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HOMOTOPY AND STATISTICS IN QUANTUM THEORY

by

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HOMOTOPY AND STATISTICS IN QUANTUM THEORY

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The description of a system of N particles is given in terms of the space of motions U introduced by Souriau. The corresponding quantizable manifold U' is introduced and its homotopy group $\pi_1(U')$ is exhibited. Some general ideas on the geometrical quantization of U' are presented and the relations of this quantization procedure with the characters of $\pi_1(U')$ are discussed.

1. INTRODUCTION

The usual form of Quantum Mechanics expresses everything in terms of three propositions: a) a state is a vector ψ in a Hilbert space; b) an observable is a selfadjoint operator A in this space; c) the connection between a) and b) being that the expectation value of an operator A in the state ψ is $(A\psi,\psi)$[1]. The dynamical aspect of the theory is brought to light by the requirement that the equations of motion shall be expressible in the Hamiltonian form. This is necessary for a transition from classical to quantum theory to be feasible.

It is possible[2] to give a complete description of Classical Mechanics in terms of symplectic geometry and Hamiltonian functions. For a dynamical system with $\underline{n}$ degrees of freedom we define the corresponding configuration space as an arbitrary differentiable manifold M of dimension $\underline{n}$. The cotangent bundle $T^* M$, of M, admits a natural symplectic structure defined by the

This work is dedicated to Prof. Mario Schoenberg on his 70th birthday. From him we have learned the most important and beautiful lesson of all: "Before being a scientist, be a man".

exact Liouville two-form. A dynamical state of the system, in the Hamiltonian formalism, is a point of $\mathcal{M} = T^* M$, which, in the usual language, is the classical phase space. But this choice of phase space is not satisfactory since, in several distinct problems, it is essential to introduce changes of the classical variables which do not respect the contangent structure of $\mathcal{M}$. With this in mind, several authors[3-4] have enlarged the definition of phase space, in order to introduce deformations of the Poisson-Lie algebras, determining interesting formal algebras. This approach, which could lead to a new description of the usual quantum mechanics, will not be followed here.

Our task in this work is to study how the decomposition of the representation Hilbert space for the symmetric group $S^{(N)}$ of N elements is related to the homotopy group of the "configuration background" of N particles. In order to focus attention on the group-theoretic essential aspects of $S^{(N)}$, unemcumbered with inconvenient observer-dependent classical phase space descriptions, we go back to the pioneering work of Souriau[5]. Probably, as has been recently emphasized[6], Souriau's method of identifying classical dynamical systems, which replaces the usual phase space by the global space of motions, perhaps marks the beginning of the breakthrough in specifying classical mechanics. Thus, for a system consisting of N independent elementary particles of mass $\underline{m}$, we consider the "configuration background" as being the space of motions U of the particles. This space is a differentiable manifold defined as the quotient $U = V/ker\ (d\tilde{\omega})$ of the evolution space V by the characteristic foliation of the presymplectic structure $d\tilde{\omega}$. The 2-form $d\tilde{\omega}$ descends to U giving rise to the symplectic structure of the space of motions U. The symplectic manifold U is homeomorphic to a sphere since, non-relativistically, for N equal mass particles the space of motions has an $\underline{a}$

priori equal population on such a sphere. This means that U is simply connected. If it is defined as the covering space of a manifold U' to be defined later as a quantizable differentiable manifold, the homotopy group $\pi_1(U')$ of U' will be identical to the permutation group $S^{(N)}$ of the definition of the covering.

It has been proposed by Souriau that the connected manifold U' has as many non-equivalent quantizations as the distinct characters of its homotopy group. Now, due to the indistinguishability of the particles, our space of motions U' which describes the system, admits a covering $(U,S^{(N)},F)$, where U, as defined above, is an open set in the quotient manifold $U_1 \times U_2 \times \ldots \times U_N = [U_1]^N/S^{(N)}$, $S^{(N)}$ being the permutation group of N elements and U_1 the space of motions of one particle. F is a differentiable mapping of U into U'.

In a preceding paper[7] we have given a representation of $S^{(N)}$ in a Hilbert space. We have also established a connection between the multidimensional irreducible subspaces and the generalized quantum statistics of Gentile[8]. For the case $N = 3$ we have shown explicitely the representative matrices of $S^{(3)}$, constructing multidimensional state vectors Y. Besides the usual completely symmetric state vector Y_S (bosons) and completely antisymmetric state vector Y_A (fermions) we have written the hybrid Y (four components), named state vector for gentileons, which could be interpreted as a new quantization for the system of 3 particles. These results can be easily generalized for N particles, giving rise to several new quantizations compatible with the postulates of quantum mechanics.

In this work, pursuing the above line of reasoning a little further, we might wish to construct the homotopy group $\pi_1(U')$, showing for the non trivial case $N = 3$ that we have 3 distinct characters χ_+, χ_- and χ. This problem is typical of those which

are extensively studied in the classical theory of homotopy; and the methods of the latter lead to the identification of the characters of $\pi_1(U')$ and $S^{(N)}$ and, consequently, to new quantizations. We do not give a complete, rigorous formulation of the problem of geometrical quantization, but indicate the direction for the construction of such a formulation in the case of General Statistics. A discussion of these results, some topological conclusions and contrasts are also given.

2. THE SPACE OF MOTIONS

In the description of a dynamical system, there are several possible choices of the representative spaces. Thus, for Newtonian mechanics, the space time background of phenomena itself is used to specify the system. Rigorously speaking, the classical space time is a fiber bundle, the time being defined as the base space. At a time t, the corresponding fiber is the instantaneous configuration space R^3, spanned by vectors $\vec{r}$. The structural group is the Euclidean group of isometries (displacements). When we pass to the Hamiltonian formulation of Classical Mechanics, the appropriate geometric background is defined as the phase space ϵ of the system. The classical phase space of a mechanical system has a natural symplectic structure. It is defined as the cotangent bundle $T^* M$ of a differentiable manifold M of dimension n. But, as has been stressed by Souriau[5], the configuration space and the phase space are not compatible with Galilei (or Lorentz) transformations of space time, in the sense that we can not give a precise definition of them which is referential independent. In order to amend this situation we introduce the time variable t as an additional coordinate axis, building the so called evolution space V of the system. In that manner, the $(6n + 1)$ vector

$$y = \begin{bmatrix} \vec{r}_1 \\ \vec{v}_1 \\ \cdot \\ \cdot \\ \cdot \\ t \end{bmatrix} \qquad (2.1)$$

will represent a point of V. Of course, V is a $(6n + 1)$ dimensional differentiable manifold. We define the phase space at a instant t_o as being the submanifold $V' \subset V$ defined by the equation $t = t_o$. It is easy to see that the mapping $[y] \to t$ is differentiable, characterizing the phase space as an imbedded manifold[2] of V.

But, if we wish to construct a canonical formalism for the relativistic (Galilean or Lorentzian) mechanics of many particles, it is not the evolution space V, but its quotient $U = V/ker\,(d\tilde{\omega})$ by the characteristic foliation of the presymplectic structure $d\tilde{\omega}$ which must be adopted for the description of the system[5]. The 2-form $d\tilde{\omega}$ descends to U and thus gives rise to the symplectic structure of the space of motions (U,σ), with $\sigma = d\tilde{\omega}$. This symplectification of a contact manifold by means of the characteristic foliation is a standard setting[2]. The integral leaves of the foliation project upon curves of the space time, the worldlines corresponding to the possible motions. With the purpose of studying the possible quantizations of a system of N identical particles, let us specify the symplectic manifold U. Of course, U is an open set of $U_1 \times U_2 \times \ldots \times U_N = [U_1]^N$. But due to the identify of the particles, the "true" space of motions is not $[U_1]^N$ but the quotient $[U_1]^N/S^{(N)}$ by the symmetric group $S^{(N)}$. Naturally, the symmetric group defines an equivalence relation:

if a typical vector of $[U_1]^N$ is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} \qquad (2.2)$$

we call $\hat{x}$ the equivalence class

$$[\hat{x} = \hat{x}'] \Longleftrightarrow \left[\text{there is a permutation } \hat{\gamma} \in S^{(N)} \text{ so that } x' = \hat{\gamma}(x)\right]$$

This characterizes the manifold of motions U of a system of N identical particles as the set of the $\hat{x}$, x being composed of non equals $x_1, x_2, \ldots, x_N$. Naturally, $dim\ U = dim\ [U_1]^N = N\ dim\ U_1$. If an internal degree of freedom is to be included for each particle, it is sufficient to enlarge the phase space by defining the direct product of the usual one with an appropriate coadjoint orbit[6].

3. THE HOMOTOPY GROUP

In section 2 we have defined the space of motions U of a system of N identical particles as the partition of $[U_1]^N$ by the permutation group $S^{(N)}$. Formally[9], this partition of $[U_1]^N$ is a disjoint family $\mathcal{D}$ of subsets of $[U_1]^N$ whose union is $[U_1]^N$. This defines an equivalence relation R[7] of the partition $\mathcal{D}$ which is the subset $[U_1]^N \times [U_1]^N$ consisting of all pairs (x_1, x_2) such that x_1 and x_2 belong to the same member of $\mathcal{D}$, or, briefly, $R = \cup\{D \times D : D \in \mathcal{D}\}$. If $\Gamma = \{\gamma_i\}_{i=1,2,\ldots,N}$ is the projection of $[U_1]^N$ into $\mathcal{D}$, then $R = \{(x_1, x_2) : \Gamma(x_1) = \Gamma(x_2)\}$.

Quantum mechanically, due to the indistinguishability, a system of N non-interacting particles must be described by a new space of motions U'. It is easy to see[5] that the triplet $(U, S^{(N)}, F)$, where F is a differentiable mapping of U into U', constitutes a covering of U'. Of course, U' is an orbit of U under the action of $S^{(N)}$. The mapping F, in a certain manner, introduces the indistinguishability in U'. $(U, S^{(N)}, F)$ is a universal covering of U' since it can be proved that U, besides being quantizable[5] is simply connected[9]. Thus, the homotopy group $\pi_1(U')$ coincides with $S^{(N)}$.

In the case $N = 3$, let us denote[13] the representative matrices of $S^{(3)}$ by

$$m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad m_3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$m_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad m_5 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad m_6 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \tag{3.1}$$

It is well known[11] that in the symmetric group $S^{(3)}$ there is the normal subgroup $A^{(3)}$ of index 2 consisting of the even permutations. Also, there exists an element $\underline{a}$ of order 2 in the coset $S^{(3)} - A^{(3)}$: $a^2 = 1$. This is obvious since all transpositions remain in $S^{(3)} - A^{(3)}$. In our notation, $A^{(3)} = \{m_1, m_2, m_3\}$, $a = m_4 = \sigma_3$, where σ_3 is a Pauli matrix and the whole group can be written as

$$S^{(3)} = A^{(3)} \otimes \Lambda \tag{3.2}$$

where $\Lambda = \{I, \sigma_3\}$. Of course, Λ is responsible for the transpositions and thus, $S^{(3)}$ is obtained from $A^{(3)}$ by adding the improper matrices ($det = -1$). From the topological point of view, $S^{(3)}$ decomposes into two pieces since the determinant changes continuously along a path in $S^{(3)}$; no path leads from $A^{(3)}$ to the region $S^{(3)} - A^{(3)}$ which contains the improper elements.

The usual approach to the problem of representing the symmetric group[12] consists in considering the factor group $S^{(3)}/A^{(3)}$ as isomorphic with $S^{(2)}$. Then, there exists an irreducible representation of $S^{(3)}$ which is induced by the single faithful irreducible representation of $S^{(2)}$. This alternating representation assigns to the identity the number 1 and to the transposition the number -1. Of course, this procedure, which is equivalent to taking the representation of $S^{(3)}$ upon the multiplicative group of complex numbers of moduli 1, can be associated to the representation used in our earlier work[13]. With the alternating representation we can construct a representation associated to the primitive one. In fact, since the characters of $S^{(3)}$ are, 2, -1 and 0, the two representations are equivalent. In this work we are not concerned with the alternating representation since it is weeded out by taking into account the physical relevance of the intermediate irreducible four dimensional subspace of U'. We have defined[13] the gentilionic state-vectors as belonging to this submanifold of U' and in this paper we consider the paths in this connected submanifold.

We now consider the product space $A^{(3)} \otimes \Lambda$. Since the homotopy group of the product space $A^{(3)} \otimes \Lambda$ is the direct sum of the homotopy groups of $A^{(3)}$ and Λ[14], our next task is to identify the corresponding elements of the two groups. First of all we note that topologically $A^{(3)}$ and the coset $S^{(3)} - A^{(3)}$ plus the identity are homeomorphic. Due to this fact and knowing that the homotopy group of $A^{(3)}$ is given by the map of the circle $Y_1^2 + Y_2^2 = 1$[14], where Y_1 and Y_2 as components of the gentilionic state-vectors are defined previously[13], the matrices m_2 and m_3 defined in (3.1) are taken as generators of the homotopy group $\pi_1(A^{(3)})$ because the map can be written in the gentilionic subspace as

$$Y' \to Y : \begin{Vmatrix} m_{2,3} & 0 \\ 0 & I_2 \end{Vmatrix} \equiv \| G \| \qquad (3.3)$$

where I_2 is the 2-rowed identity matrix. This is possible since there exists a transformation matrix B which gives the desired result

$$B^{-1} \; V \; B = G \qquad (3.4)$$

where V is any 4×4 representative matrix of $S^{(3)}$ [13]. The same occurs with Dirac equations which may be treated by decomposing the four equations into two sets of two equations and treating the invariance properties of these equations by means of two-component spinors. In so doing we restrict the spin transformations of the four-component spinors to those involving a certain pairing of the components. The Dirac equations, however, have invariance properties under the larger group of linear homogeneous transformations (projective group) which underlies the four-component theory.

In that manner, the homotopy group of U' can be written as

$$\pi_1 = \pi_1(A^{(3)}) \oplus \pi_1(S^{(3)} - A^{(3)}) = Z_2 \oplus Z_2 \qquad (3.5)$$

where Z_2 is the cyclic group of order 2. This is not a surprising result for two reasons: the first one is the presence of Λ in the decomposition of $S^{(3)}$ and the second one is the fact that U' being a non-orientable manifold admits a double covering manifold[15].

This very important result on the homotopy group of U' can be understood as being one of the topological origins of the existence of two kinds of statistics in Nature. We see that the state-vectors belonging to the gentilionic sub-manifold must, from the statistical point of view, present characteristics falling in

one of the two usual statistics.

In our preceding paper[13], we have suggested that the constituting quarks of a baryon should obey Gentile statistics. This hypothesis is not in conflict with our present results since we know that a baryon seems to obey Fermi statistics. We shall return to this point latter.

4. GEOMETRICAL QUANTIZATION

All results presented in sec. 3 sound deceptively simple enough. Nevertheless, our formulation of the representation of $S^{(3)}$ and the corresponding homotopy group will be capable of covering the whole problem of geometrical quantization developed by Souriau[5]. Let us consider the action of the torus T^2 on the manifold U [16]. T^2 is a compact Lie group of dimension 2 whose Lie algebra is the algebra of quaternions (the Clifford algebra C_2). The compactness of T^2 assures the existence of a unitary representation of $S^{(3)}$. In our problem this can be easily seen if we identify the set of representative matrices (3.1) with the operator equation

$$R(\phi,\psi) = R(\phi)\, R(\psi) = exp\left[i\, (\psi/2 + 3\, \phi/2) \right] . \; exp\left[i\, \vec{\sigma} . (\vec{n}\, \phi/2 + \vec{m}\, \psi/2) \right] \qquad (4.1)$$

where $\vec{n}$ and $\vec{m}$ are unit vectors in quaternion space, $\vec{\sigma}$ are the Pauli matrices and ϕ and ψ are the angular variables of $T^2 = S^1 \times S^1$. The first exponential is a convenient numerical phase. This corresponds to taking the representation of $S^{(3)}$ on the torus T^2 and to defining the three characters of the representation. Hence a quantization (W, P^*) of U, where W is a presymplectic manifold and P^* is a map such that $P^* : \alpha \to \hat{\alpha}$ with $\alpha \in Quant\ (W)$ (group of quantomorphisms of W) and $\hat{\alpha} \in Can\ U$

(group of symplectomorphisms of U) can be defined. Also, there exists a homomorphism K of *Can U* into *ker* (P^*)

$$K = \underline{\chi}_W \tag{4.2}$$

where χ is a character of *Can U*. This means the that isomorphic liftings of *Can U* are taken into one-to-one correspondence with the characters of *Can U*. Now, if U is monoquantizable[5], all symplectomorphisms of U can be lifted and we will be able to define the quantizations (W, P^*) of U with the quantomorphisms $\alpha \in W$ being the liftings of the elements of *Can U*.

One knows that the space of motions of N identical particles, being a direct product of N simply connected manifolds, is simply connected. Thus, U is monoquantizable and has a quantization (W, P^*), the group *Can U* having an isomorphic lifting *Quant* (W) on W. Naturally, *Quant* (W) is a discrete group of W inducing a quantization (W', P'^*) on U', where W' is the quotient manifold $W/$*Quant* (W) and where the orbits are defined as

$$\begin{aligned} \xi' &\equiv \text{orbit of } \xi \text{ under } Quant\ (W) \\ \overline{\omega}_{y'}(\delta\xi') &\equiv \overline{\omega}_y(\delta\xi) \\ P'(\xi') &\equiv F(x) \end{aligned} \tag{4.3}$$

$F(x)$ being defined by the covering $(U, Can\ U, F)$ of U'.

On the background of these general notions concerning liftings, we now state the following result due to Souriau[5]: a quantizable connected manifold U' has the same number of non-equivalent quantizations as there are distinct characters of its homotopy group π_1.

This statement, in the case $N = 3$, suggests the possibility of assigning to U' three well defined quantizations since, according to §3, the homotopy group π_1 has three characters.

The problem of quantization of U is not completely solved yet. The partition $S^{(3)}$ induces in the manifold U' a structure consisting of three irreducible manifolds. At first sight it might appear that three distinct quantizations could be defined. Nevertheless, the presence of Λ in the decomposition of $S^{(3)}$ has led us to the homotopy group Z_2 for the gentilionic submanifold of U'. Thus, some care must be taken on the definition of the quantizations. That the liftings exist and have the property of uniqueness, is assured by the global properties of U[16]. In that manner, we can define the liftings of the homotopy classes of U' and the orbits of T^2 acting on W. This permits to extend the geometrical reasoning developed by Souriau[5] when constructing the wave equations and the wave functions.

No other particle besides Fermions and Bosons has been detected in experiments. If gentilionic particles exist, for which the Exclusion Principle is not obeyed, they probably can not be detected. This astonishing confinement could be atributed, in our scheme, to the presence of Λ in the decomposition of $S^{(3)}$. The treatment of the case of N particles is essentially the same as for the case $N = 3$. Some further implications of the multidimensionality of the state-vectors will be discussed in the next section.

5. CONCLUSIONS AND COMMENTS

In this work we have outlined a set of results on the homotopy group of a quantizable manifold U' representing indistinguishable particles in the case $N = 3$. We have reached the conclusion that the homotopy group consists of three elements (one for each equivalence class). We have solved explicitly the $N = 3$ problem since, like the homology groups, the homotopy groups are topological invariants of a manifold; unlike the homology groups, however, no general method for computing them is known. Our explicit calculation of the representative matrices of $S^{(3)}$ has permitted us to draw some conclusions[13] on the peculiar nature of the statistical behavior of the 3-particle system.

The topological properties of the symmetries generated by $S^{(3)}$ on the system of 3 particles are clearly exhibited by the torus defined by (4.1). To obtain the matrices corresponding to $A^{(3)}$ we need to rotate with angles $\phi = 2\pi/3$ and $4\pi/3$ around the principal axis perpendicular to the plane containing the great parallel. The matrices representing the coset $S^{(3)} - A^{(3)}$ are obtained by rotating an angle $\psi = \pi$ on the meridians of the torus. Thus, the gauge space associated with the torus has three angular quantization numbers, $2\pi/3$, $4\pi/3$ and π which label the inequivalent states of the system.

According to our previous work[13], there are substantial differences between gentileons on one hand and bosons and fermions on the other hand. In this paper we have stressed some topological consequences of taking into account the multi-dimensional manifolds associated with the representation of $S^{(N)}$. We see that there are also very important topological distinctions between gentileons and usual Bosons and Fermions. It may be that the most important is the topological confinement of the former. It is worthwhile to note that we have not introduced specific physical models besides the three propositions enounced in the Introduction. For this reason we will not be able to write a built-in relation between some physical observables, such as the spin and the iso-spin and the characters of the homotopy group. We believe that this relationship could also be furnished by the topology. This will be our future aim.

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