$
$$[\{\vec{I}^j\},\{\vec{I}^i\}] = 0 \qquad (2.7)$$
$$[\{\vec{I}_j\},\{\vec{I}^k\}] = \delta_j^k \, 1_{L_n}$$

Equations (2.7) provide the Heisenberg commutation rules for the coordinate $\vec{Q} = Q^j\vec{q}_j$ and momentum operators $\vec{P} = P_j\vec{p}^j$, the generators of which are given by $\vec{q}_j = \{\vec{I}^j\}$ and $\vec{p}^j = i\,\hbar^{-1}\{\vec{I}^j\}$, where $\hbar$ is Plank's constant. Thus, L_n over the complex numbers is equivalent to the *Heisenberg algebra* for the operators $\vec{Q}$ and $\vec{P}$ of a quantum system with n degrees of freedom. It can also he shown that quantum kynematics is related to the *symplectic geometry* of the phase space of Hamiltonian classical mechanics through its symplectic algebra L_n[9]. Besides, the algebra L_n over the complex numbers provides the n-dimensional equivalent to the *Dirac-Jordan-Klein* algebra for the emission and absorption operators of the second quantization for bosons. In 4-dimensional space, the action algebra, obtained from $dV = dp_i \, dx^i$, i=1,2,3,4, provides a quadratic form in 8 variables. This is the only instance in which there is a triality: one vector and two half-spinors, all with 8 components and with similar properties[9,10].

3. BASIC POSTULATES

Having presented the above considerations upon the different algebraic structures generated by covariant and contravariant vectors, we may begin to assign a dynamical meaning to some of these vectors.

As we already said, the usual way of building physical models and/or theories consists in postulating a given space-time manifold, which is almost always metric (it can be shown that a differentiable manifold always admits a Riemannian metric[11,12]) and where that metric is always fixed *ab initio*. This is the fixed space-time framework upon which a certain theory is built.

Our starting point here is just the opposite: we try to determine the geometry by means of the introduction of a certain minimal number of fundamental dynamical objects. This point of view opposes the usual epistemological stand, which begins with the notion of space (of Aristotle, Newton, Minkowski, Riemann, Weyl, etc.) as the basic entity in Nature.

The only way a physicist has of interacting with Nature is by means of measuring processes (observations transmitted first to these senses and from those to the brain). The only manner of an interaction reaching the senses (and thence the brain) is by means of a signal which transfers information from the system to the observer. For this, a physical field is needed, to which a certain energy and momentum densities may be ascribed, and which are the physical agents for the transmission of the signal. Therefore, it is only through the transfer of energy and momentum that a certain knowledge of the World, that is, of natural phenomena, may be obtained; in particular, a certain knowledge of its space-time features. In other words, *the very notion of space-time is strictly dependent on the notion of energy-momentum.* In the very cosmological model most widely accepted nowadays - the big-bang model - the creation (expansion) of space-time is inextricably associated to the total initial energy-momentum

density of the universe. That is, the initial dynamical content is the only determinant on how the geometric structure unfolds.

Thus, let us consider the antisymmetrical bilinear form $dV = dp_\mu dx^\mu$, built up with the covariant momentum four-vector p_μ and the contravariant position four-vector x^μ. The hypervolume dV (physically, the *action*) is constant with respect to a variation of a parameter λ (which may be identified with the cosmological time). The universe's initial conditions were such that for $\lambda=0$, the momentum content was extremely high, whereas the space-time content was extremely low. We have here the most basic and fundamental observation refered above that the covariant vectors characterize the dynamical aspects whereas the contravariant ones characterize the geometrical aspects.

With the aim in mind, then, of trying to determine a certain geometry (i.e., a certain metric) starting from a minimal number of dynamical objects, we begin by postulating the existence of a space-time manifold, the most general possible, with the least number of predetermined geometrical properties. Next, we shall populate the naked manifold with certain dynamical objects, taken as fundamental, trying then to determine what kind of manifold is compatible with these dynamical objects.

We shall take, then, as basic postulates of all our future considerations the two following ones.

I. FUNDAMENTAL DYNAMICAL POSTULATE. The covariant 4-vector momentum p_μ is the fundamental dynamical object.

II. FUNDAMENTAL GEOMETRICAL POSTULATE. The contravariant 4-vector position x^μ is the fundamental geometrical object.

Based on this last one, we further postulate:

III. EXISTENCE OF A DIFFERENTIABLE MANIFOLD. There is a 4-dimensional differentiable manifold, $V_4(x^\mu)$, homogeneous in the (contravariant) space-time coordinates x^μ, which constitute a local system of coordinates (a chart). This parametrization need not cover the whole manifold V_A.

Following our plan, let us start trying to determine the specific nature of the manifold V_4 by means of the incorporation of specific dynamical entities. For this, we notice that *Hamiltonian dynamics* furnishes a natural way of relating dynamics and geometry, which is made possible by the fact that it is built up with the *dynamical* object p_μ and the *geometrical* object x^μ (leading to an even-dimensional manifold with a symplectic structure on it). In the suggestive words of V.I. Arnold[9]: "Hamiltonian mechanics is geometry in phase space." This built in relationship between dynamics and geometry, allowed by the Hamiltonian description, is fundamental for the establishement of the foundations of both classical and quantum mechanics, and recalls to mind the claim made by some people that perhaps the most important lesson of all from Einstein is that *geometry has its own Hamiltonian.*

Before starting, we would like to observe that, in what follows, we take, as Schönberg does[13], *particle* dynamics as the foundations of the world geometry. However, attention should be called to a recent paper by V. Kaplunovsky and M. Weinstein[14] where they formulated a quantum field theory which "abandons as superfluous the notion of the four-dimensional space-time continuum." In their own words, they developed "a framework which allows the treatment of the topology and

dimension of the space-time continuum as dynamically generated." Actually, what they did was to present examples of "quantum systems which start out with a well-defined notion of time but no notion of space, and dynamically undergo a transition to a space-time phase - a phase in which the physics of the low energy degrees of freedom of the system are best described by an effective Lagrangian written in terms of conventional relativistic fields. In this sense, the notion of the four-dimensional space-time continuum as the arena within which the game of field theory is to be played is replaced by the notion of the space-time continuum as an illusion of low-energy dynamics."

We shall analyse separately the cases of relativistics mechanics (both the general and the special theories) and of Newtonian mechanics.

4. RELATIVISTIC MECHANICS AND RELATED GEOMETRIES

Let us introduce into our "naked"(*) four dimensional

(*) The manifold $^{(r)}V_4$ is "naked", *ab initio*, due to the absence of dynamical quantities besides the momentum p_μ (Postulate I), and to the absence of any geometrical structure besides the existence of coordinates x^μ (Postulate II).

differentiable manifold ${}^{(r)}V_4(x^\mu)$ a particle of four-momentum p_μ (the fundamental dynamical object, according to Postulate I), describing a world-line Γ characterized by x^μ (the fundamental geometrical object).

If we now associate to this particle its Hamiltonian state function of general relativity, $H(p_\rho, x^\rho)$, this allows the definition of a contravariant vector p^μ,

$$p^\mu \equiv (1/2)\frac{\partial}{\partial p_\mu} H(p_\rho, x^\rho) \qquad (4.1)$$

to which no dynamical meaning is assigned *a priori*. The Hamiltonian H is the energy function of the particle. Imposing that this energy is given by the usual square of the four-momentum, this automatically endows the manifold ${}^{(r)}V_4$ with an inner product

$$p_\mu p^\mu = p^\mu p_\mu = 2H(p_\rho, x^\rho) \qquad (4.2)$$

This atribution of an inner product to ${}^{(r)}V_4$ is of course equivalent to this manifold being both:

(a) *metric*

$$p^\mu = g^{\mu\nu}(x^\lambda) p_\nu \qquad (4.3)$$

where $g^{\mu\nu}(x^\lambda)$ is the contravariant metric tensor of ${}^{(r)}V_4$, satisfying the orthogonality conditions $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$, and

(b) *Riemannian*

$$2H = g^{\mu\nu} p_\mu p_\nu \qquad (4.4)$$

Moreover, since we imposed that the inner product (4.2) or (4.4) must represent an invariant (the energy scalar function)

of the general relativistic dynamical group, the metric has to be indefinite, with signature of absolute value 2; in other words, the metric of ${}^{(r)}V_4$ has to be *pseudo*-Riemannian.

We conclude, therefore, that resorting to the dynamical momentum p_μ (Dynamical Postulate I), and ascribing to the dynamical function $H(p_\rho, x^\rho)$ the meaning of the particle's Hamiltonian state function of general relativity, it is possible to endow the manifold ${}^{(r)}V_4$ with a pseudo Riemannian metric[13].

It should be pointed out that (more than anything else) we are really dealing here with an epistemological option (Leibniz inspired), which involves an opposite reading to the usual one. Actually, there is nothing remarkable here: we are merely stressing that the fact of the energy being given by the square of the 4-momentum may be taken as automatically leading to the Riemannian character of the manifold.

Next we observe that in (4.1) H was differentiated with respect to its covariant variables p_μ, defining thus a contravariant vector p^μ. Obviously, the energy function may also be partially differentiated with respect to its contravariant variables x^μ, defining then a covariant vector ϕ_μ:

$$\phi_\mu \equiv \frac{\partial H(p_\rho, x^\rho)}{\partial x^\mu} \qquad (4.5)$$

We recall that from the theory of first order partial differential equations there is the Cauchy system of ordinary differential equations for the characteristic lines associated to such equations. In the case of Eq. (4.2), $p_\mu p^\mu = p^\mu p_\mu = 2H(p_\rho, x^\rho)$, the Cauchy system is:

$$\frac{dx^\mu}{p^\mu} = -\frac{dp_\mu}{\phi_\mu} \equiv d\sigma \qquad (4.6)$$

with p^μ and ϕ_μ denoting (as already said) partial derivatives of H, and $d\sigma$ denoting the common value of the eight ratios above. The infinitesimal proper time ds, corresponding to the infinitesimal displacement dx^μ on the world-lines of the particle, is defined as $ds^2 = g_{\mu\nu}(x^\rho)dx^\mu dx^\nu$, which, according to the Cauchy system (4.6) may be written as:

$$ds^2 = g_{\mu\nu}p^\mu p^\nu d\sigma^2 \tag{4.7}$$

or, by (4.4), as:

$$ds^2 = 2H\, d\sigma^2 \tag{4.8}$$

Hence, it follows from (4.6) and (4.8) that at any point x^μ of a word-line Γ of the particle, the vector p^μ is tangent to that world-line: $p^\mu = dx^\mu/d\sigma = (dx^\mu/ds)(1/\sqrt{2H})$.

Special Relativity. In general relativity, a particle moving in a given gravitational field is always an inertial system, that is, it always follows a geodesic of the geometry related to that gravitational field. In that framework, there are no external "potentials" acting on the particle (the only "potentials" present are the metric coefficients $g_{\mu\nu}(x)$ which determine the geometry in question). The picture is quite different in special relativity, though. There, we can talk of forces and of potentials, and, therefore, we can interpret the 4-vector ϕ_μ, defined in (4.5), as an *external potential function* to which a particle may be submitted. In special relativity, then, a particle in inertial motion may be characterized as one for which ϕ_μ=0 over all the particle's world-line Γ. This corresponds to having H independent of x^μ

over Γ, that is, to having H=constant=E over $\Gamma^{(*)}$. Since, from (4.4) and (4.5), we have that

$$\phi_\mu = \frac{\partial}{\partial x^\mu}\left\{\frac{1}{2}\, g^{\rho\sigma}p_\rho p_\sigma\right\} = \frac{1}{2}\,\frac{\partial g^{\rho\sigma}}{\partial x^\mu}\, p_\rho p_\sigma \tag{4.10}$$

we see from this equation that if ϕ_μ=0 over the particle's world-line Γ, then we must have $\partial g^{\rho\sigma}/\partial x^\mu=0$, that is, $g^{\mu\nu}$= constant over Γ. Since this world-line is arbitrary, this means that $g^{\mu\nu}$ must be constant over the manifold ${}^{(r)}V_4$. In summary, characterizing a particle in inertial motion by the condition ϕ_μ=0 over any of its world-lines, we conclude that the geometry of ${}^{(r)}V_4$ is flat with signature of absolute value 2. On the other hand, if we had admitted in our flat manifold that 2H = $= g^{\mu\nu}p_\mu p_\nu$ had a *positive definite* metric, it can be easily shown[4,15] that this is equivalent to admit that there is no upper bound to the velocity: an infinite value for the speed of particles would be physically realizable. This, in turn, is equivalent to admit that the space-like and time-like components of the four-momentum are entirely interchangeable, a possibility which is completely foreign to our experience. We must, therefore, impose the dynamical principle that there is a limiting velocity for the propagation of physical signals.

5. NEWTONIAN MECHANICS AND RELATED GEOMETRY

Here, in the Newtonian case, we shall take ${}^{(n)}V_4$ as our differentiable four-dimensional space-time manifold. According to Postulate I, let us introduce again into this

(*) This implies that we can define the energy E over all the manifold, which, in turn is equivalent to stating that we can now build, in special relativity, a *global* inertial frame over all of ${}^{(r)}V_4$.

manifold a particle of three-momentum p_i, $i=1,2,3$, which will once more determine the geometrical features of the manifold.

Let our particle be moving with three-velocity defined by $\dot{x}^i \equiv dx^i/dx^4$, where x^i are the space variables and x^4 is the time variable. We next associate to this moving particle what Poincaré called its *mass of Maupertuis*[16], m. Hence, here in Newtonian physics, we take as one of the particle's accessory essential dynamical characteristics (besides the basic property of possessing three-momentum p_i), not its intertial mass, but the mass associated to its state of motion. With these definitions of mass (of Maupertuis) and three-velocity we can define the *contravariant* three-vector

$$p^i \equiv m\dot{x}^i \tag{5.1}$$

which, as before in the relativistic case, cannot, *a priori*, be related with the covariant fundamental dynamical three-momentum p_i. This identification of $p^i = m\dot{x}^i$ with p_i (which, for instance, enables us the identifiaation of the time derivative of (5.1) - as it is usually done - with the Newtonian concept of force) is possible if the three-dimensional spatial hypersurface ${}^{(n)}V$ of the entire space-time manifold ${}^{(n)}V_3$ is *metric*. That is,

$$p^i = g^{ij}(x^k,x^4)p_j \tag{5.2}$$

where $g^{ij}(x^k,x^4)$ is the contravariant metric tensor of ${}^{(n)}V_3$, satisfying the orthogonality relation $g^{ij}\, g_{jk} = \delta^i_k$. That is, the identification of the three-momentum p_i with the three contravariant vector $p^i \equiv m\dot{x}^i$ obviously does not make the entire four-dimensional space-time ${}^{(n)}V_4$ a metric manifold[2],

but only its three-space hypersurface ${}^{(n)}V_3$. In this three-space metric manifold we can then define an inner product and hence the bilinear symmetric form

$$(2m)^{-1}p^ip_i = (2m)^{-1}p_ip^i = (2m)^{-1}g^{ij}p_ip_j \tag{5.3}$$

This implies that the metric of ${}^{(n)}V_3$ is symmetric, $g_{ij} = g_{ji}$.

Introducing then the *energy* concept into our three-dimensional spatial manifold ${}^{(n)}V_3$ and imposing that the particle's kynetic energy T be given by:

$$T \equiv (2m)^{-1}p^2 = (2m)^{-1}g^{ij}p_ip_j \tag{5.4}$$

We define a free particle as one having for state function $H(p_i,x^i,x^4)$ its kynetic energy, which obeys the dynamical equations of motion

$$-\dot{p}_i = \frac{\partial T}{\partial x^i} \tag{5.5a}$$

$$\dot{x}^i = \frac{\partial T}{\partial p_i} \tag{5.5b}$$

From (5.4) and (5.5a) we see tht $\dot{p}_i = 0$, that is, the three-momentum of our free particle is a constant of motion. Hence:

$$-\dot{p}_i = \frac{\partial T}{\partial x^i} = \frac{1}{2m}\frac{\partial}{\partial x^i}(g^{jk}p_jp_k) =$$

$$= \frac{1}{2m}(\partial g^{jk}/\partial x^i)p_jp_k = 0 \quad ,$$

that is,

$$\partial g^{jk}/\partial x^i = 0 \tag{5.6}$$

Moreover, since in Newtonian physics, $\dot{p}^i = m\ddot{x}^i$ is identified as the force felt by the particle, we must have $\dot{x}^i$ = constant. Therefore, from (5.5b)

$$\dot{x}^i = \frac{\partial T}{\partial p_i} = \frac{1}{2m}\frac{\partial}{\partial p_i}(g^{jk}p_j p_k) = \frac{g^{jk}}{2m}(\delta^i_j p_k + p_j \delta^i_k) =$$

$$= \frac{1}{2m}(g^{ik}p_k + g^{ji}p_j) = \frac{1}{m}g^{ij}p_j = \text{const.}$$

and since p_j is a constant of motion, g^{ij} must be time independent:

$$\frac{\partial}{\partial x^4} g^{ij} = 0 \qquad (5.7)$$

From (5.6) and (5.7) we conclude that the metric of the three-dimensional spatial hypersurface ${}^{(n)}V_3$ is flat:

$$g^{ij} = \eta^{ij} = \text{const.} \qquad (5.8)$$

We see, therefore, that endowing our particle with the concept of mass (of Maupertuis) we are able to introduce the three contravariant vector $p^i = m\dot{x}^i$, which can be associated to the dynamical covariant three-momentum p_i only if the three-spatial manifold ${}^{(n)}V_3$ is metric, Eq.(5.2). We can then build the symmetric bilinear form $T = (2m)^{-1}p^i p_i = (2m)^{-1}g^{ij}p_i p_j$. Imposing that this dynamical function is the free particle's Hamiltonian state function, we were able to reach the conclusion that the spatial part of our four-dimensional manifold is flat.

On the other hand, in the basic dynamical equations which we considered - Hamilton equations of motion (5.5) - the time coordinate x^4 plays the role of an independent parameter with respect to the space coordinates x^i. This means that the time axis has to be orthogonal to the three dimensional spatial manifold[3].

Contrary to the relativistic case, in which for the determination of the related geometry we resorted to only *one* auxiliary dynamical function - the Hamiltonian state function $H(p_\mu, x^\mu)$ - here in the Newtonian case, we needed to introduce

separately the *two* concepts of mass and energy. With the aid of the former, we defined the contravariant quantity $p^i = m\dot{x}^i$, whilst, with the help of the latter, we wrote down the particle's kynetic energy (its Hamiltonian state function).

6. CONCLUSIONS

Contrary to the costumary way of doing physics, we were presently able to show that starting from a few given dynamical quantities we can arrive at certain specific geometries. Thus, general relativistic physics implies general Riemannian geometry (Einstein space-time), while the physics of the special theory of relativity is tied up with a flat Riemann manifold (Minkowski space-time). Finally, Newtonian particle dynamics is bound to Newtonian space-time.

What this clearly seems to indicate is that the connection between physics and geometry is even more profound than is commonly considered. By this, we mean that not only particle dynamics and certain space-times are closely interconnected, as stated above, but, also more important, that maybe the point of view taken here is perhaps the most fundamental. Namely, that instead of departing from a given postulated space-time and then infer the associated particle dynamics, we should start by postulating a certain physics and then try to determine its related geometry. In other words: geometry should be considered as an aspect of dynamics[14]. This point of view reminds us of Leibniz conception of dynamics[17].

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