UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

# PUBLICAÇÕES

IFUSP/P-521



## SCALAR FIELDS IN KASNER-TYPE SPACE-TIMES: TIME DEPENDENT VACUUM EXPECTATION VALUES

by

H. Fleming
Instituto de Física, Universidade de São Paulo

and

V.L.R. da Silveira
Departamento de Física, Universidade de Brasília, Brasil

Março/1985

SCALAR FIELDS IN KASNER-TYPE SPACE-TIMES:
TIME DEPENDENT VACUUM EXPECTATION VALUES

H. Fleming*
Instituto de Física, Universidade de São Paulo,
São Paulo, Brasil

and

V.L.R. da Silveira
Departamento de Física, Universidade de Brasília,
Brasil

ABSTRACT

Non-constant vacuum expectation values of scalar fields in curved space-times generate important physical consequences. We here show this to be the case for a class of spatially homogeneous and anisotropic metrics.

*FAPESP fellow.



The vast literature on field theories in curved spaces[1] is justified by the great difficulty of the equations of gravity coupled to matter fields. In fact, even in this approximation (gravity treated as a background metric), life is not easy. On the other hand, quite interesting and unusual phenomena appear in rather simple settings. This is, for instance, the case of massless, scalar, conformal-coupled fields propagating on a Friedmann universe of the open type[2], where the non-constancy (i.e., time dependence) of the vacuum expectation value of the scalar field gives rise to a host of phenomena, and even to conceivable phenomenological consequences[3].

We start by examining, quite generally, how these abnormal vacuum expectation values can appear in General Relativity or other theories of gravity. Subsequently it will be shown that this is the case for a class of homogeneous and anisotropic expanding universes which have been studied by Kasner[4] and Lifshitz et al.[5], and are potentially useful in cosmology.

We characterized the (curved) vacuum as follows[6], taking Minkowski vacuum as a model. The vacuum should be invariant under the transformations generated by the Killing fields[7] of its metric tensor. That is, given a coordinate transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \qquad (1)$$

where $\xi^{\mu}$ is an infinitesimal 4-vector field, $g_{\mu\nu}$ must obey the equation

$$L_{\xi}\, g_{\mu\nu} = 0 \qquad (2)$$

where $L_{\xi}$ stands for the Lie derivative with respect to the vector field $\xi^{\mu}(x)$. This is the same as requiring that

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \quad , \tag{3}$$

and, in this situation, one says that $g_{\mu\nu}$ is form-invariant under the transformations (1). The complete characterization is this: every tensor field present in the vacuum must be form invariant. In particular,

$$L_{\xi} \langle 0|T_{\mu\nu}(x)|0\rangle = 0 \tag{4}$$

$T_{\mu\nu}$ being the "matter" energy-momentum tensor. This immediately suggests that

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \Lambda(x) g_{\mu\nu}(x) \tag{5}$$

where $\Lambda(x)$ is a scalar field such that

$$L_{\xi} \Lambda(x) \equiv \xi^{\mu}(x)\, \partial_{\mu} \Lambda(x) = 0 \quad . \tag{6}$$

The connection between the "cosmological constant" $\Lambda(x)$ and $v \equiv \langle 0|\phi(x)|0\rangle$, $\phi(x)$ being a scalar ("Higgs") field present in the matter Lagrangian, is known since a long time[8]. A time dependent $v$ can only occur when $\Lambda$ is itself time dependent.

Now, if General Relativity is used in its full attire, a consequence of Einstein's equation is that

$$T^{\mu\nu}{}_{;\mu} = 0 \tag{7}$$

which, combined with $g^{\mu\nu}{}_{;\mu} = 0$, leads, used in Eq.(5), to

$$\Lambda(x) = \text{constant} \quad . \tag{8}$$



This, however, is not the case when $g_{\mu\nu}$ is a priori fixed, as in the case under analysis. This approximation truncates General Relativity, in the sense that $T^{\mu\nu}{}_{;\mu}$ does not necessarily vanish, as the matter Lagrangian in question does not participate of Einstein's equation. Therefore the restrictions on $\Lambda(x)$ come only from the symmetries (Killing fields) of the metric. Suppose the metric is such that no time-like Killing field exists, whereas three independent space-tike ones do exist. In this case one can, in principle, have

$$\Lambda = \Lambda(t) \quad . \tag{9}$$

Whether the time dependence actually exists must be decided by closer examination of the equations of motion.

Kasner spaces have 3-dimensional sections that are homogeneous but anisotropic. Near the singularity ($t=0$) it suffices to consider the empty space field equations[5]. The metric is given by

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2 \tag{10a}$$

$$p_1 \neq p_2 \neq p_3 \tag{10b}$$

$$p_1 + p_2 + p_3 = 1 \tag{10c}$$

$$p_1^2 + p_2^2 + p_3^3 = 1 \tag{10d}$$

the space being anisotropic because of Eq.(10b). We are using the notation and conventions of Ref.(5), and putting $c=1$.

We now consider a $\lambda\phi^4$ scalar field theory defined on this space-time. If $\underline{t}$ is kept small, the result is independent of the matter distribution. However, the case of the void

Kasner space-time is interesting in its own right. The reader should keep in mind that the Kasner space, in this paper, is a background field; its metric is not due to the $\lambda\phi^4$ energy-momentum tensor, but to some other source.

The lagrangian is written as

$$L = \sqrt{-g}\left\{g^{ik}\frac{\partial\phi^*}{\partial x^i}\frac{\partial\phi}{\partial x^k} + \frac{R}{6}\phi^*\phi - \frac{\lambda}{6}(\phi^*\phi)^2\right\} \tag{11}$$

$R$ standing for the scalar curvature of space-time. For arguments favoring the presence of the term $\frac{R}{6}$, see Ref. (2) and references cited therein. In Eq. (11), $R=0$, as the space is void. However, the presence of the term $\frac{R}{6}\phi^*\phi$ is far from trivial, as the term $\frac{\delta R}{\delta g^{ik}}$, which contributes a term to the momentum-energy tensor, does not vanish for $R=0$.

One has

$$\delta R = \left\{R_{ik} - \frac{1}{2}(\nabla_i\nabla_k + \nabla_k\nabla_i) + g_{ik}\Box\right\}\delta g^{ik} \tag{12}$$

where $\nabla_i$ is the i-th component of the covariant derivative, and

$$\Box \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\ g^{ik}\ \frac{\partial}{\partial x^k}\right) . \tag{13}$$

The energy-momentum tensor can now be computed and reads

$$T_{ik} = \frac{\partial\phi^*}{\partial x^i}\frac{\partial\phi}{\partial x^k} + \frac{\partial\phi^*}{\partial x^k}\frac{\partial\phi}{\partial x^i} - g_{ik}\frac{L}{\sqrt{-g}} + \frac{1}{4}\left[\nabla_i\nabla_k - g_{ik}\Box\right](\phi^*\phi) . \tag{14}$$

This derivation follows rather closely the one



found in Ref. (9), and is omitted here.

Corresponding to the lagrangian of Eq. (11) one has the Euler-Lagrange equations

$$\Box\phi + \frac{\lambda}{3}\phi^*(x)\,\phi^2(x) = 0 . \tag{15}$$

Homogeneity of 3-space means that there are Killing fields corresponding to (infinitesimal) translations along three independent spatial directions. One expects the vacuum to be invariant under these translations.

A completely different situation is met when we examine translations along the time direction, the metric being time-dependent. In fact, as shown in Ref. (5), the Kasner universe is an expanding one.

This means that

$$\langle 0|\phi(x)|0\rangle = v(t) \tag{16}$$

where the time dependence of $v$ must the determined by the equations of motion. Using the invariance of the theory under charge conjugation we get the following, semi-classical approximation for Eq. (15):

$$\Box v(t) + \frac{\lambda}{3}v^3(t) = 0 \tag{17}$$

that is,

$$\frac{d^2v}{dt^2} + \frac{1}{t}\frac{dv}{dt} + \frac{\lambda}{3}v^3 = 0 . \tag{18}$$

Putting $v = \frac{u}{t}$, and $t = e^x$, one has, instead of (18),

$$\frac{d^2u}{dx^2} - 2\,\frac{du}{dx} + u + \frac{\lambda}{3}\,u^3 = 0 \quad . \tag{19}$$

This is Duffing's equation, and its stability (in the Liapunov sense) has been studied in detail. In Ref.(10) it is shown that the solution $u=0$ (that is, $\langle 0|\phi(x)|0\rangle = 0$) is unstable, as the coefficient of the "dissipative" term $\frac{du}{dx}$ is negative. So, the vacuum expectation value of the scalar field cannot be zero, and the vacuum does not preserve the invariance $\phi \to -\phi$ of the lagrangian. No other solutions of Eq.(19) exist corresponding to constant $\underline{v}$. Therefore, $\underline{v}$ must be a function of the time.

To get some more information about $v(t)$, let us examine the energy density of the field. This is given by $T_{oo}$ calculated with the vacuum field configuration $v(t)$. Using (14) one gets for the energy density

$$\varepsilon(t) = \dot{v}^2 + \frac{\lambda}{6}\,v^4 + \frac{2}{3}\,v\dot{v}\,\frac{1}{\sqrt{-g}}\,\frac{d}{dt}\sqrt{-g} \quad . \tag{12}$$

As the universe is an expanding one, $\sqrt{-g}$, the volume element, increases with time. So,

$$f^2(t) \equiv \frac{d}{dt}\log\sqrt{-g} > 0 \tag{13}$$

and

$$\varepsilon(t) = \dot{v}^2 + \frac{\lambda}{6}\,v^4 + \frac{2}{3}\,v\dot{v}\,f^2(t) \quad . \tag{14}$$

Suppose $v\dot{v}$ is not negative. Then, $\varepsilon(t)$ has its minimum (as a function of $v$) for $v=0$. But this would mean $\langle 0|\phi(x)|0\rangle = 0$, which we have seen not to be stable. So, $v\dot{v}$



must be negative. Therefore, $|v|$ decreases with $t$, a result analogous to that of Ref.(2).

For gravity theories like the ones of Ref.(11), this would imply, under certain circumstances, a Newton's "constant" that increases with time; when symmetry restoration is possible, our results (both here and in Ref.(2)) imply a time dependence also of the "critical temperature", as discussed in Ref.(12).

In conclusion, time-dependent vacuum expectation values of scalar fields propagating on homogeneous space-time fields are not exotic objects, but can happen in quite usual circumstances.

REFERENCES

(1) N.D. Birrell, P.C.W. Davies, "Quantum Fields in Curved Space", Cambridge University Press, 1982.

(2) A.A. Grib, V.M. Mostepanenko, JETP Lett. 25, 277 (1977); V.M. Frolov, A.A. Grib, V.M. Mostepanenko, Theor. Math. Phys. (USSR) 33, 42 (1977); H. Fleming, J.L. Lyra, C.P.C. Prado, V.L.R. Silveira, Lett. Nuovo Cimento 27, 261 (1980).

(3) H. Fleming, Physics Letters B (1984).

(4) E. Kasner (1922), as cited in Ref. (5), pg. 388 .

(5) L.D. Landau, E.M. Lifshitz, "The Classical Theory of Fields", 4th revised English edition, Pergamon Press (1975).

(6) H. Fleming, "The Symmetries of the Vacuum", São Paulo preprint (1985).

(7) See, for instance, S. Weinberg, "Gravitation and Cosmology", Wiley, 1972.

(8) See, for instance, Ya.B. Zel'dovich, I.D. Novikov, "Structure and Evolution of the Universe", Chicago University Press (1984).

(9) N.A. Chernikov, E.A. Tagirov, Ann. Inst. Henri Poincaré 9A, 109 (1968).

(10) R.A. Struble, "Nonlinear Differential Equations", McGraw-Hill, New York (1962), page 135.

(11) A. Zee, Phys. Rev. Lett. 42, 417 (1979); 44, 703 (1980); L. Smolin, Nucl. Phys. B160, 253 (1979).

(12) H. Fleming, V.L.R. da Silveira, Nuovo Cimento 58B, 208 (1980).