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by

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ABSTRACT

We present a simple argument which shows that the dynamical mass induced by interactions of massless quarks with pseudo-particle configurations, behaves like p^{-6} for asymptotically large quark momenta.

The idea that instanton dynamics may be responsible for chiral symmetry breaking was first put forward by Callan, Dashen and Gross[1]. Subsequently, quantitative calculations by Caldi[2] indicated that interactions of massless quarks with instanton gauge field configurations were capable of generating dynamical quark masses. A detailed investigation was carried out by Carlitz and Creamer[3] for quarks in the fundamental representation, and then by Carneiro and McDougall[4] for the adjoint representation, who found non-trivial solutions of the equation which generates the dynamical quark masses. These investigations, done in the dilute gas approximation, found that chiral symmetry breaking scales for these representations were rather similar, the dynamical masses behaving in both cases for asymptotically large momenta p, like p^{-6}. However these results emerged only in consequence of very long and involved calculations, and it was not clear whether this behaviour was accidental or not.

In this letter, we present a very simple argument, valid for any representation of the quark system, which shows that the dynamical masses induced by the instantons must necessarily behave like p^{-6} for large quark momenta. To this end, we consider a multiplet of massless fermion fields ψ_u, $1<u<2T+1$, having isospin T, coupled to $SU(2)$ gauge fields $A_{\rho c}$ through the interaction:

$$L_I(x) = i\,\overline{\Psi}(x)\gamma_\rho D_\rho \Psi(x) \tag{1a}$$

where D_ρ denotes the covariant derivative:

$$(D_\rho)_{uv} = \partial_\rho\, \delta_{uv} - i A_{\rho c}(T_c)_{uv} \quad . \tag{1b}$$

To compute the fermion induced masses we examine the propagator $S(x-y)$ defined by the Euclidean functional integral:

$$S(x-y) = N^{-1} \int \mathcal{D}A_\mu \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \; \Psi(x)\, \overline{\Psi}(y) \; \exp \int d^4x\,(L_o + L_I) \qquad (2a)$$

Here N denotes gluon and quark determinants chosen so that the contribution from the perturbative sector is normalized to unity. The Lagrange function L_o includes contributions from vector and ghost fields. The contribution of a given field configuration $A_\rho(x)$ to this functional can be expressed in terms of the eigenfunctions ψ^i of the operator $\gamma . D$: $- i\,\gamma.D\, \psi^i(x) = \lambda_i\, \psi^i(x)$, with eigenvalues λ_i . The integral over the fermionic fields in (2.a) will be then determined essentially by the eigenfunctions with zero eigenvalue through the expression[(5)]:

$$S(x-y) = N^{-1} \int \mathcal{D}A_\mu \sum_{i=1}^{C(T)} \psi_u^i(x)\, \psi_u^{i\,\dagger}(y) \; \times$$

$$\times \prod_{\substack{j=1\\ j\neq i}}^{C(T)} m_{jj} \prod_{k\neq i,j} \lambda_k \qquad (2b)$$

[where sum over the isotopic index $1 < u < 2T+1$ is to be understood]. Here $C(T)$ determines the number of zero modes of a fermion with isospin T in the field of a pseudoparticle: $C(T) = \frac{2}{3}\, T(T+1)(2T+1)$. Choosing a basis which diagonalizes the dynamical mass m induced by the instantons, m_{jj} is given by:

$$m_{jj} = \int d^4x\; d^4y\; \psi_u^{j\,\dagger}(x)\, m(x-y)\, \psi_u^j(y) \quad . \qquad (2c)$$

Next we consider the integral over the gluon fields in (2b), using the semiclassical approximation. Since the zero eigenmodes of the Dirac equation have the form: $\psi_u^i(x) = \psi_u^i(x-z,\rho)$, where ρ denotes the pseudoparticle size centered around z , one obtains an integration over z which has the form of a convolution. Consequently the pseudoparticle contribution to the momentum-space propagator $\tilde{S}(p)$ will have the form:

$$\tilde{S}(p) = \int d\rho \left[\sum_{i=1}^{C(T)} \tilde{\psi}_u^i(p,\rho) \; \tilde{\psi}_u^{i\dagger}(p,\rho) \right] \prod_{\substack{j=1 \\ j\neq i}}^{C(T)} m_{jj} \; F(\rho) \; . \tag{3}$$

In this expression $F(\rho)$ is proportional to the density of the instantons and has the important property that it vanishes as ρ goes to zero. It may be shown, in the dilute gas approximation, that the contribution to the induced mass is effectively determined by the single instanton and anti-instanton configurations, via the relation:

$$m(p^2) \cong p^2 \left[\tilde{S}^+(p) + \tilde{S}^-(p) \right] \; . \tag{4}$$

Comparing equations (3) and (4) we see the one obtains essentially a complicated integral equation. However, it is crucial to realize that the momentum dependence is determined solely by the expression in brackets in equation (3). We will now show that $\tilde{\psi}(p)$ behaves asymptotically like p^{-4} , yielding with the help of (3) and (4) a behaviour proportional to p^{-6} for the induced dynamical mass.

To this end we consider the solution of the Dirac equation:

$$\gamma_\mu (\partial_\mu - i\, A_{\mu a}\, T_a)\psi = 0 \tag{5a}$$

in the background of an instanton or anti-instanton given by:

$$A^{\pm}_{\mu a} = -\,\eta^{\mp}_{\mu\nu a}\, \partial_\nu\, \ell n \left[1 + \frac{\rho^2}{(x-z)^2} \right] \tag{5b}$$

where $\eta^{\mp}_{\mu\nu a}$ are t'Hooft[6] antisymmetric symbols: $\eta^{\mp}_{k\ell a} = \epsilon_{k\ell a}$, $\eta^{\mp}_{k4a} = \mp\,\delta_{ka}$. Defining $\psi \equiv \begin{pmatrix} \psi^R \\ \psi^L \end{pmatrix}$, we can rewrite (5), in terms of the 2×2 matrices $\alpha_\nu = (-i\vec{\sigma},1)$ as follows:

$$\alpha^\dagger_\mu (\partial_\mu - i\, A_{\mu c}\, T_c)\psi^L = 0 \tag{6a}$$

$$\alpha_\mu (\partial_\mu - i\, A_{\mu c}\, T_c)\psi^R = 0 \quad . \tag{6b}$$

As shown by Jackiw and Rebbi[7], in order to obtain normalizable eigenfunctions we must have $\psi^L = 0$ for self-dual fields, and $\psi^R = 0$ for antiself-dual configurations. We will concentrate from now on, for definitness, on the solutions of equation (6.b) corresponding to instanton fields.

Since we are interested in determining the behavior of $\tilde{\psi}(p)$ at asymptotic momenta, this means that we must look in configuration space at the behaviour of ψ^R for values of x such that $|x-z|^2 \ll \rho^2$. [Although instantons of all sizes ρ may occur, it is important to recall that $F(\rho\to 0) \to 0$ in equation (3) so the above inequality is consistent]. Because of translational invariance, we can choose the center of the instanton at the origin and then, from (5b) and (6b) we obtain:

$$\sqrt{x^2}\ (\alpha.\partial)_{AA'}\ \psi^{A'}_{u} + 2\left(\sigma_a \frac{x.\alpha}{\sqrt{x^2}}\right)_{AA'} (Ta)_{uu'}\ \psi^{A'}_{u'} = 0 \qquad (7a)$$

where A is the index describing the two spacial components of ψ^R.

It is important to realize that this equation is invariant under scale transformations $x \to \lambda x$ and consequently its solution will be a function of zero degree in x. Introducing a dimensionless parameter $y_\nu \equiv x_\nu / \sqrt{x^2}$, we can rewrite (7a) in the following convenient form:

$$\left[\left(\alpha\cdot\frac{\partial}{\partial y}\right)_{AA'} - (\alpha.y)_{AA'}\ y\cdot\frac{\partial}{\partial y}\right]\psi^{A'}_{u} + 2(\sigma_a\ \alpha.y)_{AA'}\ (T_a)_{uu'}\ \psi^{A'}_{u'} = 0\ . \qquad (7b)$$

We will next show that the solutions of this equation, for fermions with isospin T, are polinomials of degree $2T$ in y. To this end we shall employ the spinorial formulation of the Dirac equation(7), where ψ_{Au} is represented by a spinor $\psi_{A;U_1\ldots U_{2T}}$ which is entirely symmetric in its $2T$ indices. These indices refer to the $2T+1$ components of the isospin T. In this formulation, using $\partial \equiv \partial/\partial y$, equation (7b) can be written as follows:

$$(\alpha^\dagger_\mu)_{A'A}\left[\partial_\mu - y_\mu\, y.\partial\right]\psi_{A';U_1\ldots U_{2T}} -$$

$$-\left[(\alpha^\dagger.y\ \sigma_a)_{A'A}\ (\sigma_a)_{U_1V}\ \psi_{A';V\,U_2\ldots U_{2T}} + \text{permutations}\right] = 0 \qquad (8)$$

In the case of isospin $T = 1/2$, it is readily verified that the solution is:

$$\psi_{A;U_1} = (\alpha.y)_{U_1A} = \frac{(\alpha.x)_{U_1A}}{\sqrt{x^2}} \qquad (9a)$$

which represents the small x limit of the exact solution[7] corresponding to $T = 1/2$. Before describing the general solution, let us also consider the $T = 1$ case. Making the ansatz:

$$\psi_{A;U_1U_2} = \psi_{A;U_1} X_{U_2} + \psi_{A;U_2} X_{U_1} \tag{9b}$$

we find that it will satisfy (8) provided X obeys the equation:

$$\psi_{A';U_1} \Big[(\alpha^\dagger_\mu)_{A'A} \, (\partial_\mu - y_\mu \, y\cdot\partial) \, X_{U_2} -$$

$$- (\alpha^\dagger \cdot y \, \sigma_a)_{A'A} \, (\sigma_a)_{U_2V} X_V \Big] + 1 \leftrightarrow 2 = 0 \quad . \tag{10a}$$

The solution to this equation is given by:

$$X_U = C_1 (\alpha\cdot y \; \eta_1)_U + C_2 (\alpha\cdot y \; \eta_2)_U \tag{10b}$$

where C_1 and C_2 are constants with $\eta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Substituting (9a) and (10b) into the ansatz (9b), will yield in this case 2 independent solutions, which again correspond to the $x^2 << \rho^2$ limit of the exact solution for isospin $T = 1$ [7].

The general idea is now clear. We make the ansatz:

$$\psi_{A;U_1 \ldots U_{2T}} = \psi_{A;U_1} F_{U_2 \ldots U_{2T}} + \text{permutations} \tag{11a}$$

and substitute it into the spinorial form of the Dirac equation (8). A detailed analysis shows that this will be a solution provided F factorizes into functions of the form:

$$F_{U_2U_3 \ldots U_{2T}} = X_{U_2} \, Y_{U_3} \, \cdots \, Z_{U_{2T}} \tag{11b}$$

where each of the $X, Y, \ldots Z$ satisfies equations analogous to (10a). Therefore, from (11), and making also use of relations (9a) and (10b), we see that a general solution has the form:

$$\psi_{A;U_1\ldots U_{2T}} = (\alpha.y)_{U_1A}\,(\alpha.y\,\eta_i)_{U_2}\cdots(\alpha.y\,\eta_j)_{U_{2T}} + \text{permutations} \qquad (12)$$

where η_i , η_j can be any of the constant spinors η_1 or η_2 described in (10b). One can readily verify that there will be in general 2T independent solutions, each one characterized by the numbers ℓ_1 and ℓ_2 with $\ell_1 + \ell_2 = 2T - 1$, which describe how many times η_1 and η_2 appear in (12).

We are now in a position to analyse the momentum-space propagator $\tilde{S}(p)$. As we have seen, the above 2T functions reflect the properties of the zero modes at small distances and will therefore determine, via equation (3), the behaviour of the dynamical mass for large quark momenta. [All other $C(T) - 2T$ zero modes appearing in $\tilde{S}(p)$ vanish faster for asymptotic momenta and hence will not be relevant for our purposes]. We proceed now to calculate the Fourier transform of the 2T functions described by equation (12). To this end we use, with $r \equiv \sqrt{x^2}$, the expression

$$\int d^4x\; e^{ipx}\; F(r) = \frac{4\pi^2}{p}\int_0^\infty dr\; r^2\; J_1(pr)F(r) \qquad (13a)$$

as well as known integrals[8] involving Bessel functions. After a straightforward calculation we find that:

$$\int d^4x\; e^{ipx}\; \frac{x_{\nu_1}\cdots x_{\nu_{2T}}}{(x^2)^T} = 8\pi^2(i^{2T})\,\frac{1}{p^4}\times$$

$$\times\left\{2T(T+1)\,\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_{2T}} - T\,\delta_{\nu_1\nu_2}\,\hat{p}_{\nu_3}\cdots\hat{p}_{\nu_{2T}} + \text{permutations} + \ldots\right\} \qquad (13b)$$

where $\hat{p}_\nu \equiv p_\nu/p$ and ... denote contributions involving two or more delta functions. This yields for the Fourier transform of (12):

$$\tilde{\psi}_{A;U_1\ldots U_{2T}} = 16\pi^2\, i^{2T}\, T(T+1)\, \frac{1}{p^4} \left\{(\alpha.\hat{p})_{U_1 A} \cdots (\alpha.\hat{p}\,\eta_j)_{U_{2T}} + \right.$$

$$\left. + \text{ permutations}\right\} + \ldots \qquad (14)$$

where we have written explicitly only the contribution arising from the first term in (13b). [This is the only one contributing to $\tilde{S}(p)$ for isospins $T = \frac{1}{2}$ and $T = 1$].

Clearly, the p^{-4} behaviour of $\tilde{\psi}_{A;U_1 \ldots U_{2T}}$ is a direct consequence of the fact that $\psi_{A;U_1 \ldots U_{2T}}$ is a function of zero degree in x. Consequently, at large momenta $\tilde{S}(p)$ will be proportional to p^{-8}, making the dynamical mass $m(p^2)$ in (4) behave asymptotically like p^{-6}. Finally, it is perhaps worthwhile to remark that the coefficient of the factor p^{-4} appearing in (14) rises with T, a feature that may explain the fact[3,4] that the scale for the dynamical mass seems to increase with the representation to which the fermion fields belong.

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