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PERIOD DOUBLING PHENOMENON IN A CLASS OF TIME DELAY EQUATIONS

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ABSTRACT

We investigate the properties of the solutions of a nonlinear time delayed differential equation (infinite dimension) as function of two parameters: the time delay τ and another parameter A (nonlinearity). After a Hopf bifurcation period doubling may occur and is characterized by Feigenbaum's δ. A strange attractor is obtained after the period doubling cascade and the largest Lyapunov exponent is calculated indicating that the attractor has low dimension. The behaviour of this Liapunov exponent as function of τ is different from its behaviour as function of A.

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Rich dynamical behaviour is a common feature of nonlinear differential equations and iterations of endomorphisms depending on parameters[1-3]. Period doubling bifurcation is a common instability found in nonlinear dynamical systems: by varying a parameter a stable periodic orbit becomes unstable and gives rise to a new periodic orbit having twice its period. A cascade of period doubling is observed as a function of the parameter until a critical value of the parameter is reached beyond which "chaos" takes place for a certain range of parameter.

The universality of the period doubling bifurcation sequence is not only qualitative, it also is quantitative[2-5]. Feigenbaum[4] has found that the period doubling sequence for unimodal maps is characterized by two universal constants, $\alpha = 2.5029078...$ and $\delta = 4.6692016...$. The constant α is the asymptotic value of the scaling of the transformation while δ is the asymptotic value of the ratio between the ranges of parameter values in which successive periodic solutions appear and then become unstable.

First order nonlinear differential equations with time delay may exhibit even richer dynamical behaviour (they are infinite dimensional system) as they depend on at leat two parameters. Sometime ago, Perez, Malta and Coutinho[6] proposed the following equation, with time delay τ,

$$\frac{dN_t}{dt} = b(N_{t-\tau})N_{t-\tau} - m(N_t)N_t \qquad (1)$$

to describe isolated population of Drosophila Sturtevantis flies

(N_t is the number of flies per unit volume at time t).

The functions $b(N_t)$ (birth rate per capita) and $m(N_t)$ (death rate per capita) have the general behaviour shown in figure 1a and the functions $f(N_t) = b(N_t)N_t$ and $g(N_t) = m(N_t)N_t$ should have the behaviour in figure 1b. In terms of $f(N_t)$ and $g(N_t)$ equation (1) becomes

$$\frac{dN_t}{dt} = f(N_{t-\tau}) - g(N_t) \quad . \tag{2}$$

Sufficient conditions[6] for the stability of the equilibrium population $\bar{N}(f(\bar{N}) = g(\bar{N}))$ are obtained from a linearization procedure around this equilibrium point. Violation of these sufficient conditions provides necessary conditions[6] for an oscillatory behaviour of the population. It was established in reference 6 that as a the time delay parameter τ is varied the solution $\bar{N}$ becomes unstable via a Hopf[8] bifurcation at a value τ_H and gives rise to an oscillatory solution (this type of behaviour was observed experimentally by Tadei and Mourão[7]).

The value τ_H is given by[6] (the prime indicates derivative with respect to N_t)

$$\tau_H = \frac{\theta}{\sqrt{(f'(\bar{N}))^2 - (g'(\bar{N}))^2}} \quad , \tag{3a}$$

with θ given by

$$\theta = \sin^{-1}\left[-\sqrt{1 - \left(\frac{g'(\bar{N})}{f'(\bar{N})}\right)^2}\right] \quad , \quad \frac{\pi}{2} < \theta < \pi \quad . \tag{3b}$$

Equation (2) has been numerically solved for $g(N_t) = N_t$ and the following two functions $f(N_t)$:

$$\begin{aligned} f_1(N_t) &= A \sin N_t \quad , \\ f_2(N_t) &= N_t(A - .01\, N_t) \quad . \end{aligned} \tag{4}$$

The result of the calculations for several sets of parameters (A,τ) confirmed that the equilibrium solution undergoes a Hopf bifurcation at the parameter value τ_H generating an oscillatory solution. Continuing the variation of the parameter τ beyond τ_H, the oscillations seems to have a single frequency for a certain range of τ but, in general, it starts to dèvelop a second bump at certain parameter value suggesting the presence of period doubling phenomenon.

In this work we present the results of a Fourier analysis of the solutions of equation (2) (with $g(N_t) = N_t$) for f_1 and f_2 given in (4) in the following cases:

i) fixing parameter A and varying the time delay τ;

ii) fixing τ and varying A.

From the numerical analysis it is observed that fixing τ and varying A, period doubling always occurs after the Hopf bifurcation but when A is fixed and τ varied, there exists a minimum value $A = A_{min}$ for period doubling to occur after the Hopf bifurcation. For $A < A_{min}$ as τ is increased beyond τ_H the Fourier analysis of the solution shows that there is always a single dominant frequency present and the period T of the oscillations is approximately equal to the corresponding (2τ) value (see figures 2a and 2b). For function f_1 (4), $A_{min} \cong 2.70$ and for f_2 (4), $A_{min} \cong 3.25$.

Results with τ varying are displayed in tables 1a and 1b for two different values of A (both $> A_{min}$). The values τ_k given in the 3rd column of these tables correspond to the values of τ at which the k^{th} period doubling occurs (see the power spectrum shown in figures 3a and 3b and curves $N_t \times N_{t-\tau}$ in figures 3c and 3d). The ratios,

$$\delta_k^\tau = \frac{\tau_{k+1} - \tau_k}{\tau_{k+2} - \tau_{k+1}} , \tag{5}$$

(given in the 4th column of tables 1 and 2) are very close to the universal (asymptotic) value $\delta \approx 4.6692016...$[4] (in fact we should have $\delta = \lim_{k\to\infty} \delta_k^\tau$). The value of τ at which "chaos" starts is τ_∞ and is given in tables 1a and 1b for each A value considered.

Results with A varying are given in table 2. After the Hopf bifurcation at A_H period doubling is always observed and occurs at the values A_k given in the 3rd column of table 2. The ratios,

$$\delta_k^A = \frac{A_{k+1} - A_k}{A_{k+2} - A_{k+1}} , \tag{6}$$

are given in the 4th column. Again, we expect that $\delta = \lim_{k\to\infty} \delta_k^A$.

The values A_∞ given in tables 2 correspond to the values at which "chaos" starts.

It should be remarked that the critical values (A_∞ and τ_∞) were determined using the asymptotic δ value (see figures 4a and 4b).

The word "chaos" has been employed up to now without specifying its meaning precisely. We have adopted the most widely accepted concept at the moment: a deterministic system is chaotic if nearby points in phase space separate at an exponential rate on average. This property of sensitivity on initial conditions gives rise to large positive Lyapunov exponents[9]. The largest exponent λ_L (also called characteristic exponent) is used to characterize the chaos. Other useful and related properties are positive metric entropy and nonintegral dimension[10]. In figures 5a and 5b we show the behaviour of λ_L as function of A and τ respectively in the case of function f_1 given in (4). For $\tau(A)$ in the interval $\tau_H < \tau < \tau_\infty$ ($A_H < A < A_\infty$), λ_L vanishes and for $\tau < \tau_H$ ($A < A_H$) λ_L is negative. Figures 5 are qualitatively reproduced if f_2 is used instead of f_1 (or any function having the same characteristics of f_1 and f_2).

From figures 5a and 5b we see that the behaviour of λ_L as function of τ is completely different from the behaviour of λ_L as function of A and this arises the question: what is the behaviour of λ_L for infinite dimensional systems depending on a single parameter? Also, what is the difference between the chaos for parameter A varying and the chaos for parameter τ varying? In order to clarify these points a study is being made concerning creation of information (metric entropy) and the dimension of chaotic attractors for the equations considered here.

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TABLE CAPTIONS

Table 1 - 1a) $f = f_2$, $A = 4.0$ fixed, $\tau_H = 1.209$.

1b) $f = f_1$, $A = 3.3$ fixed, $\tau_H = 0.965$.

Table 2 - 2a) $f = f_1$, $\tau = 5.0$ fixed, $A_H = 2.360$.

2b) $f = f_2$, $\tau = 30.0$ fixed, $A_H = 3.005$.

FIGURE CAPTIONS

Figure 1 - 1a) Birth rate per head $b(N)$ and death rate per head $m(N)$. The point $\overline{N}$ is the equilibrium point.

1b) The general behaviour of $f(N)$ and $g(N)$.

Figure 2 - 2a) Power spectrum for $f = f_2$, $\tau = 50.0$, $A = 3.20$.

2b) $x(t) \times x(t-\tau)$ for $f = f_2$, $\tau = 50.0$, $A = 3.20$.

Figure 3 - 3a) Power spectrum for $f = f_2$, $A = 4.0$, and $\tau_1 < \tau = 2.85 < \tau_2$.

3b) Power spectrum for $f = f_2$, $A = 4.0$, and $\tau_2 < \tau = 3.16 < \tau_3$.

3c) $N_t \times N_{t-\tau}$ for $f = f_2$, $A = 4.0$, and $\tau_1 < \tau = 2.85 < \tau_2$.

3d) $N_t \times N_{t-\tau}$ for $f = f_2$, $A = 4.0$, and $\tau_2 < \tau = 3.16 < \tau_3$.

Figure 4 - 4a) Power spectrum for $A = 2.90 > A_\infty$ and $\tau = 5.0$.

4b) $N_t \times N_{t-\tau}$, $A = 2.90$ and $\tau = 5.0$.

Figure 5 - 5a) Largest Lyapunov exponent $\times \tau$ for $f = f_1$, $A = 3.3$.

5b) Largest Lyapunov exponent $\times A$ for $f = f_1$, $\tau = 5.0$.

TABLE 1a

τ	T	τ_k	δ_k^τ
1.2100	3.6		
2.4000	6.4		
2.5000	6.6	$\tau_1 = 2.5111$	
2.6000	13.6		
3.0900	15.8	$\tau_2 = 3.1000$	$\delta_1 \cong 4.730$
3.1060	31.8		
3.2240	33.6	$\tau_3 = 3.2245$	$\delta_2 \cong 4.560$
3.2280	64.6		
3.2510	65.0	$\tau_4 = 3.2518$	$\delta_3 \cong 4.550$
3.2525	133.2		
3.25783	242.6	$\tau_5 = 3.25782$	$\rightarrow \tau_\infty \cong 3.26$

TABLE 1b

τ	T	τ_k	δ_k^τ
1.00	3.1		
1.80	5.1	$\tau_1 \cong 1.8719$	
2.0	11.1		
2.36	25.5	$\tau_2 \cong 2.2792$	$\delta_1 \cong 3.56$
2.405	51.9	$\tau_3 \cong 2.3937$	$\delta_2 \cong 5.02$
2.419	104.3	$\tau_4 \cong 2.4165$	$\rightarrow \tau_\infty \cong 2.42$

TABLE 2a

A	T	A_k	δ_k^A
2.500	11.8		
2.7505	11.7	$A_1 = 2.7501$	
2.8400	23.3		
2.8500	46.6	$A_2 = 2.8411$	$\delta_1 \cong 4.29$
2.8670	93.1	$A_3 = 2.8623$	$\delta_2 \cong 4.08$
2.8680	186.2	$A_4 = 2.8675$	$\delta_3 \cong 4.75$
2.86885	372.5	$A_5 = 2.86855$	$\rightarrow A_\infty \cong 2.87$

TABLE 2b

A	T	A_k	δ_1^A
3.2500	61.8		
3.5000	123.2	$A_1 = 3.4396$	~ 4.659
3.5500	245.8	$A_2 = 3.5449$	
3.5700	491.6	$A_3 = 3.5675$	$\rightarrow A_\infty \cong 3.57$

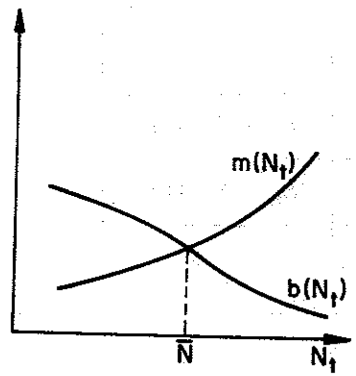

FIGURE 1a

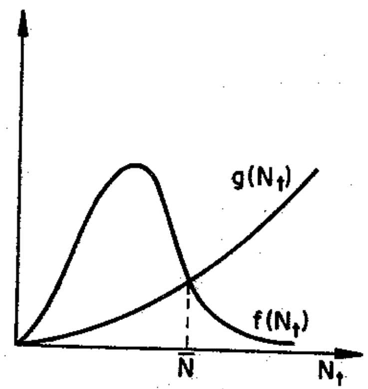

FIGURE 1b

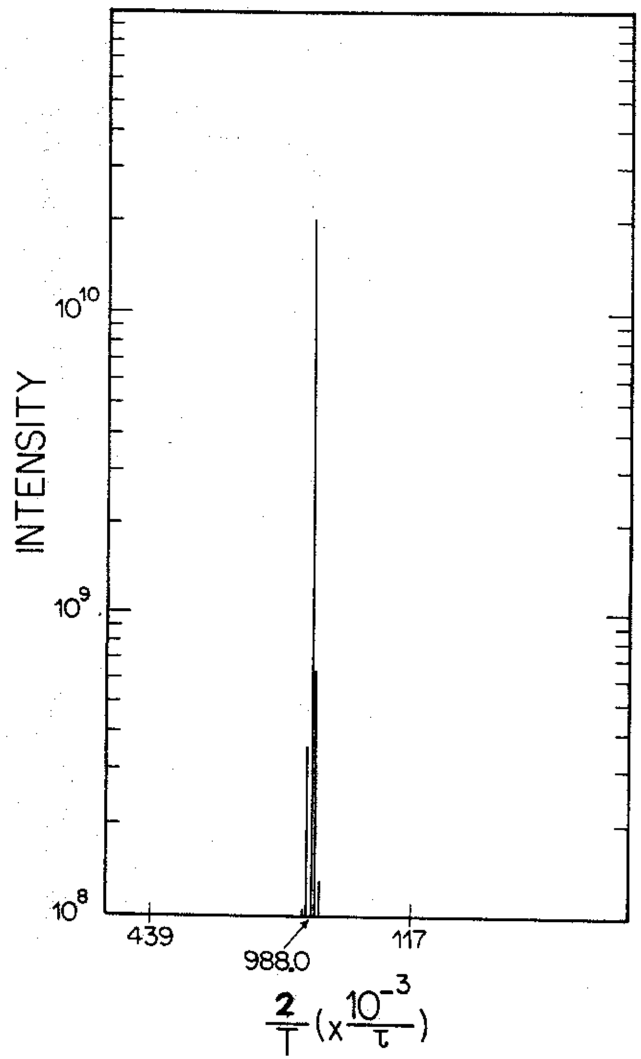

FIGURE 2a

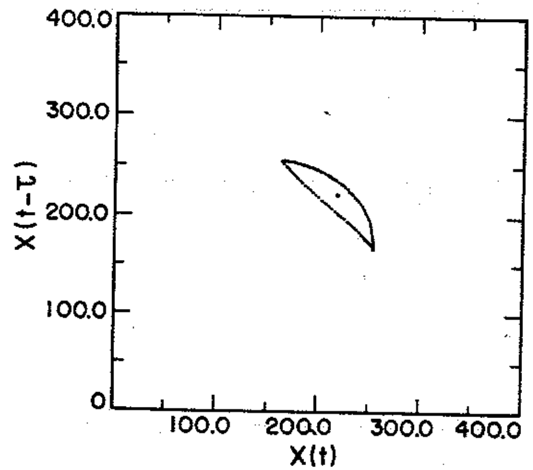

FIGURE 2b

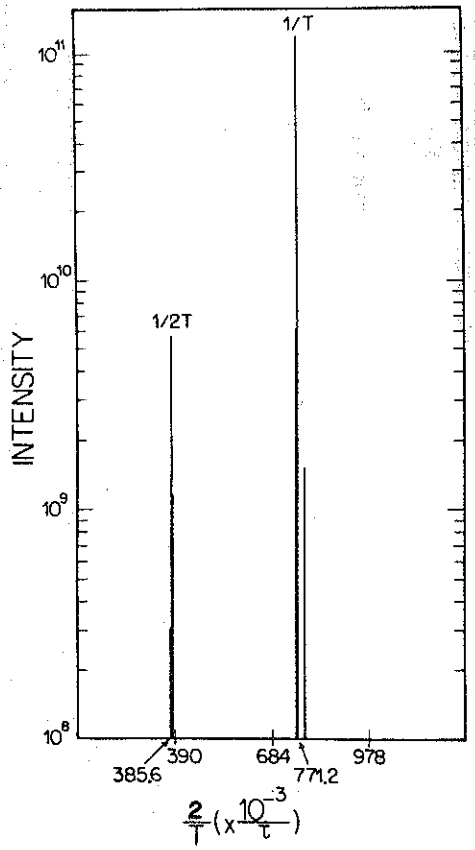

FIGURE 3a

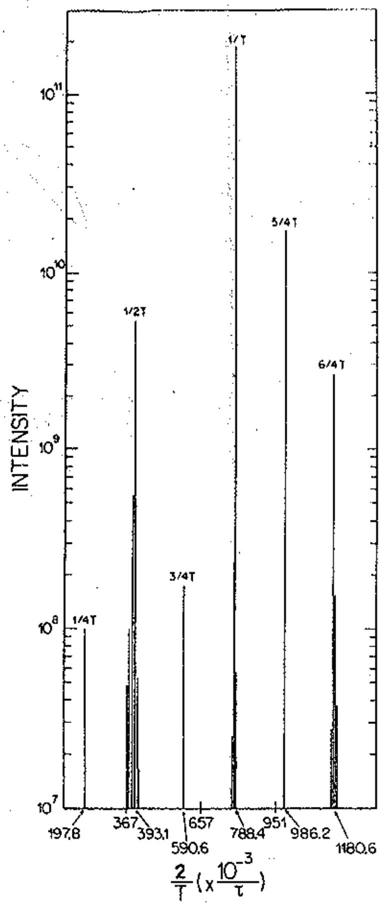

FIGURE 3b

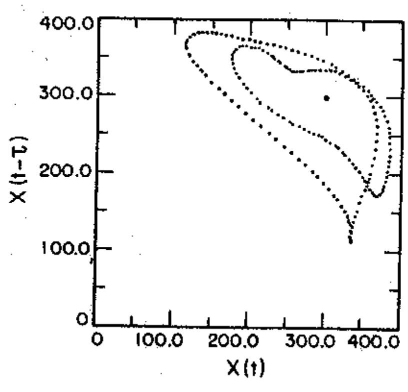

FIGURE 3c

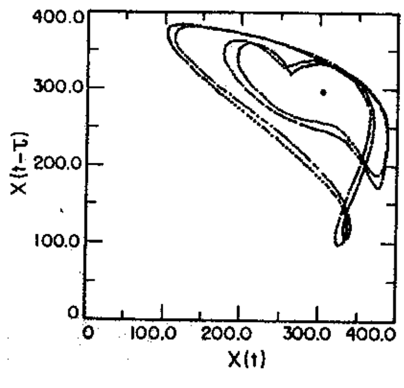

FIGURE 3d

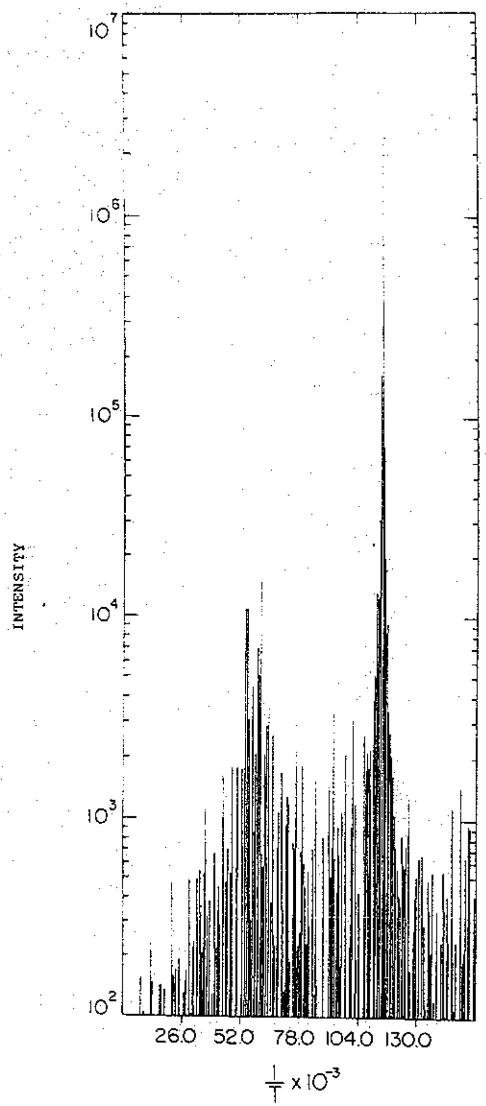

FIGURE 4a

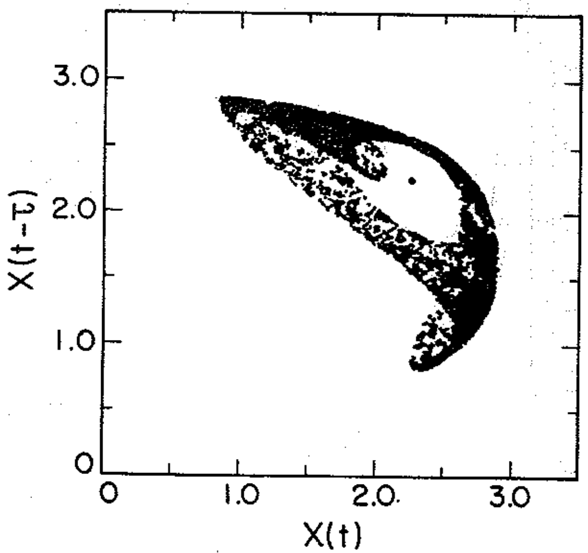

FIGURE 4b

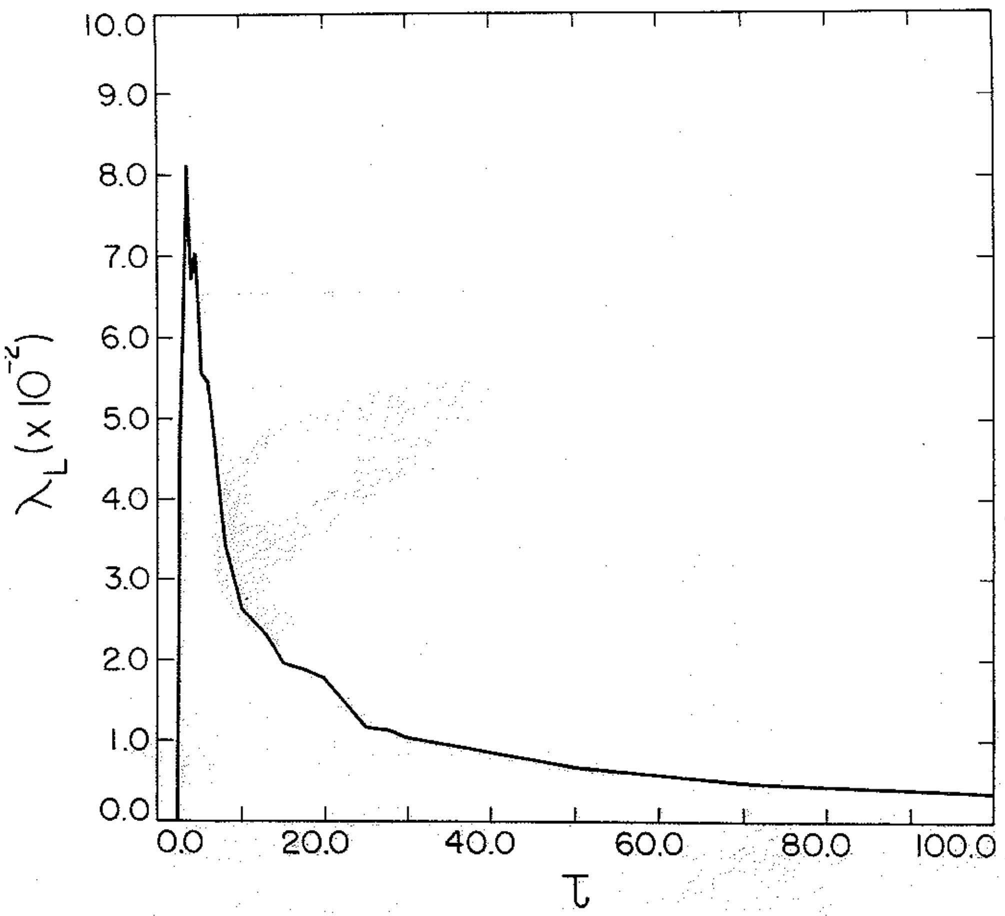

FIGURE 5a

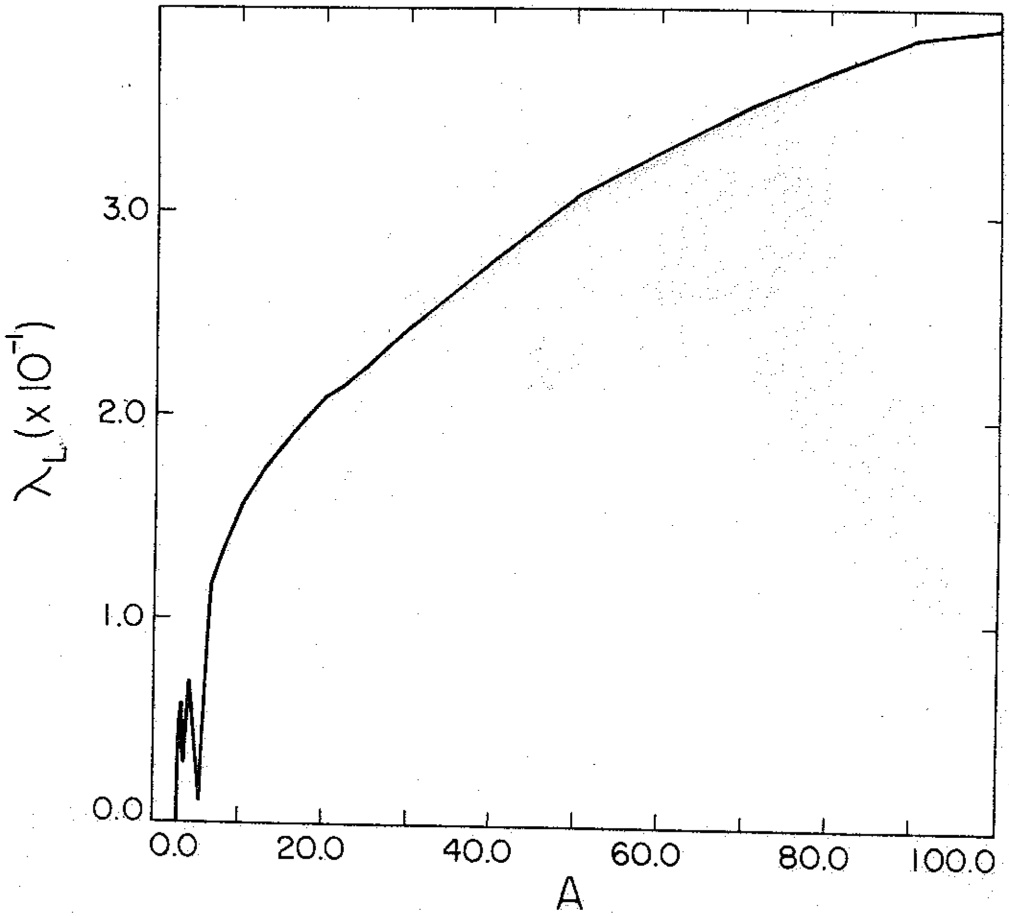

FIGURE 5b