UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

publicações

IFUSP/P-479

RESOLUTION OF HYDRODYNAMICAL EQUATIONS FOR TRANSVERSE EXPANSIONS - I

by

Y. Hama and F.W. Pottag

Instituto de Física, Universidade de São Paulo

Julho/1984

RESOLUTION OF HYDRODYNAMICAL EQUATIONS

FOR TRANSVERSE EXPANSIONS - I

Y. Hama and F.W. Pottag

Instituto de Física, Universidade de São Paulo,
São Paulo, Brasil

ABSTRACT

The three-dimensional hydrodynamical expansion is treated with a method similar to that of Milekhin, but more explicit. Although in the final stage we have to appeal to numerical calculation, the partial differential equations governing the transverse expansions are treated without transforming them into ordinary equations with an introduction of averaged quantities. The present paper is concerned with the formalism and we will report the numerical results in the next paper.

I. INTRODUCTION

The hydrodynamical model for high-energy multiparticle production has been proposed by Landau [1] a long time ago and revived by some modern researchers, under a more current point of view, which has been acquired with recent progresses in particle physics*. So far, due to their large mathematical complexity, the complete hydrodynamical equations have never been solved, unless in a very important case of one-dimensional motion for which Khalatnikov found an elegant exact solution [3]. That this solution is approximately valid in actual high-energy multiparticle production (evidently here we are not questionning the validity of the hydrodynamical model itself) follows from the flatness of the initial fluid due to Lorentz contraction of the incident particles, so that the expansion occurs mainly in the incident direction. However, since the transverse dimensions of such a state of fluid are, although large, finite, transverse expansion certainly exists and there are indeed some empirical evidences of this phenomenon. Thus, in previous works [4], we have shown that within the framework of hydrodynamical description, the observed flattening of the large-$p_\perp$ inclusive pion distribution $E\left.\frac{d\sigma}{d\vec{p}}\right|_{\frac{\pi}{2}}$ with the energy increase might well be attributed to the transverse expansion of the pre-hadronic gas, which would be larger the larger were the energy. In Ref. [5], as a byproduct of our large missing-mass--cluster analysis, we concluded that the influence of the transverse expansion on the longitudinal-momentum distribution is not negligible. This influence appears because it causes

*See, for instance, a review given in Ref. [2].

an additional cooling and so smaller acceleration along the longitudinal direction before the dissociation takes place. Finally, an analysis of recent $\overline{p}p$ collider's data [6] showed that the observed strong correlation between the average transverse momentum $\langle p_\perp \rangle$ and the multiplicity is well understood [7] in terms of the transverse expansion of an initially flat quark-gluon plasma.

The considerations above indicate that it is of great interest to obtain solutions of the hydrodynamical equations which take into account also the transverse motions. In the existent literature first we could find the old Milekhin's work [8], in which he does not properly solve the system of partial differential equations, but avoid the mathematical complexities by transforming it into a system of ordinary equations, with an introduction of averaged quantities. In Ref. [7], we used one of his result obtained in this way, which gave a good account of the empirical behaviour of $\langle p_\perp \rangle$ as a function of the multiplicity. It is clear, on the other side, that his treatment is incomplete in the sense that it cannot evidently provide a more detailed $p_\perp$ distribution. Besides the introduction of the averaged quantities, Milekhin's method consists fundamentally in separating out the transverse expansion from the longitudinal one, by assuming that the latter may be approximated fy the asymptotic one-dimensional solution and that the transverse expansion gives but a small change in the longitudinal motion*. Some years ago, Yotsuyanagi has studied the same problem, by developping Milekhin's idea [9].

*Observe that even though, the final particle distribution may be quite different, due to an additional cooling as mentioned above.

In his paper, he obtains an analytical solution of the system of partial differential equations in the large transverse-rapidity limit. However, as will be discussed in section III, we think his choice of boundary conditions is not correct. While we were writing this paper, we took notice of a recent work by G. Baym et al. [10], where one of the topics treated by them is precisely the transverse expansion accompanying longitudinal flows in nucleus-nucleus collisions. Although it is not clearly referred in their paper, fundamentally they follow Milekhin's prescription described above, so their method is similar to the one we are going to describe in this paper. As for the choice of the initial conditions, we see some ambiguities there, beside those characteristic of Bjorken-Kajantie's version [11] they follow, with respect to the longitudinal distribution of $\varepsilon, T, s, \ldots$ Namely, we are now considering also the transverse flows, so we must know how to specify the transverse distribution of thermodynamic quantities at $\tau = \tau_0$? Apparently, they neglect any transverse motion of fragments for $\tau < \tau_0$, but since they assume a non-zero probability of forming fragments entirely at rest, so with large momentum-transfer, it is hard to understand why transverse motions of fragments are forbidden. In our work, we prefer to be more orthodox and to state without ambiguity all the initial conditions and then to solve the equations. There will appear some differences in the final results (which are not reported in this paper) which follow from the difference in the choice of the initial conditions.

The purpose of the present paper is to present the formalism used by us to solving explicitly the system of hydrodynamical equations including the transverse motions, applicable especially in the large-angle region. The method we have employed is fundamentally the one proposed by Milekhin, except evidently for the final part, where we have explicitly integrated the system of partial equations. Although our method and the results which follow apply to the original Landau's model, we have in mind a model which we have been studying [5,7,12,13], in which during a collision one or two large fireballs with masses M_1 and M_2 are formed, around each of the incident particles*. So, we will specify everything with the mass M of such a fireball, which is reduced to the total energy $\sqrt{s}$ in the case of Landau's model.

An additional remark regarding the applicability of hydrodynamical concepts to hadron-hadron collision is in order. Several authors criticise the use of hydrodynamics in processes such as hadron-hadron collisions, but others advocate its use even without local thermal equilibrium. One of the latter is P. Carruthers who says "hydrodynamic behavior may exist without thermodynamic equilibrium" [14]. He argues that local thermal equilibrium is not a prerequisit to the use of collective variables, so formal hydrodynamic structures may exist even in the absence of this equilibrium and could provide useful information. In a recent paper [15], B. Lukács and K. Martinás have shown how to extend the thermodynamic formalism for situations where the distribution function deviates from equilibrium in momentum space. They conclude that the results are compatible with continuum mechanics. We accept these opinions in the present paper for our hydrodynamical study of

*One may imagine each incident hadron as a superposition of virtual states with a variable number of components, but having a definite mass. During the collision, one of these states would be materialized with a larger mass M. We think this is a way to take the event-by-event fluctuation of n, $\langle p_\perp \rangle$, $\frac{d\sigma}{d\eta}$, ... into account.

hadron-hadron collisions.

In what follows, we present the method of resolution by starting from the choice of the coordinate system given in the next Section. In Sec. III, we write down the equations of motion for transverse flow in terms of it and discuss the boundary conditions. In Sec. IV, these equations are solved both in the "trivial" as well as in the "non-trivial" regions, by reducing the equations to canonical form. Contrary to the case of longitudinal expansion, the trivial region in the transverse expansion is much larger due to the initial dimensions $R \gg \Delta$ and so much important in the latter as compared with the former case. We explain, in Sec. V, how to compute the physically observable quantities such as the inclusive distribution $E\frac{d\sigma}{d\vec{p}}$ from the knowledge of the solution of the hydrodynamical equations obtained above. We give additional remarks in Sec. VI and some mathematical details are gathered in the Appendices.

II. COORDINATE SYSTEM

The object whose expansion we would like to study is a flat disc of thickness 2ℓ, radius $R \gg \ell$, initial temperature $T_0 \gg T_d$, where T_d is the temperature when the dissociation into the final particles takes place. The expansion is assumed to be axially as well as forward-backward symmetrical, just for the sake of simplicity. This is a quite natural assumption in terms of the large-cluster model we have been studying [12,5,7,13]. So, in the center-of-mass system of the fireball, the four velocity may conveniently be parametrized in terms of rapidity variables (α,ξ) as

$$\overline{u}^{\mu}(\overline{x}) = (\mathrm{ch}\alpha\mathrm{ch}\xi\ ,\ \mathrm{sh}\alpha\mathrm{ch}\xi\ ,\ \mathrm{sh}\xi\cos\phi\ ,\ \mathrm{sh}\xi\,\mathrm{sen}\phi) \quad , \qquad (2.1)$$

where ϕ is the azimutal angle around the symmetry (x-) axis. The equations of relativistic hydrodynamics are [1]

$$\partial_{\mu} T^{\mu\nu} = 0 \quad , \qquad (2.2)$$

where

$$\begin{cases} T^{\mu\nu} = (\varepsilon+p)u^{\mu}u^{\nu} - pg^{\mu\nu} \quad \text{and} \\ p = c_0^2\,\varepsilon \quad , \end{cases} \qquad (2.3)$$

which have been exactly solved by Khalatnikov in the case of the one-dimensional expansion. If

$$\begin{cases} y^2 - c_0^2\alpha^2 >> 1 \quad \text{and} \\ y^2 >> \alpha^2 \quad , \end{cases} \qquad (2.4)$$

where

$$y = \ell n \frac{T}{T_0} \quad , \qquad (2.5)$$

his solution may be approximately written as

$$\begin{cases} \alpha \cong \frac{1}{2}\,\ell n\,\frac{t+x}{t-x} \quad , & (2.6) \\ y \cong -\frac{1+c_0^2}{4}\,\ell n\,\frac{t^2-x^2}{\Delta^2} + \frac{1-c_0^2}{4}\left[\ell n^2\,\frac{t^2-x^2}{\Delta^2} - \ell n^2\,\frac{t+x}{t-x}\right]^{1/2} \quad , & (2.7) \end{cases}$$

where $$\Delta = \sqrt{\frac{1-c_0^2}{\pi}}\ \ell \quad . \qquad (2.8)$$

If $\alpha << \ell n \frac{\sqrt{t^2-x^2}}{\Delta}$, we may rewrite (2.7) as

$$y \cong -c_0^2\ \ell n\,\frac{\sqrt{t^2-x^2}}{\Delta} \quad . \qquad (2.9)$$

Here, eq. (2.6) and (2.9) appear as the solution of (2.2), showing an approximate scale invariance. In Ref. [11], the scale invariance is instead imposed as an external condition.

Now, in accordance with Milekhin's method, we introduce the following system of coordinates, which is suggested by the solution above for the one-dimensional motion and will show useful in solving three-dimensional problems:

$$\begin{cases} \tau = \sqrt{t^2-x^2} \quad , \\ \alpha_0 = \mathrm{th}^{-1}\,\frac{x}{t} \quad , \\ r = \sqrt{y^2+z^2} \quad , \\ \phi = \tan^{-1}\frac{z}{y} \quad , \end{cases} \qquad (2.10)$$

in terms of which we have

$$\begin{cases} t = \tau\,\mathrm{ch}\,\alpha_0 \quad , \\ x = \tau\,\mathrm{sh}\,\alpha_0 \quad , \\ y = r\cos\phi \quad , \\ z = r\,\mathrm{sen}\,\phi \quad . \end{cases} \qquad (2.11)$$

In (2.10), α_0 represents the rapidity of a fluid element in the absence of the transverse expansion and τ is the corresponding proper time. The introduction of these variables reflects our expectation that the radial motion is much smaller than the longitudinal one and so does not modify the latter in any considerable amount. Actually, we will assume $\alpha = \alpha_0$ in the derivation of the radial equations below (3.2). The metric tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ in the new coordinate system write

$$\begin{cases} g^{00} = 1 \ , \ g^{11} = -\dfrac{1}{\tau^2} \ , \ g^{22} = -1 \ , \ g^{33} = -\dfrac{1}{r^2} \ \text{ and} \\ g^{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \end{cases} \tag{2.12}$$

and

$$\begin{cases} g_{00} = 1 \ , \ g_{11} = -\tau^2 \ , \ g_{22} = -1 \ , \ g_{33} = -r^2 \ \text{ and} \\ g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \ . \end{cases} \tag{2.13}$$

So,

$$g = \det|g_{\mu\nu}| = -\tau^2 r^2 \ . \tag{2.14}$$

Let us now rewrite the four velocity given by (2.1) in the new coordinate system. We have

$$\begin{cases} u^\mu(x) = (\mathrm{ch}(\alpha-\alpha_0)\mathrm{ch}\xi \ , \ \dfrac{\mathrm{sh}(\alpha-\alpha_0)}{\tau}\mathrm{ch}\xi \ , \ \mathrm{sh}\xi \ , \ 0) \\ \text{and} \\ u_\mu(x) = g_{\mu\nu}u^\nu = (\mathrm{ch}(\alpha-\alpha_0)\mathrm{ch}\xi \ , \ -\tau\,\mathrm{sh}(\alpha-\alpha_0)\mathrm{ch}\xi \ , \ -\mathrm{sh}\xi \ , \ 0) \ . \end{cases} \tag{2.15}$$

III. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

EQUATIONS OF MOTION

In curvilinear coordinates, the equations of hydrodynamics (2.2) must be rewritten by replacing the

derivatives which appear there by the covariant derivatives. More explicitly, the generalization of (2.2) is

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,T^\mu_\nu)}{\partial x^\mu} - \frac{1}{2}\frac{\partial g_{\mu\lambda}}{\partial x^\nu}T^{\mu\lambda} = 0 \ . \tag{3.1}$$

By putting eqs. (2.3) into this equation and using thermodynamical relations, we may rewrite it as

$$\frac{\partial y}{\partial x^\nu} = -\frac{c_0^2 u_\nu}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\,u^\mu\right] + u^\mu\frac{\partial u_\nu}{\partial x^\mu} - \frac{u^\mu u^\lambda}{2}\frac{\partial g_{\mu\lambda}}{\partial x^\nu} \ . \tag{3.2}$$

In the specific case of our interest, if we assume

$$\alpha = \alpha_0 \tag{3.3}$$

as discussed before, the introduction of (2.10), (2.13–15) into eq. (3.2) leads to

$$\begin{cases} \dfrac{\partial y}{\partial \tau} = -\dfrac{c_0^2\mathrm{ch}^2\xi}{\tau} - \dfrac{c_0^2\,\mathrm{sh}\xi\,\mathrm{ch}\xi}{r} + (1-c_0^2)\mathrm{sh}\xi\mathrm{ch}\xi\,\dfrac{\partial\xi}{\partial\tau} + (\mathrm{sh}^2\xi - c_0^2\mathrm{ch}^2\xi)\dfrac{\partial\xi}{\partial r} \ , \\ \dfrac{\partial y}{\partial \alpha_0} = 0 \ , \\ \dfrac{\partial y}{\partial r} = \dfrac{c_0^2\,\mathrm{sh}\xi\,\mathrm{ch}\xi}{r} + \dfrac{c_0^2\mathrm{sh}^2\xi}{r} - (\mathrm{ch}^2\xi - c_0^2\mathrm{sh}^2\xi)\,\dfrac{\partial\xi}{\partial\tau} - (1-c_0^2)\mathrm{sh}\xi\,\mathrm{ch}\xi\,\dfrac{\partial\xi}{\partial r} \ , \\ \dfrac{\partial y}{\partial \phi} = 0 \ . \end{cases} \tag{3.4}$$

The second of eqs. (3.4) is actually not entirelly compatible with (2.6,7) in the one-dimensional flow limit and the origin of this inconsistency is traced back to our assumption (3.3). However, the main part of the entropy is concentrated in the region $\alpha_0 \sim 1$ and one may show that there

$$\alpha - \alpha_0 \sim \frac{1}{\ell n \frac{\tau}{\Delta}} \tag{3.5}$$

when $\ell n \frac{\tau}{\Delta} >> \alpha_0$. Since we will mainly be concerned with this region, we may neglect, in this paper, the small difference given above.

The system of equations (3.4) represents an enormous simplification as compared with (3.1). We have now a system of two equations in two independent variables (τ, r) and the unknown functions are y and ξ. Thus, the transverse motions have been separated from the longitudinal ones. In order to solve this system, we must now specify the boundary conditions.

BOUNDARY CONDITIONS

All our approximation scheme is based on a fundamental assumption that, as far as the central region of the disc is concerned and for $t \lesssim R$, the one-dimensional solution is a good description of the phenomena and that the deviation from this appears first at the boundary of the disc and it propagates from outside to the center of the disc. This is Milekhin's picture and, in our opinion, is both an intuitive and correct image of the phenomenon. Accordingly, the fluid in three-dimensional flow would be bounded by the surface

$$r = R + \tau \tag{3.6}$$

on the vacuum side and would contact the one-dimensional flow region on

$$r = R - c_0 \tau \tag{3.7}$$

(see Appendix A for the derivation of this equation, although

it is more or less self-evident). We illustrate this picture by Figs. 1 and 2.

However, Yotsuyanagi in his paper [9] gives another version for the boundaries. His argument goes as follows. At the moment when the fireball is formed, a weak discontinuity leaves the initial surface and goes to the inside the fluid. At a very early stage $t \sim \frac{\Delta}{c_0}$, this discontinuity reaches the symmetry plane $x=0$ and vanishes so that it cannot be the surface of separation between the region of one-dimensional and three-dimensional flows. Notice that if one assumes a very flat spheroidal fireball as he did and observe the motion of the above mentioned discontinuity at 90° in the center of mass frame, eq. (3.7) would describe its motion. His proposal is then taking the surface of separation as starting from the origin, $r=0$, at $t \sim \frac{\Delta}{c_0}$, which would travel outward in transverse direction as t increases. We cannot, nevertheless, agree with this view, because we think it is in contradiction with the very basic assumption which lies under this kind of approximation and which has been stated at the top of this subsection. For, according to his version, all the fluid would be in three-dimensional flow at the beginning (here we neglect a very small interval of time $t \sim \Delta$). As t increases, the purely one-dimensional-flow region would appear behind a surface of discontinuity and would increase indefinitely.

In our opinion, $t \lesssim \Delta << R$ is a too small interval of time where we do not even know whether it is justifiable treating our fireball using eq. (3.4)*. We would reserve their use only for $t \gtrsim \Delta$. Observe that the weak discontinuity

*As mentioned in the Introduction, there are criticisms to applying hydrodynamics even at $t \sim R$.

mentioned above does not reach the center of the spheroid in the transverse direction but in the longitudinal direction (because $\Delta \ll R$) and along the axis, so when it reaches the symmetry plane, a new discontinuity begins to travel outward in the longitudinal direction. It is clear that in this treatment, a small transverse inhomogeneity is, as usual, completely neglected and within this approximation, the central fluid will continue to expand longitudinally, until the surface given by eq. (3.7) reaches the fluid element in question. Remark that what we are considering is not a discontinuity in mathematically strict sense but has a certain width $\sim \Delta$, which is neglected for mathematical simplicity of treatment.

The boundary conditions of our problem may now be written

$$\begin{cases} \xi = \infty & \text{and} \\ y = -\infty & , \end{cases} \tag{3.8}$$

when $r = R + \tau$,

and

$$\begin{cases} \xi = 0 & \text{and} \\ y = -c_0^2 \ln \frac{\tau}{\Delta} \end{cases} \tag{3.9}$$

when $r = R - c_0\tau \quad (\tau \leq \frac{R}{c_0})$.

Along the axis and for $\tau \geq \frac{R}{c_0}$, we have

$$\xi = 0 \quad , \tag{3.10}$$

which we will see below in Sec. IV that implies

$$\frac{\partial y}{\partial r} = 0 \quad . \tag{3.11}$$

As referred to in the Introduction, the initial conditions are not clearly stated in [10], but it seems that they assume transverse expansion starts only after $\tau = \tau_0 \sim 1$ fm, when the materialization would occur and, at that instant, ξ and y would approximately be distributed as in our case, given by (3.8) and (3.9) at $\tau \sim \Delta$.

IV. RESOLUTION OF HYDRODYNAMICAL EQUATIONS

REDUCTION TO CANONICAL FORMS

We are now ready to solve the system of equations (3.4), satisfying the boundary conditions specified in the last Section. This will be done by the method of characteristics. First of all, write

$$y = y_\perp - c_0^2 \ln \frac{\tau}{\Delta} \quad , \tag{4.1}$$

which separates from y the purely longitudinal contribution as given by eq. (2.9). The new variable $y_\perp$ satisfies a boundary condition

$$y_\perp = 0 \quad \text{on} \quad r = R - c_0\tau \quad , \tag{4.2}$$

which replaces the second of eqs. (3.9).

Define now combinations of $y_\perp$ and ξ

$$\begin{cases} \psi = y_\perp + c_0\xi & \text{and} \\ \phi = y_\perp - c_0\xi & , \end{cases} \tag{4.3}$$

so that

$$\begin{cases} y_\perp = \frac{1}{2}(\psi+\phi) \quad \text{and} \\ \xi = \frac{1}{2c_0}(\psi-\phi) \quad . \end{cases} \tag{4.4}$$

With the help of eqs. (4.1) through (4.4) and after an appropriate recombination, we may rewrite the system of eqs. (3.4) in the canonical form as

$$\begin{cases} \frac{\partial\psi}{\partial\tau} + \frac{v_\perp + c_0}{1+c_0 v_\perp}\frac{\partial\psi}{\partial r} + \frac{c_0^2 v_\perp}{1+c_0 v_\perp}\left[\frac{1}{r} - \frac{c_0}{\tau}\right] = 0 \quad , \\ \frac{\partial\phi}{\partial\tau} + \frac{v_\perp - c_0}{1-c_0 v_\perp}\frac{\partial\phi}{\partial r} + \frac{c_0^2 v_\perp}{1-c_0 v_\perp}\left[\frac{1}{r} + \frac{c_0}{\tau}\right] = 0 \quad , \end{cases} \tag{4.5}$$

where

$$v_\perp = \text{th}\,\frac{\psi-\phi}{2c_0} \quad . \tag{4.6}$$

This is a hyperbolic system of quasi-linear equations. It has the following family of characteristics, which we illustrate in Fig. 2:

$$\begin{cases} \text{(a)} \quad \frac{dr}{d\tau} = \frac{v_\perp + c_0}{1+c_0 v_\perp} \quad , \\ \text{(b)} \quad \frac{dr}{d\tau} = \frac{v_\perp - c_0}{1-c_0 v_\perp} \quad . \end{cases} \tag{4.7}$$

These equations are precisely (A.4) of Appendix A and may be obtained directly from (4.5) using the same procedure (observe that the changes of variables, (4.1) and (4.4) do not affect

the results). From (4.5), it follows that, along each family (a) and (b) given by (4.7), we have

$$\begin{cases} \text{along (a):} \quad d\psi = \frac{c_0^2 v_\perp}{1+c_0 v_\perp}\left[\frac{c_0}{\tau} - \frac{1}{r}\right] d\tau \quad , & (4.8) \\ \text{along (b):} \quad d\phi = -\frac{c_0^2 v_\perp}{1-c_0 v_\perp}\left[\frac{c_0}{\tau} + \frac{1}{r}\right] d\tau \quad . & (4.9) \end{cases}$$

Therefore, our procedure in solving the transverse part of hydrodynamical equations is to integrate (4.8,9) along the characteristics (4.7), using the boundary conditions (3.8-10) and (4.2). This will be done numerically, so, in principle, it is possible. However, as will be explained below, it will present some difficulties in applying the boundary conditions, requiring a special care.

ULTRARELATIVISTIC APPROXIMATION

Let us first consider the integration in the trivial region or region II of Fig. 2. In what the ψ-integration is concerned, everything goes as indicated above since its boundary value is well defined on the curve (3.7), where (4.8) is regular except at $\tau=0$. On the contrary, the ϕ-integration is problematical because, as shown in Fig. 2, all the characteristics (b) start at $(\tau=0\ ,\ r=R)$, where the corresponding differential $d\phi$ is singular. They continue beyond the region II and end at (or reflect from) the straight line $r=0$, $\tau > \frac{R}{c_0}$, where we do not have the boundary value of ϕ. Thus, we cannot integrate in ϕ neither starting from $\tau=0$, nor backward starting from $r=0$. Some other procedure is required to treat it. For doing so, we make use of the circumstances that, due to their form given by (4.7) and illustrated in Fig. 2, the most part of the curves (b) pass through the ultrarelativistic region at the beginning of the expansion. So we try to find the ultrarelativistic solution of (4.5) to describe the initial flow. By putting $v_\perp \to 1$, we may rewrite (4.5) as

$$\left\{\begin{array}{l} \dfrac{\partial\psi}{\partial\tau} + \dfrac{\partial\psi}{\partial r} + \dfrac{c_0^2}{1+c_0}\left[\dfrac{1}{r} - \dfrac{c_0}{\tau}\right] \cong 0 \quad , \\[2ex] \dfrac{\partial\phi}{\partial\tau} + \dfrac{\partial\phi}{\partial r} + \dfrac{c_0^2}{1-c_0}\left[\dfrac{1}{r} + \dfrac{c_0}{\tau}\right] \cong 0 \quad , \end{array}\right. \tag{4.10}$$

that is, in this limit the equations become decoupled in ψ and ϕ, so are easily integrated. The solution which satisfies the boundary condition

$$\psi = \phi = 0 \quad \text{on} \quad r = R - c_0\tau \quad , \tag{4.11}$$

which follows from (3.9), (4.2) and (4.3) is, as will be computed in Appendix B,

$$\left\{\begin{array}{l} \psi \cong \dfrac{c_0^2}{1+c_0}\,\ell n\left[\dfrac{R-c_0(\tau-r)}{(1+c_0)r}\left(\dfrac{(1+c_0)\tau}{R+\tau-r}\right)^{c_0}\right] \quad , \\[2ex] \phi \cong \dfrac{c_0^2}{1-c_0}\,\ell n\left[\dfrac{R-c_0(\tau-r)}{(1+c_0)r}\left(\dfrac{R+\tau-r}{(1+c_0)\tau}\right)^{c_0}\right] \quad . \end{array}\right. \tag{4.12}$$

Strictly speaking, since eqs. (4.10) are ultrarelativistic, they are not valid close to the curve (3.7) where $\xi \cong 0$ and so imposing the condition (4.11) to their solution is actually not correct. It is however a good approximation, which we will take as the boundary conditions close to (3.6), replacing (3.8).

Anyhow, from the physical point of view, it is intuitive that in solving the system (4.5), influences coming from the entire boundary (3.7) is much more important than those coming from $\tau=0$ region, or in other worths from the vacuum side boundary, thus justifying our approximation.

BOUNDARY CONDITION ON $r=0$, $\tau > R/c_0$

In the last subsection, we have explained how to solve eqs. (4.5) in the trivial region. Now, we shall turn our attention to the region III, where the characteristics (b) arriving at $r=0$ (τ axis) suffer a reflection and leave it as characteristics (a). This domain has two boundaries, namely, one which separates it from region II (curve A), where ψ and ϕ are in principle known and the other which is the straight line $r=0$, $\tau > R/c_0$, where $\xi=0$, according to (3.10), but we do not know which is the value of $y_\perp$. In terms of ψ and ϕ, this means that we know a particular combination of these functions there, but not ψ itself whose value is needed there in order to carry the ψ-integration out.

For treating this problem, rewrite (4.5) for the neighbouring points of τ-axis ($r \cong 0$), where $v_\perp \cong \xi \cong 0$:

$$\left\{\begin{array}{l} \dfrac{\partial\psi}{\partial\tau} + c_0\dfrac{\partial\psi}{\partial r} + \dfrac{c_0^2\xi}{r} \cong 0 \quad , \\[2ex] \dfrac{\partial\phi}{\partial\tau} - c_0\dfrac{\partial\phi}{\partial r} + \dfrac{c_0^2\xi}{r} \cong 0 \quad . \end{array}\right. \tag{4.13}$$

Now, along τ-axis ($r=0$, $\tau > R/c_0$), it follows from (3.10)

$$\frac{\partial\xi}{\partial\tau} = 0 \quad , \tag{4.14}$$

or with the help of (4.4), this is rewritten as

$$\begin{cases} \psi = \phi , \\ \dfrac{\partial\psi}{\partial\tau} = \dfrac{\partial\phi}{\partial\tau} \qquad (\text{for} \quad r=0 , \quad \tau > R/c_0) . \end{cases} \tag{4.15}$$

By subtracting eqs. (4.13) from each other and using (4.15) we have also

$$\frac{\partial\psi}{\partial r} = -\frac{\partial\phi}{\partial r} , \qquad (r=0 , \quad \tau > R/c_0) . \tag{4.16}$$

Now, making use of (3.10) and (4.14), expand ξ in (4.13) in power series around a point $(\tau,0)$:

$$\xi(\tau,r) \cong \frac{\partial\xi}{\partial r}(\tau,0)r \cong \frac{1}{c_0}\frac{\partial\psi}{\partial r}(\tau,0)r , \tag{4.17}$$

valid up to the first order. In the last step, we have used (4.4) and (4.16). Add (4.13) each other and with the help of (4.15) and (4.16) obtain finally

$$\frac{\partial\psi}{\partial\tau} = -2c_0\frac{\partial\psi}{\partial r} = 2c_0\frac{\partial\phi}{\partial r} \tag{4.18}$$

(for $r=0$, $\tau > R/c_0$). This is a relation between the time variation of ψ and the space variation of ψ or ϕ, so once ψ or ϕ is given for a fixed τ and in a small neighbourhood of $r=0$, it allows us to compute its value in the vicinity along τ-axis. Thus, (4.18) is our boundary condition to be used in ψ-integration.

V. TRANSVERSE RAPIDITY DISTRIBUTION OF THE HADRONIC FLUID AND THE INCLUSIVE PARTICLE DISTRIBUTIONS

In the last Section, we have shown how to solve the hydrodynamical equations for transverse flows and to obtain ψ and ϕ, and so y and ξ by means of eqs. (4.4) and (4.1), as functions of τ and r (in our approximation, the solution is independent of α_0 and of the azimutal angle ϕ). Now, let us obtain the momentum distribution of particles to allow a comparison with the experimental data. Although it is not indispensable for this end, first we will derive the transverse rapidity distribution of the hadronic fluid and then the inclusive particle distributions.

TRANSVERSE RAPIDITY DISTRIBUTIONS OF THE HADRONIC FLUID

In the original version of hydrodynamical model, the initially hot, pancake-shaped blob expands until each part of it reaches some critical temperature, called dissociation temperature T_d, after which particles appear as independent, non-interacting objects. In terms of a current view, we would initially have a hot quark-gluon plasma which would suffer a phase transition as the fluid expands and the temperature decreases. In any case, the final particle distribution as well as the rapidity distribution* of the hadronic fluid must be calculated on the hypersurface where $T = T_d$. We have, thus

---

*We observe, however, that in the expansion of the quark-gluon plasma, an additional complication appears which is related to what happens with the system during the phase transition. It will surely continue to expand but, as far as we know, this problem has not yet been treated in the literature.

$$dN = n\, u^{\mu} \sqrt{-g}\; d\sigma_{\mu}\Big|_{T=T_d} \quad , \tag{5.1}$$

where n is the particle density and $d\sigma_{\mu}$ are the components of the surface element (inclusive the normal direction). The meaning of eq. (5.1) is clear, especially if one uses the Cartesian coordinates when $\sqrt{-g}$ will be reduced to 1. In our coordinate system, u^{μ} are given by (2.15) and, with the approximation (3.3), they become

$$u^{\mu} = (\mathrm{ch}\xi\,,\,0\,,\,\mathrm{sh}\xi\,,\,0) \quad . \tag{5.2}$$

By using the usual procedure*, we obtain for $d\sigma_{\mu}$ in the same approximation

$$d\sigma_{\mu} = d\alpha_0\, d\phi\, (-dr, 0, d\tau, 0) \quad . \tag{5.3}$$

So, by putting (5.2), (5.3) and (2.14) into (5.1) and considering the axial symmetry of the problem, we have

$$dN = 2\pi\, n\, \tau r (\mathrm{sh}\xi d\tau - \mathrm{ch}\xi dr) d\alpha_0 \Big|_{T=T_d} \quad , \tag{5.4}$$

where the signs have been chosen so that (τ,r) integration starts from the point $(0,R)$. It is convenient to rewrite (5.4) in terms of $d\xi$ instead of $d\tau$ and dr, because we are interested in ξ distribution. Then

$$dN = -2\pi\, n \frac{\tau r}{\left|\frac{\partial(\xi,y)}{\partial(\tau,r)}\right|} \left(\mathrm{sh}\xi \frac{\partial y}{\partial r} + \mathrm{ch}\xi \frac{\partial y}{\partial \tau} \right)\Big|_{T=T_d} d\xi\, d\alpha \quad . \tag{5.5}$$

---

*See for instance Ref. [16].

where $\frac{\partial(\xi,y)}{\partial(\tau,r)}$ is Jacobian of the transformation $(\xi,y) \to (\tau,r)$ and we have also replaced α_0 by α, with the help of eq. (3.3). It is clear that the distribution given above is independent of the longitudinal rapidity α, because of our approximation (3.3). More correct α-distribution would indeed be an approximate Gaussian as it follows from eqs. (2.6) and (2.7).

INCLUSIVE PARTICLE DISTRIBUTIONS (FOR FIXED M)

In order to obtain the (semi-)inclusive particle distribution, one must further consider the thermal fluctuation at $T=T_d$. The correct receipt for this, consistent with energy conservation, has been obtained by Cooper and Frye [17] starting from the transport theory of a relativistic gas and reads

$$E \frac{dN}{d\vec{p}} = \frac{w}{(2\pi)^3} \int \frac{p^{\mu} \sqrt{-g}\, d\sigma_{\mu}}{\exp(\bar{E}/T_d) \pm 1} \quad , \tag{5.6}$$

where the integration is taken over the hypersurface $T = T_d$, w is the statistical weight and p^{μ} is the four momentum of the particle to be observed, which may be written in our coordinate system (with $\alpha=\alpha_0$ and using the usual rapidity variables $y_{\parallel}, y_{\perp}$) as

$$p^{\mu} = m(\mathrm{ch}(y_{\parallel}-\alpha)\mathrm{ch}\, y_{\perp}\,,\, \frac{1}{\tau}\mathrm{sh}(y_{\parallel}-\alpha)\mathrm{ch}\, y_{\perp}\,,\, \mathrm{sh}\, y_{\perp}\,, 0) \quad . \tag{5.7}$$

The proper-frame energy $\bar{E}$ is expressed in terms of the center-of-mass frame (of M) variable as

$$\bar{E} = m\left[\mathrm{ch}(y_{\parallel}-\alpha)\mathrm{ch}\, y_{\perp}\, \mathrm{ch}\xi - \mathrm{sh}\, y_{\perp}\, \mathrm{sh}\xi \cos\phi\right] \quad . \tag{5.8}$$

By putting (5.3), (5.7) and (5.8) into (5.6), one obtains

$$E\frac{dN}{d\vec{p}} = \frac{wm}{(2\pi)^3}\iiint_{T=T_d} \frac{\tau r\left[\operatorname{sh} y_\perp d\tau - \operatorname{ch}(y_\parallel-\alpha)\operatorname{ch} y_\perp dr\right] d\alpha\, d\phi}{\exp\left\{\frac{m}{T_d}\left[\operatorname{ch}(y_\parallel-\alpha)\operatorname{ch} y_\perp \operatorname{ch}\xi - \operatorname{sh} y_\perp \operatorname{sh}\xi\cos\phi\right]\right\}\pm 1} . \tag{5.9}$$

Despite the α-independence of $\frac{d^2N}{d\alpha d\xi}$ as given by eq. (5.5), the integrand above contains α-dependence which predominates over the actual α-dependence of $\frac{d^2N}{d\alpha d\xi}$ as discussed below (5.5), because of its sharp form.

VI. CONCLUDING REMARKS

In the present paper, we have developped an algorithm for solving the hydrodynamical transverse expansion of an initially flat ($\Delta << R$) and hot ($T_0 >> m_\pi$) disc of large mass M and obtained both the rapidity distribution of the fluid and the inclusive particle distributions which emerge from the expansion. Although our aim is applying this prescription first to studying hadron-hadron collision, it may evidently be used also for nucleus-nucleus collisions.

The basic assumption of the present receipt is the approximate validity of the one-dimensional solution as given by eq. (2.6) [and eq. (2.9)], which allows to separates the transverse from the longitudinal flows. The resulting couple of equations (3.4) for transverse flows are then put in the canonical form (4.5), which allows the integration along the characteristics, eq. (4.7), with the use of eqs. (4.8) and (4.9). This result is always valid as far as the assumption

above remains valid.

As for the initial conditions, we have assumed the fluid begins to expand at a time $t \cong \Delta \cong 0$, in accordance to the most orthodox view. If one assumes a model as discussed in [11], we think natural to include also the transverse rapidity distribution of fragments or of ε at $\tau=\tau_0$. Such a distribution would probably be more or less constant over R, but with a surface with a finite thickness which would increase with τ_0 (if $\vec{v}$ = const., as assumed by those authors). Although there is no apparent reason to being so, the expansion for $\tau < \tau_0 \sim 1\,\mathrm{fm}$, as calculated here may well simulate this thickness-widening effect. This is certainly the case in the one-dimensional approximation to treating nucleus-nucleus collisions in the central rapidity region, when $v = \frac{x}{t}$ is usually assumed and then the source is guessed by using $\frac{dN}{dy_\parallel}$ for nucleon-nucleon collision [11].

An explicit numerical computation will be reported in the forthcoming paper.

APPENDIX A

In this appendix, we calculate the surface of separation between regions of one-dimensional and three-dimensional flows. Such a surface is given as a characteristic surface of the system of equations (3.4), which describe three-dimensional flows. Following the standard method*, the equation of a characteristic surface is written

$$F(\tau,r) = 0 \quad , \tag{A.1}$$

where F is a function satisfying

$$\begin{vmatrix} \frac{\partial F}{\partial \tau} & -(1-c_0^2)\,\mathrm{sh}\xi\,\mathrm{ch}\xi\,\frac{\partial F}{\partial \tau} - (\mathrm{sh}^2\xi - c_0^2\mathrm{ch}^2\xi)\,\frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial r} & (\mathrm{ch}^2\xi - c_0^2\mathrm{sh}^2\xi)\,\frac{\partial F}{\partial \tau} + (1-c_0^2)\,\mathrm{sh}\xi\,\mathrm{ch}\xi\,\frac{\partial F}{\partial r} \end{vmatrix} = 0 \ . \tag{A.2}$$

By developping this equation and factorizing it, we may rewrite it as

$$\left[(\mathrm{ch}\xi - c_0\,\mathrm{sh}\xi)\,\frac{\partial F}{\partial \tau} + (\mathrm{sh}\xi - c_0\,\mathrm{ch}\xi)\,\frac{\partial F}{\partial r}\right]$$

$$\times\left[(\mathrm{ch}\xi + c_0\,\mathrm{sh}\xi)\,\frac{\partial F}{\partial \tau} + (\mathrm{sh}\xi + c_0\,\mathrm{ch}\xi)\,\frac{\partial F}{\partial r}\right] = 0 \quad ,$$

so

$$(\mathrm{ch}\xi \pm c_0\,\mathrm{sh}\xi)\,\frac{\partial F}{\partial \tau} \pm (c_0\,\mathrm{ch}\xi \pm \mathrm{sh}\xi)\,\frac{\partial F}{\partial r} = 0 \quad . \tag{A.3}$$

By dividing each of these equations by $\frac{\partial F}{\partial r}$, we obtain

*See for instance Ref. [18].

$$\frac{dr}{d\tau} = \pm\,\frac{c_0 \pm \mathrm{th}\xi}{1 \pm c_0\,\mathrm{th}\xi} \quad . \tag{A.4}$$

So, we have two characteristics of eqs. (3.4), which pass on each point. The first of these equations represents a surface which moves outward, whereas the second one corresponds to the one which goes inward (with respect to the fluid element). In Sec. IV, the system of eqs. (4.5), which is just another form of (3.4), will be integrated following these characteristics. In the particular problem of surface of separation that we are considering presently, we choose the minus sign in (A.4) and, by using the boundary condition, put $\xi=0$. Taking also the initial condition into account, (A.4) may be easily integrated and gives

$$r = -c_0\tau + R \quad , \tag{A.5}$$

which is our eq. (3.7) of Sec. III. Although it is clear enough that the other boundary of the three-dimensional-flow region is given by (3.6), remark that it may be obtained in a similar way, by taking the limit $\xi\to\infty$ of (A.4) with plus sign.

APPENDIX B

Consider the equations

$$\begin{cases} \dfrac{\partial\psi}{\partial\tau} + \dfrac{\partial\psi}{\partial r} + \dfrac{c_0^2}{1+c_0}\left[\dfrac{1}{r} - \dfrac{c_0}{\tau}\right] = 0 \quad , & \text{(B.1)} \\ \dfrac{\partial\phi}{\partial\tau} + \dfrac{\partial\phi}{\partial r} + \dfrac{c_0^2}{1+c_0}\left[\dfrac{1}{r} + \dfrac{c_0}{\tau}\right] = 0 \quad , & \text{(B.2)} \end{cases}$$

which we would like to solve assuming

$$\psi = \phi = 0 \quad \text{on} \quad r = R - c_0\tau \quad . \tag{B.3}$$

Take (B.1) first. Its characteristic as well as it solution ψ satisfy the following system of ordinary equations (in q):

$$\frac{d\tau}{1} = \frac{dr}{1} = \frac{d\psi}{-\dfrac{c_0^2}{1+c_0}\left(\dfrac{1}{r} - \dfrac{c_0}{\tau}\right)} = dq \quad . \tag{B.4}$$

Upon integration, this will give

$$\begin{cases} \tau = q + \tau_0 \quad , \\ r = q + r_0 \quad , \\ \psi = -\dfrac{c_0^2}{1+c_0}\left[\ell n(q+r_0) - c_0 \ell n(q+\tau_0)\right] + \psi_0 \quad , \end{cases} \tag{B.5}$$

or by eliminating the auxiliary variable q and now taking τ as the independent variable

$$\begin{cases} r = \tau - \tau_0 + r_0 \\ \psi = -\dfrac{c_0^2}{1+c_0}\, \ell n \dfrac{r}{\tau^{c_0}} + \psi_0 \quad . \end{cases} \tag{B.6}$$

Due to the boundary condition (B.3), we have

$$\begin{cases} r_0 = R - c_0\tau_0 \quad , \\ \psi = 0 \quad , \text{ when } \quad \tau = \tau_0 \quad . \end{cases} \tag{B.7}$$

From (B.6) and (B.7), it follows

$$\begin{cases} \tau_0 = \dfrac{\tau + R - r}{1 + c_0} \quad , \\ \psi_0 = \dfrac{c_0^2}{1+c_0}\, \ell n \dfrac{R - c_0\tau_0}{\tau_0^{c_0}} \quad . \end{cases} \tag{B.8}$$

The insertion of this into (B.6) gives finally

$$\psi = \frac{c_0^2}{1+c_0}\, \ell n \left[\frac{R - c_0(\tau - r)}{(1+c_0)r}\left(\frac{(1+c_0)\tau}{R+\tau-r}\right)^{c_0}\right] \quad . \tag{B.9}$$

The integration of (B.2) is entirely similar so we do not repeat it here. The result is

$$\phi = \frac{c_0^2}{1-c_0}\, \ell n \left[\frac{R - c_0(\tau - r)}{(1+c_0)r}\left(\frac{R+\tau-r}{(1+c_0)\tau}\right)^{c_0}\right] \quad . \tag{B.10}$$

(B.9) and (B.10) are precisely the ultrarelativistic solution given in Section IV.

REFERENCES

[1] L.D. Landau, Izv. Akad. Nauk SSSR Ser. Fiz. 17 (1953) 51; Collected papers, ed. D. Ter Haar (Pergamon, Oxford, 1965) p. 569.

[2] E.V. Shuryak, Phys. Rep. 61 (1980) 71.

[3] I.M. Khalatnikov, Zhur. Eksp. Teor. Fiz. 27 (1954) 529.

[4] Y. Hama, Nuovo Cimento 46A (1978) 569; Y. Hama, Lett. Nuovo Cimento 24 (1979) 530.

[5] Y. Hama and F.W. Pottag, Rev. Bras. Fís. 12 (1982) 247.

[6] UA1 Collab., G. Arnison et al., Phys. Lett. 118B (1982) 167.

[7] Y. Hama and F.S. Navarra, Phys. Lett. 129B (1983) 251.

[8] G.A. Milekhin, Sov. Phys. JETP 35 (1959) 829.

[9] I. Yotsuyanagi, Prog. Theor. Phys. 55 (1976) 539.

[10] G. Baym et al., Nucl. Phys. A407 (1983) 541.

[11] K. Kajantie and L. McLerran, Phys. Lett. 119B (1982) 203; J.D. Bjorken, Phys. Rev. D27 (1983) 140.

[12] Y. Hama, Phys. Rev. D19 (1979) 2623.

[13] Y. Hama and F.S. Navarra, Correlation between Charged--Particle Multiplicities and Pseudo-Rapidity Distributions in Hydrodynamical Cluster Model, to be published on Zeit. Phys. C.

[14] P. Carruthers, Nucl. Phys. A418 (1984) 501c.

[15] B. Lukács and K. Martinás, Thermodynamics of systems of non-equilibrium momentum distribution, preprint KFKI-1984-25, Budapest (Hungary).

[16] L.D. Landau and E.M. Lifshitz, Théorie du Champ, Moscow: Editions Mir, 1966.

[17] F. Cooper and G. Frye, Phys. Rev. D10 (1974) 186.

[18] I.G. Petrovskii, Partial Differential Equations, Philadelphia: W.B. Saunders Company, 1967.

FIGURE CAPTIONS

Fig. 1 - Contour of the fluid as seen in the center-of-mass frame of M, at an instant $t < R/c_0$. The broken lines indicate the boundary between the three-dimensional-flow region (outside) and the one-dimensional-flow region (inside).

Fig. 2 - (Proper-) time evolution of the boundaries among the vacuum, the "trivial" three-dimensional-flow region (II), non-trivial three-dimensional-flow region (III) and the one-dimensional-flow region (I). The two families of characteristics given by eqs. (4.7) are also schematically shown.

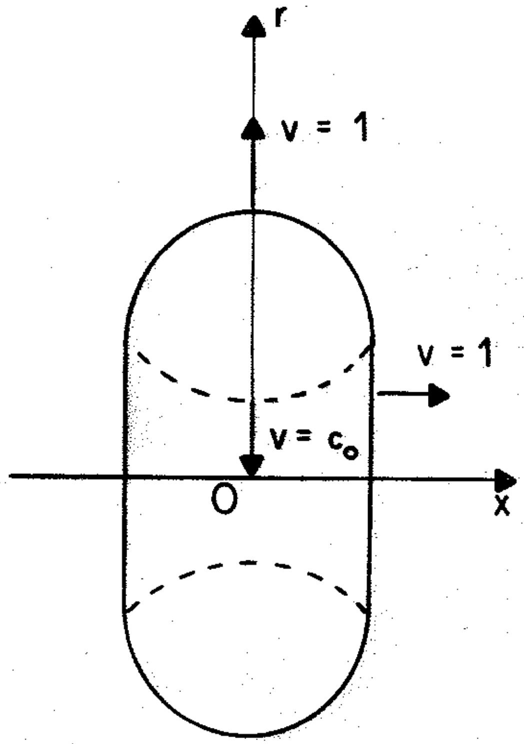

Fig. 1

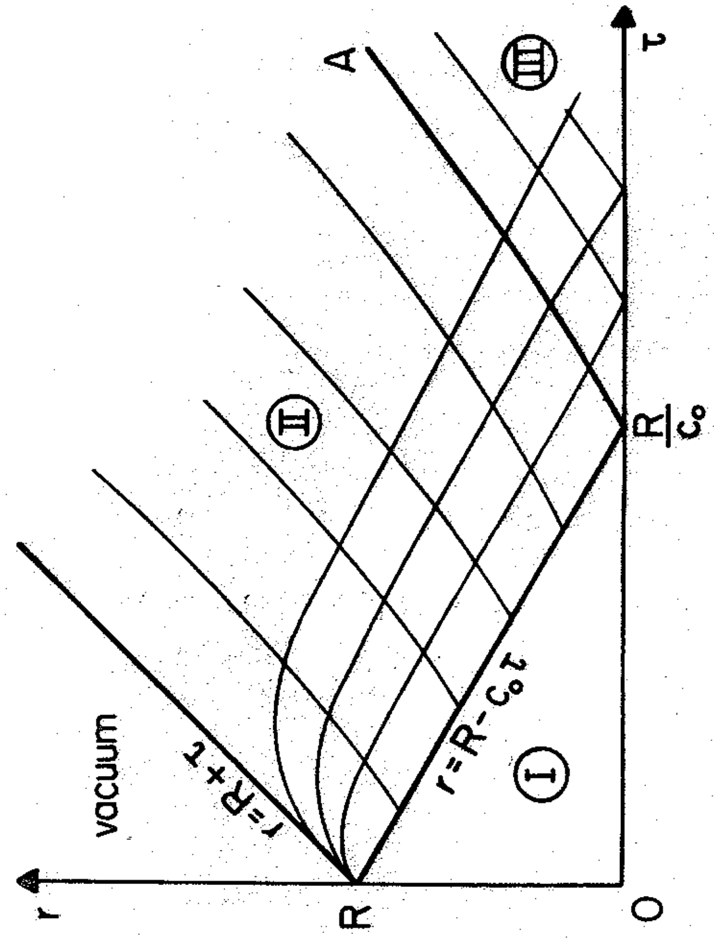

Fig. 2