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INFRARED BEHAVIOUR OF THREE AND FOUR GLUON VERTICES IN YANG-MILLS THEORY

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ABSTRACT

We study in a general covariant gauge the structure of the 3-point function with one and two external gluons on-shell. The contributions which result in one loop approximation are expressed in terms of simple functions containing collinear and soft singularities. Furthermore we analyse the contributions associated with the 4-point vertex when all external gluons are on-shell. As an application of these results, we study the infrared structure of gluon-gluon scattering amplitude.

I - INTRODUCTION

As is well known, the infrared singularities do not cancel in non-abelian gauge theories [1] for processes containing at least two coloured particles in the initial state. This property was first discovered in connection with the Drell-Yan process evaluated to 4^{th} order [2] and subsequently confirmed also by other investigations [3]. Afterwards this result was generalized to all leading orders [4] and recently it has been proved to all orders in perturbative QCD [5]. It was in connection with this work that a study of the infrared structure of Green's functions became relevant, since these appear as subdiagrams of higher order QCD graphs contributing to the Drell-Yan process.

Of course, the study of Yang-Mills vertex functions is interesting also in other connections and for this reason it has been considered by other authors. In reference [6] the 3-point function was evaluated at a particular off-shell symmetric point in a general covariant gauge and in reference [7] it has been calculated in the Feynman gauge when two of its external gluons are on shell. These calculations were done in the one-loop approximation using the dimensional regularization scheme [8] in a space-time of dimension $n=4+2\epsilon$. In general for arbitrary ϵ this vertex cannot be evaluated in terms of known functions when all gluons are off-shell. Similarly, many aspects of the contributions relevant to the calculation of the 4-point Green's function have been investigated in the literature [9].

In this paper we consider in the one-loop approximation the 3-point vertex in a general covariant gauge for the case when at least one external gluon is on-shell, which is relevant to the study of its infrared structure. Its contributions can be expressed analytically for arbitrary values of the parameter ε in terms of elementary functions. In section II we discuss our approach and organization of the calculation, which was done using the algebraic computational method SCHOONSHIP, in terms of a basic set of parametric integrals. In section III we evaluate explicitly these integrals when one external gluon is on shell. Some aspects of the results obtained in this case are analysed in connection with the Slavnov-Taylor identities for the 3-gluon vertex [10]. In section IV we discuss the structure of its infrared singularities in all relevant cases. When only one external gluon is on shell we obtain single poles which result from collinear singularities. With two gluons on shell these singularities operate together with infrared soft divergences yielding double poles. Finaly, in section V, we use these results to study the infrared structure of gluon-gluon scatering amplitude, with all external momenta on shell and transverse, which represents a gauge invariant process. To this end, we investigate first the structure of the corresponding reducible 4-point function and then we extend the analysis to the case involving the irreducible 4-point vertex. We find for the amplitude a Bose and gauge invariant expression which exhibits single and double pole infrared singularities.

II - THREE-GLUON FUNCTION IN A GENERAL COVARIANT GAUGE

The Feynman diagrams contributing in the one-loop approximation to the 3-gluon irreducible vertex are shown in Figure I.

The Feynman rules [11] for calculating the contributions of these diagrams are given in appendix A. In a general covariant gauge characterized by a gauge parameter λ, these yield a very large number of terms. We have found that the simplest way to express the results was in terms of the independent types of integrals which appear in the calculations.

Graph 1 gives rise to integrals of the following type:

$$I^{a_1a_2a_3}_{\mu_1\cdots\mu_\ell} = \int \frac{d^mQ}{(2\pi)^m}\,\frac{Q_{\mu_1}\dots Q_{\mu_\ell}}{\left[(Q+p)^2-i\eta\right]^{a_1}\left[(Q-q)^2-i\eta\right]^{a_2}\left(Q^2-i\eta\right)^{a_3}} \tag{1}$$

where $a_1=a_2=a_3=1$ and $1\leq \ell \leq 3$. On the other hand diagram 2 yields integrals where $1\leq a_1,a_2,a_3\leq 2$ and $0\leq \ell\leq 5$. Finally the graph shown in figure 3 produces basic integrals of the form:

$$J^{a_1a_2}_{\mu_1\cdots\mu_\ell} = \int \frac{d^mQ}{(2\pi)^m}\,\frac{Q_{\mu_1}\dots Q_{\mu_\ell}}{\left[(Q-k)^2-i\eta\right]^{a_1}\left(Q^2-i\eta\right)^{a_2}} \tag{2}$$

where $1\leq a_1,a_2\leq 2$ and $0\leq \ell\leq 2$.

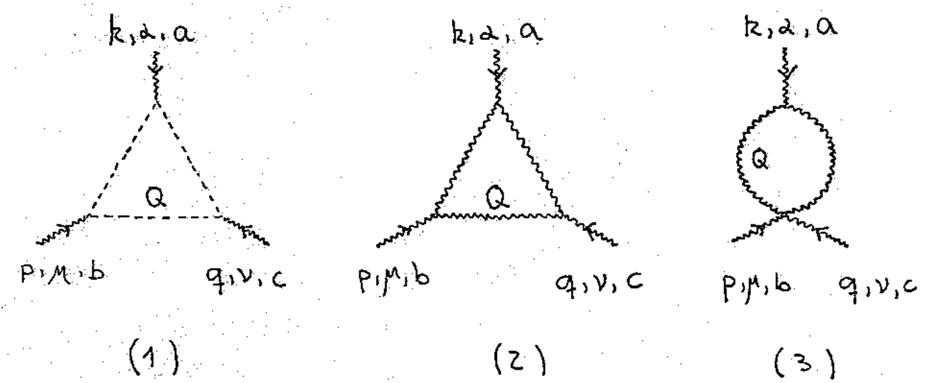

Figure I - Wavy lines represent gluons and dashed lines stand for ghost particles. Graphs obtained by cyclic permutations of external gluons in diagram (3) are to be understood.

After a straightforward application of the Feynman rules, we obtain that the diagram 1 yields the following contribution for the SU(N) Yang-Mills theory:

$$ {}^{1}D^{\alpha\mu\nu}_{abc} = i\bar{g}^3 \frac{N}{2} f_{abc} \sum_{\ell=1}^{3} K^{\alpha\mu\nu;\mu_1\cdots\mu_\ell} I^{111}_{\mu_1\cdots\mu_\ell} \tag{3} $$

where f_{abc} are structure constants of the theory and $K^{\alpha\mu\nu;\mu_1\cdots\mu_\ell}$ denote Lorentz tensors given by:

$$ K^{\alpha\mu\nu;\beta} = -p_\alpha q_\nu \delta_{\mu\beta} + (\mu \leftrightarrow \nu;\ p \leftrightarrow q) $$

$$ K^{\alpha\mu\nu;\beta\gamma} = p_\alpha \delta_{\mu\beta}\delta_{\nu\gamma} + p_\mu \delta_{\alpha\beta}\delta_{\nu\gamma} + (\mu \leftrightarrow \nu;\ p \leftrightarrow q) $$

$$ K^{\alpha\mu\nu;\beta\gamma\rho} = 2\delta_{\alpha\rho}\delta_{\mu\gamma}\delta_{\nu\rho} \tag{4} $$

with $p \leftrightarrow q$ representing permutations of the fourvectors p and $-q$ as well as of q and $-p$.

For the contribution resulting from graph 2 the situation is much more complicated in a general gauge, where it gives rise to terms proportional to various powers of the gauge parameter: λ^j with j =0,1,2,3. We characterize the corresponding Lorentz tensors in this case by an index j and by the various indices a_1, a_2, a_3 which appear in the calculation: $K^{\alpha\mu\nu;\mu_1\cdots\mu_\ell}_{a_1a_2a_3;j}$. These tensors are presented explicitly in Appendix B. In terms of these functions we obtain:

$$ {}^{2}D^{\alpha\mu\nu}_{abc} = i\bar{g}^3 \frac{N}{2} f_{abc} \sum_{\ell=0}^{5} \sum_{a_1,a_2,a_3=1}^{2} \sum_{j=0}^{3} K^{\alpha\mu\nu;\mu_1\cdots\mu_\ell}_{a_1a_2a_3;j} I^{a_1a_2a_3}_{\mu_1\cdots\mu_\ell} \lambda^j \tag{5} $$

Finally, the diagrams in figure 3 give the contribution:

$$ {}^{3}D^{\alpha\mu\nu}_{abc} = i\bar{g}^3 \frac{N}{2} f_{abc} \sum_{\ell=0}^{2} \sum_{a_1,a_2=1}^{2} \sum_{j=0}^{2} K^{\alpha\mu\nu;\mu_1\cdots\mu_\ell}_{a_1a_2;j} J^{a_1a_2}_{\mu_1\cdots\mu_\ell} \lambda^j \tag{6} $$

where cyclic permutations are to be understood, with the non-vanishing Lorentz tensors $K^{\alpha\mu\nu;\mu_1\cdots\mu_\ell}_{a_1a_2;j}$ given by:

$$ K^{\alpha\mu\nu}_{11;0} = \frac{9}{2}\left[k_\mu \delta_{\nu\alpha}\right];\ K^{\alpha\mu\nu;\beta}_{11;0} = \frac{3}{2}\left[\delta_{\mu\beta}\delta_{\nu\alpha}\right];\ K^{\alpha\mu\nu}_{21;1} = \frac{3}{2}k^2\left[k_\nu \delta_{\mu\alpha}\right] $$

$$ K^{\alpha\mu\nu;\beta\gamma}_{12;1} = \frac{3}{2}\left[k_\nu \delta_{\alpha\beta}\delta_{\mu\gamma} + 2k_\beta \delta_{\alpha\mu}\delta_{\nu\gamma}\right];\ K^{\alpha\mu\nu;\beta\gamma\rho}_{21;1} = \frac{3}{2}\left[\delta_{\mu\alpha}\delta_{\nu\rho}\delta_{\beta\gamma}\right] $$

$$ K^{\alpha\mu\nu;\beta}_{21;1} = \frac{3}{2}\left[k_\alpha k_\mu \delta_{\nu\beta} + k^2 \delta_{\nu\alpha}\delta_{\mu\beta}\right] $$

$$ K^{\alpha\mu\nu;\beta\gamma}_{21;1} = \frac{3}{2}\left[k_\mu \delta_{\gamma\alpha}\delta_{\beta\gamma} + k_\nu \delta_{\alpha\gamma}\delta_{\mu\beta}\right] $$

$$ K^{\alpha\mu\nu;\beta\gamma}_{22;2} = \frac{3}{2}\left[k_\alpha k_\nu k_\beta \delta_{\mu\gamma} + k^2 k_\mu \delta_{\alpha\gamma}\delta_{\nu\beta}\right] \tag{7} $$

where the square brackets imply that one must antisymmetrize with respect to $(\mu \leftrightarrow \nu)$.

Our basic task consists now in evaluating the set of integrals described by equations (1) and (2). To this end we combine first the denominators using the Feynman parametrization given in (A4). We next perform appropriate shifts in the integration 4-momenta dropping odd powers which vanish by symmetry. The resulting dimensional integrals containing even powers of the momenta are carried out with the help of (A5). In this way, after a straightforward algebra, we can express the set $I^{a_1a_2a_3}_{\mu_1\cdots\mu_\ell}$ in terms of the quantities:

$$s = px - qy \;\; ; \;\; m^2 = p^2 x(1-x-y) + q^2 y(1-x-y) + k^2 xy - i\eta \tag{8}$$

in the following form:

$$I^{a_1 a_2 a_3}_{\mu_1 \cdots \mu_\ell} = \frac{i(4\pi)^{-\varepsilon-2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{P(\mu_1 \cdots \mu_\ell)} \sum_{r=0}^{\langle \ell \rangle} \sum_{P(\mu_1 \cdots \mu_{2r})} (-1)^\ell$$

$$\frac{\Gamma(-\varepsilon - r + a_1 + a_2 + a_3 - 2)}{(2r)!\,(\ell-2r)!\,r!\,2^{2r}} \int_0^1 dx \int_0^{1-x} dy \; s_{\mu_{2r+1}} \cdots s_{\mu_\ell} \; \frac{x^{a_1-1} y^{a_2-1} (1-x-y)^{a_3-1}}{(m^2)^{-\varepsilon - r + a_1 + a_2 + a_3 - 2}} \, .$$

$$\delta_{\mu_1 \mu_2} \cdots \delta_{\mu_{2r-1} \mu_{2r}} \tag{9}$$

where P denotes permutations over the corresponding indices and $\langle \ell \rangle$ is given by $(\ell-1)/2$ or $\ell/2$ respectively, when ℓ is odd or even. Proceeding in a similar way, we obtain for the set of integrals given in equation (2) the result:

$$J^{a_1 a_2}_{\mu_1 \cdots \mu_\ell} = \frac{i(4\pi)^{-\varepsilon-2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{P(\mu_1 \cdots \mu_\ell)} \sum_{r=0}^{\langle \ell \rangle} \sum_{P(\mu_1 \cdots \mu_{2r})}$$

$$\frac{\Gamma(-\varepsilon - r + a_1 + a_2 - 2)}{(2r)!\,(\ell-2r)!\,r!\,2^{2r}} \int_0^1 dx \; x^{a_1-1+\ell-2r} (1-x)^{a_2-1} \left[k^2 x(1-x) - i\eta \right]^{\varepsilon + r - a_1 - a_2 + 2}$$

$$\delta_{\mu_1 \mu_2} \cdots \delta_{\mu_{2r-1} \mu_{2r}} \, k_{\mu_{2r+1}} \cdots k_{\mu_\ell} \tag{10}$$

These expressions contain ultraviolet divergences which manifest themselves for $\varepsilon > 0$ when the arguments of the gamma functions become vanishingly small. In order to renormalize the vertex function it is convenient to use the $\overline{MS}$ scheme [12] where one substracts, in addition to the pure pole parts, also some finite quantities which always appear in the dimensional

regularization method. This prescription can be implemented by taking [see A2] $\bar{\mu}^2 = \mu^2 e^\gamma / 4\pi$ and substracting $1/\varepsilon$ poles only, where γ denotes the Euler constant. In this way we obtain from equations (3), (5) and (6) the following renormalized quantities:

$${}^1_r D^{\alpha\mu\nu}_{abc} = \frac{-ig^2}{(4\pi)^2} \frac{N}{2} \frac{1}{\varepsilon} \frac{1}{12} V^{\alpha\mu\nu}_{abc} + {}^1 D^{\alpha\mu\nu}_{abc} \tag{11}$$

$${}^2_r D^{\alpha\mu\nu}_{abc} = \frac{ig^2}{(4\pi)^2} \frac{N}{2} \frac{1}{\varepsilon} \left(\frac{13}{4} - \frac{9}{4}\lambda \right) V^{\alpha\mu\nu}_{abc} + {}^2 D^{\alpha\mu\nu}_{abc} \tag{12}$$

$${}^3_r D^{\alpha\mu\nu}_{abc} = \frac{ig^2}{(4\pi)^2} \frac{N}{2} \frac{1}{\varepsilon} \left(-\frac{9}{2} + \frac{3}{4}\lambda \right) V^{\alpha\mu\nu}_{abc} + {}^3 D^{\alpha\mu\nu}_{abc} \tag{13}$$

where $V^{\alpha\mu\nu}_{abc}(k,p,q)$ is given by the lowest order 3-gluon vertex (A2). In order to determine the above renormalized expressions, we must calculate the parametric integrals (9) and (10) to which we now turn.

III – EVALUATION OF PARAMETRIC INTEGRALS AND S-T IDENTITIES

The integral which appears in equation (10) can be expressed in terms of Euler's beta function $B(a,b)$ [13]. We then obtain for $J^{a_1 a_2}_{\mu_1 \cdots \mu_\ell}$ the result:

$$J^{a_1 a_2}_{\mu_1 \cdots \mu_\ell} = \frac{i(4\pi)^{-\varepsilon-2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{P(\mu_1 \cdots \mu_\ell)} \sum_{r=0}^{\langle \ell \rangle} \sum_{P(\mu_1 \cdots \mu_{2r})} \frac{\Gamma(-\varepsilon - r + a_1 + a_2 - 2)}{(2r)!\,(\ell-2r)!\,r!\,2^{2r}} \, .$$

$$B(\varepsilon - r + \ell - a_2 + 2,\; \varepsilon + r - a_1 + 2) \left(k^2 - i\eta \right)^{\varepsilon + r - a_1 - a_2 + 2}$$

$$\delta_{\mu_1 \mu_2} \cdots \delta_{\mu_{2r-1} \mu_{2r}} \cdot k_{\mu_{2r+1}} \cdots k_{\mu_\ell} \tag{14}$$

On the other hand, it is not possible in general, to evaluate analitically the integrals appearing in equation (9). Only when one gluon is on-shell we can evaluate them explicity in terms of known (and simple) functions. To this end we proceed now by making an analytic continuation of the parameter ϵ to positive values in order to regulate the infrared singularities and letting for definiteness $k^2 = 0$. Then, by making a change of variable to $z = x + y$ with $x < z < 1$, we can express all relevant integrals in terms of:

$$I^{s}_{mn}\left(p^2, q^2\right) = \int_0^1 dx \int_x^1 dz\; x^m (z-x)^n \left[(p^2-q^2)x + q^2 z\right]^s (1-z)^s \tag{15}$$

where an $-i\eta$ is understood to be associated with p^2 and q^2. Working in the equivalent region $0 < z < 1,\ 0 < x < z$, the above integrations can be easily performed with the result that:

$$I^{s}_{mn}\left(p^2, q^2\right) = \frac{\Gamma^2(1-s)}{\Gamma(m+n-2s+3)} \frac{\partial^{m+n}}{\partial(p^2)^m \partial(q^2)^n} \frac{(p^2)^{-s+m+n+1} - (q^2)^{-s+m+n+1}}{p^2 - q^2} \tag{16}$$

Finally, with the help of this equation, we obtain after some calculations that the set of integrals $I^{a_1 a_2 a_3}_{\mu_1 \cdots \mu_\ell}$ can be expressed in the following form:

$$I^{a_1 a_2 a_3}_{\mu_1 \cdots \mu_\ell} = \frac{i}{(4\pi)^{\epsilon+2}} \sum_{P(\mu_1 \cdots \mu_\ell)} \sum_{r=0}^{\langle \ell \rangle} \sum_{P(\mu_1 \cdots \mu_{2r})} \sum_{P(\mu_{2r+1} \cdots \mu_\ell)} \sum_{j=2r}^{\ell} (-1)^j$$

$$\frac{\Gamma(-\epsilon - r + a_1 + a_2 + a_3 - 2)}{(2r)!\,(\ell-2r)!\,r!\,(j-2r)!\,(\ell-j)!} \frac{\Gamma^2(\epsilon + r - a_1 - a_2 - a_3 + 3)}{2^{2r}\,\Gamma(2\epsilon - a_1 - a_2 - 2a_3 + \ell + 5)} \cdot$$

$$\frac{\partial^{a_1+a_2-2r+\ell-2}}{\partial(p^2)^{a_1+j-2r-1}\,\partial(q^2)^{a_2+\ell-j-1}} \left\{ \frac{(p^2)^{\epsilon-a_3+\ell-r+1} - (q^2)^{\epsilon-a_3+\ell-r+1}}{p^2 - q^2} - \right.$$

$$\left. - \frac{\delta_{a_3 2}}{2\epsilon - a_1 - a_2 - 2a_3 + \ell + 5} \left[\left(\frac{\partial}{\partial p^2} + \frac{\partial}{\partial q^2} \right) \frac{(p^2)^{\epsilon+\ell-r} - (q^2)^{\epsilon+\ell-r}}{p^2 - q^2} \right] \right\} \cdot$$

$$\delta_{\mu_1 \mu_2} \cdots \delta_{\mu_{2r-1} \mu_{2r}}\, p_{\mu_{2r+1}} \cdots p_{\mu_j}\, q_{\mu_{j+1}} \cdots q_{\mu_\ell} \tag{17}$$

Therefore, the results given by the forms (14) and (17) allow in this case an explicit determination, in terms of elementary functions, of the renormalized 3-gluon vertex expressed via the contributions (11), (12) and (13).

In order to check these expressions and obtain a better understanding of their content, we will consider now the Slavnov-Taylor identities for the 3-gluon vertex $\Gamma^{\alpha\mu\nu}_{abc}$. These identities can be represented diagramatically [14] in our case as indicated in Figure II, where the permutations $[p \leftrightarrow q;\ \mu \leftrightarrow \nu;\ b \leftrightarrow c]$ are to be understood on the right hand side.

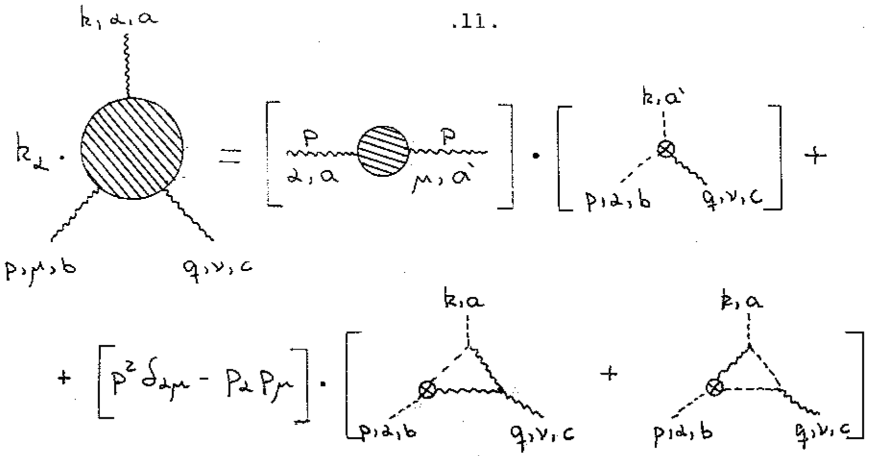

Figure II - The cross $\otimes$ denotes that the momenta associated with the ghost-gluon vertex is absent, being replaced by a Kronecker delta function.

Using the explicit results previously derived in the case $k^2=0$, we calculated $k_\alpha\,{}^r\Gamma^{\alpha\mu\nu}_{abc}(k,p,q)$. Similarly, with the help of the algebraic method SCHOONSHIP we calculated in a general covariant gauge the right hand side, which we denote $R^{\mu\nu}_{abc}(k,p,q)$ The explicit result is rather involved, being a polynomial in the gauge parameter λ . However, it is worth noticing that due to the structure of the graphs appearing on the right hand side , λ^3 contributions are absent. We obtained as expected, an identity which represents a check on our calculations, namely:

$$k_\alpha\,{}^r\Gamma^{\alpha\mu\nu}_{abc}(k,p,q) = R^{\mu\nu}_{abc}(k,p,q) \qquad (18)$$

It is interesting to study, with the help of this relation, the behaviour of the 3-gluon vertex in the limit $k\to 0$,when as a consequence of momentum conservation we have $p+q=0$. Using

equations (14) and (17) it can be verified that $k_\rho \frac{\partial}{\partial k_\alpha}\,{}^r\Gamma^{\rho\mu\nu}_{abc}(k,p,q)$ vanishes in this limit. Hence, taking the derivative on both sides of (18) with respect to k_α , we obtain in this case that:

$${}^r\Gamma^{\alpha\mu\nu}_{abc}(k=0,p,-p) = \frac{\partial}{\partial k_\alpha} R^{\mu\nu}_{abc}(k,p,q)\Big|_{k=0} \qquad (19)$$

With the help of the result obtained for $R^{\mu\nu}_{abc}$, we then find explicitly:

$$\begin{aligned}{}^r\Gamma^{\alpha\mu\nu}_{abc}(k=0,p,-p) = {} & \frac{g^3}{16\pi^2}\,\frac{N}{2}\,f_{abc}\Big\{\left(\frac{8}{3}+4\lambda-\frac{1}{2}\lambda^2\right)p_\alpha p_\mu p_\nu/p^2 \\ & + \left[\frac{46}{9}-\frac{8}{3}\ln\frac{p^2}{\mu^2}-\left(6+3\ln\frac{p^2}{\mu^2}\right)\lambda+\frac{3}{2}\lambda^2\right]\delta_{\mu\nu}p_\alpha + \\ & + \left[-\frac{35}{9}+\frac{4}{3}\ln\frac{p^2}{\mu^2}+\left(1+\frac{3}{2}\ln\frac{p^2}{\mu^2}\right)\lambda-\frac{1}{2}\lambda^2\right]\left(\delta_{\mu\alpha}p_\nu+\delta_{\nu\alpha}p_\mu\right)\Big\}\end{aligned} \qquad (20)$$

As we have seen, although in general ${}^r\Gamma^{\alpha\mu\nu}_{abc}(k,p,q)$ contains contributions proportional to λ^3 , the fact that in ${}^r\Gamma^{\alpha\mu\nu}_{abc}(k=0,p,-p)$ only first and second powers of the gauge parameter appear can be understood as a consequence of the S-T identities. We notice furthermore that ${}^r\Gamma^{\alpha\mu\nu}_{abc}(k=0,p,-p)$ is a completely finite quantity . This may be understood if we think of this 3-gluon vertex as resulting from inserting an external gluon with zero momenta into the self-energy gluon function: in both cases the off-shell gluon is controlling yielding an infrared finite contribution. However, with $k^2=0$, but $k\neq 0$, there will appear as we shall next show, additional singularities resulting from configurations where k is parallel to an internal gluon .

IV - INFRARED BEHAVIOUR OF THE 3-GLUON VERTEX

In this section we will concentrate on the infrared structure of the contributions given explicitly in (11), (12) and (13) via the expressions (14) and (17). We will begin by analysing qualitatively the structure expected from the diagrams of Figure I which are associated with these expressions. As we have seen, graph 1 gives a contribution expressed with the help of equation (1) with $a_1 = a_2 = a_3 = 1$. Clearly when all momenta are off- shell there are no infrared singularities present. When one of the external momenta goes on-shell, we see by power counting that also in this case there will be no soft singularities. However there will appear now a colinear singularity when the internal momenta becomes parallel to this external momenta. Indeed, as expected from this discussion we obtain for the infrared divergent contribution associated with this graph the result:

$$
\begin{aligned}
{}_{ir}^{\;1}D^{\alpha\mu\nu}_{abc}(k^2=0) = {} & \frac{g^3}{16\pi^2}\,\frac{N}{2}\,f_{abc}\,\frac{1}{\varepsilon}\,\frac{q_\alpha}{q^2}\Big\{\Big[2f^3(f+1)\ell + \tfrac{1}{3}f - f^2 - 2f^3\Big]\cdot \\
& (p_\mu p_\nu - q_\mu q_\nu) + \Big[-f^2(2f^2+3f+1)\ell + 2f^3 + 2f^2 + \tfrac{1}{6}f\Big]\cdot \\
& (-2q_\mu q_\nu - q_\mu p_\nu - p_\nu q_\mu)\Big\} + (\mu \leftrightarrow \nu\,;\; p \leftrightarrow q\,;\; b \leftrightarrow c)
\end{aligned}
\tag{21}
$$

where $f \equiv q^2/(p^2-q^2)$; $\ell = \ln(p^2/q^2)$. The contribution associated with diagram (2) is qualitatively similar, although in this case the number of relevant contributions is much higher, and will be discussed below.

Consider now the contribution associated with graph 3. Since in a general covariant gauge the gluon propagator is a

function of degree -2 in its momenta, like it is in the Feynman gauge, it is simpler and sufficient for our qualitative analysis to restrict ourselves to this gauge. We then see from equation (2) that in this case, with $a_1 = a_2 = 1$ and $\lambda = 0$, there are no soft divergences even when $k^2 = 0$. The collinear divergences which potentially could be presente vanish by symmetric integration in this case. Actually, as can be clearly seen from equation (14) the whole contribution vanishes in this case. The only singularity surviving, results from the ultra violet substracted pole left over in equation (13). So we obtain in this case an infrared contribution given by:

$$
{}_{ir}^{\;3}D^{\alpha\mu\nu}_{abc}(k^2=0) = -\frac{g^3}{16\pi^2}\,\frac{N}{2}\,\frac{1}{\varepsilon}\,f_{abc}\left(-\frac{9}{2} + \frac{3}{4}\lambda\right)\left(\delta_{\mu\alpha}k_\nu - \delta_{\nu\alpha}k_\mu\right)
\tag{22}
$$

The relevant contributions resulting from diagram 2 when $k^2 = 0$ are most conveniently expressed in terms of the following independent Lorentz tensors:

$$
\begin{aligned}
& T_1^{\alpha\mu\nu} = -q_\alpha p_\mu p_\nu/q^2 \;;\; T_2^{\alpha\mu\nu} = q_\alpha p_\mu q_\nu/q^2 \\
& T_3^{\alpha\mu\nu} = q_\alpha q_\mu p_\nu/q^2 \;;\; T_4^{\alpha\mu\nu} = -q_\alpha q_\mu q_\nu/q^2 \\
& T_5^{\alpha\mu\nu} = -\delta_{\mu\nu} q_\alpha \;;\; T_6^{\alpha\mu\nu} = -\delta_{\nu\alpha} q_\mu \;;\; T_7^{\alpha\mu\nu} = -\delta_{\mu\alpha} q_\nu
\end{aligned}
\tag{23}
$$

Including the contributions associated with graphs 1 and 3, the sum of all infrared divergent contributions can then be expressed in the form:

$$i\Gamma^{\alpha\mu\nu}_{abc}(k^2=0) = -\frac{g^3}{16\pi^2}\frac{N}{2} f_{abc}\frac{1}{\varepsilon}\left[\sum_{j=1}^{7} T_j^{\alpha\mu\nu}\, G^j - (\mu \leftrightarrow \nu;\ p \leftrightarrow q)\right] \tag{24}$$

where the factors G^j are given in terms of a serie in the gauge parameter λ as follows:

$$\begin{aligned} G^1 = {} & -\frac{17}{3}f + 8f^2 + 16f^3 + \ell\left(3f^2 - 16f^3 - 16f^4\right) + \\ & + \lambda\left[-\frac{3}{2}\frac{f^2}{1+f} - 3f - 10f^2 - 24f^3 + \ell\left(-\frac{1}{2}f + \frac{15}{2}f^2 + 22f^3 + 24f^4\right)\right] + \\ & + \lambda^2\left[\frac{3}{4}\frac{f^2}{1+f} + \frac{7}{4}\frac{f^3}{1+f} - f - \frac{67}{4}f^2 - 30f^3 + \ell\left(7f^2 + 30f^3 + 30f^4\right)\right] + \\ & + \lambda^3\left[-\frac{1}{2}\frac{f^3}{1+f} - \frac{1}{2}\frac{f^4}{1+f} - 3f^2 - \frac{11}{2}f^3 + \ell\left(f^2 + 6f^3 + 6f^4\right)\right] \end{aligned} \tag{25.a}$$

$$\begin{aligned} G^2 = {} & \frac{10}{6}f - 16f^2 - 16f^3 + \ell\left(5f^2 + 24f^3 + 16f^4\right) + \\ & + \lambda\left[\frac{3}{2} + \frac{25}{2}f + 26f^2 + 24f^3 + \ell\left(-\frac{11}{2}f - \frac{47}{2}f^2 - 38f^3 - 24f^4\right)\right] + \\ & + \lambda^2\left[\frac{1}{2} + \frac{19}{2}f + \frac{63}{2}f^2 + 30f^3 - \frac{1}{2}\frac{f^3}{1+f} + \ell\left(-\frac{15}{4}f - 23f^2 - 46f^3 - 30f^4\right)\right] + \\ & + \lambda^3\left[\frac{1}{4}\frac{f^3}{1+f} + \frac{1}{4}\frac{f^4}{1+f} + \frac{3}{2}f + 6f^2 + \frac{23}{4}f^3 + \right. \\ & \left. \ell\left(-\frac{1}{2}f - 4f^2 - 9f^3 - 6f^4\right)\right] \end{aligned} \tag{25.b}$$

$$\begin{aligned} G^3 = {} & -\frac{44}{6}f - 16f^2 - 16f^3 + \ell\left(3f + 14f^2 + 24f^3 + 16f^4\right) + \\ & + \lambda\left[4f + 26f^2 + 24f^3 + \ell\left(-f - 15f^2 - 38f^3 - 24f^4\right)\right] + \\ & + \lambda^2\left[4f + 31f^2 + 30f^3 + \ell\left(-\frac{1}{2}f - 17f^2 - 46f^3 - 30f^4\right)\right] + \\ & + \lambda^3\left[\frac{1}{2}f + 6f^2 + 6f^3 + \ell\left(-3f^2 - 9f^3 - 6f^4\right)\right] \end{aligned} \tag{25.c}$$

$$\begin{aligned} G^4 = {} & \frac{16}{3}f + 24f^2 + 16f^3 + \ell\left(-16f^2 - 32f^3 - 16f^4\right) + \\ & + \lambda\left[-\frac{5}{2} - 21f - 42f^2 - 24f^3 + \ell\left(11f + 40f^2 + 54f^3 + 24f^4\right)\right] + \\ & + \lambda^2\left[-1 - \frac{37}{2}f - 47f^2 - 30f^3 + \ell\left(\frac{15}{2}f + \frac{79}{2}f^2 + 62f^3 + 30f^4\right)\right] + \\ & + \lambda^3\left[-3f - 9f^2 - 6f^3 + \ell\left(f + 7f^2 + 12f^3 + 6f^4\right)\right] \end{aligned} \tag{25.d}$$

$$\begin{aligned} G^5 = {} & 7 + 10f + \ell\left(-2 - 12f - 10f^2\right) + \lambda\left[-\frac{1}{2} - f + \ell\left(f + f^2 + 1\right)\right] + \\ & \lambda^2\left[-1 - 2f + \ell\left(\frac{1}{4} + 2f + 2f^2\right)\right] + \lambda^3\left[-\frac{1}{4} - \frac{1}{2}f + \ell\left(\frac{1}{2}f + \frac{1}{2}f^2\right)\right] \end{aligned} \tag{25.e}$$

$$\begin{aligned} G^6 = {} & -3 - 2f + \ell\left(3 + 7f + 2f^2\right) + \lambda\left[-\frac{1}{2} - 2f + \ell\left(\frac{1}{2} + \frac{5}{2}f + 2f^2\right)\right] + \\ & + \lambda^2\left[-\frac{1}{2} - \frac{1}{2}f + \ell\left(\frac{1}{2}f + \frac{1}{2}f^2\right)\right] \end{aligned} \tag{25.f}$$

$$\begin{aligned} G^7 = {} & -\frac{1}{2} - 2f + \ell\left(-2 - \frac{3}{2}f + 2f^2\right)] + \lambda\left[-\frac{11}{4} - \frac{1}{f} - 2f + \right. \\ & \left. + \ell\left(1 + \frac{11}{4} + 2f^2\right)\right] + \lambda^2\left[-\frac{1}{4} - \frac{1}{2}f + \ell\left(\frac{1}{4} + \frac{3}{4}f + \frac{1}{2}f^2\right)\right] \end{aligned} \tag{25.g}$$

It can be verified that as $p \to -q$, all the above infrared contributions cancel out, in accordance with the finiteness expected from (20) for the renormalized 3-gluon vertex.

We consider now the situation when two external gluons are on-shell. To understand what happens in this case let us look for instance at equation (1). We obtain, as before, a collinear singularity which results when the internal gluons becomes parallel to one or another on-shell external gluon. However we see that in this case will appear in addition in the denominators more powers of the internal momenta, which can potentially yield soft infrared divergences. This is precisely the case with the contributions associated with graph 2, where we obtain a double pole structure resulting from the superposition of these two effects.

For the contributions given by diagram 1, it turns out that due to the appearence of additional powers of Q also in the numerator, we obtain in this case only single collinear singularities. Taking for definitness k and p, on-shell, these are given by:

$$ {}^{1}_{ir}D^{\alpha\mu\nu}_{abc}\left(k^2=p^2=0\right) = -\frac{g^3}{16\pi^2}\frac{N}{2} f_{abc}\frac{1}{\varepsilon}\frac{1}{q^2}\left[\frac{1}{3}p_\alpha p_\mu p_\nu - \right. $$
$$ \left. -\frac{1}{2}k_\alpha\left(p_\mu k_\nu + p_\nu k_\mu\right) - \left(\mu \leftrightarrow \alpha;\ k \leftrightarrow p\right)\right] \tag{26} $$

The singular contributions associated with graph 3 have a similar origin as in the previous case: they result from the uncancelled substracted ultraviolet poles connected with letting $p^2 = k^2 = 0$

in equation (13). We obtain then:

$$ {}^{3}_{ir}D^{\alpha\mu\nu}_{abc}\left(k^2=p^2=0\right) = \frac{-g^3}{16\pi^2}\frac{N}{2} f_{abc}\frac{1}{\varepsilon}\left(-\frac{9}{2}+\frac{3}{4}\lambda\right) \times $$
$$ \left[-p_\nu\delta_{\mu\alpha} + p_\alpha\delta_{\mu\nu} + k_\nu\delta_{\mu\alpha} - k_\mu\delta_{\alpha\nu}\right] \tag{27} $$

The contributions connected with diagram 2 are most conveniently expressed in terms of the following Lorentz tensors:

$$ L_1^{\alpha\mu\nu} = p_\mu\delta_{\nu\alpha}\ ;\ L_2^{\alpha\mu\nu} = p_\nu\delta_{\mu\alpha}\ ;\ L_3^{\alpha\mu\nu} = p_\alpha\delta_{\mu\nu} $$
$$ L_4^{\alpha\mu\nu} = \frac{p_\alpha p_\mu p_\nu}{q^2}\ ;\ L_5^{\alpha\mu\nu} = \frac{p_\alpha k_\mu k_\nu}{q^2}\ ;\ L_6^{\alpha\mu\nu} = \frac{k_\alpha p_\mu k_\nu}{q^2}\ ;\ L_7^{\alpha\mu\nu} = \frac{k_\alpha k_\mu p_\nu}{q^2} \tag{28} $$

Then the infrared divergent contributions to the renormalized vertex corresponding to the case when k and p are on-shell can be expressed as:

$$ {}_{ir}\Gamma^{\alpha\mu\nu}_{abc}\left(k^2=p^2=0\right) = \frac{-g^3}{16\pi^2}\frac{N f_{abc}}{2}\sum_{\lambda=1}^{7}\left[L_i^{\alpha\mu\nu}(k,p) - L_i^{\mu\alpha\nu}(p,k)\right]F_i\left(q^2/\mu^2\right) \tag{29} $$

where the factors F_i are gauge dependent and exhibit double and single infrared poles given by:

$$ F_1\left(\frac{q^2}{\mu^2}\right) = \frac{3}{2}\frac{1}{\varepsilon} + \lambda\left[\left(\frac{1}{4}+\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\right] + $$
$$ + \lambda^2\left[\left(-\frac{1}{2}+\frac{1}{4}\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon} + \frac{1}{4}\frac{1}{\varepsilon^2}\right] \tag{30.a} $$

$$ F_2\left(\frac{q^2}{\mu^2}\right) = \left(-3+\frac{3}{2}\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon} + \frac{3}{2}\frac{1}{\varepsilon^2} + $$
$$ + \lambda\left[\left(-\frac{1}{2}+\frac{1}{4}\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon} + \frac{1}{4}\frac{1}{\varepsilon^2}\right] \tag{30.b} $$

$$ F_3\left(\frac{q^2}{\mu^2}\right) = \left(5-2\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon} - \frac{2}{\varepsilon^2} - \lambda\left(\frac{1}{2\varepsilon}\right) \tag{30.c} $$

$$F_4\left(\frac{q^2}{\mu^2}\right) = -\frac{8}{3}\frac{1}{\varepsilon} - \lambda\left[\left(\frac{3}{2}+\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\right] \tag{30.d}$$

$$F_5\left(\frac{q^2}{\mu^2}\right) = \lambda\left[\left(-\frac{3}{2}+\frac{1}{2}\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{1}{2}\frac{1}{\varepsilon^2}\right] \tag{30.e}$$

$$F_6\left(\frac{q^2}{\mu^2}\right) = \left(-7+3\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{3}{\varepsilon^2}+\lambda\left[\left(-\frac{13}{2}+5\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{5}{\varepsilon^2}\right]+ \\ +\lambda^2\left[\left(-\frac{7}{2}+\frac{13}{4}\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{13}{4}\frac{1}{\varepsilon^2}\right]+\lambda^3\left[\left(\frac{1}{4}+\frac{1}{2}\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{1}{2\varepsilon^2}\right] \tag{30.f}$$

$$F_7\left(\frac{q^2}{\mu^2}\right) = \left(13-3\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}-\frac{3}{\varepsilon^2}+\lambda\left[\left(\frac{1}{2}+\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\right]+ \\ +\lambda^2\left[\left(-1+\frac{1}{2}\ln\frac{q^2}{\mu^2}\right)\frac{1}{\varepsilon}+\frac{1}{2}\frac{1}{\varepsilon^2}\right] \tag{30.g}$$

Finaly we mention that the case when all gluons are on-shell would require them to be in a parallel configuration, which is not allowed by angular momentum conservation.

Thereby, we have obtained an analytic expression in terms of simple functions for the renormalized 3-gluon vertex, in the case when at least one external gluon is on-shell ,We have determined, by consistently expanding it in powers of ε ,all the infrared singularities as well as the finite contributions, which involve in a general covariant gauge an extremely large number of terms. The calculation is straightforward and, in particular, we find in the Feynman gauge, for the case when two gluons are on-shel,an expression in agreement with reference [7]

V - INFRARED STRUCTURE OF THE 4-GLUON FUNCTION

In this section we wish to study, in one loop aproximation, the structure of the gluon-gluon scattering amplitude, when all external gluon are on-shell and transverse. Since this is a gauge invariant process, it is sufficient for our purpose to perform the analysis in the Feynman gauge. We can write this amplitude as:

$$\mathcal{A}(k,p,q,r) = V_\alpha^a(k)\,V_\mu^b(p)\,V_\nu^c(q)\,V_\beta^d(r)\,M^{abcd}_{\alpha\mu\nu\beta}(k,p,q,r) \tag{31}$$

where V denote the transverse polarization vectors of the four gluons with momenta k_α , p_μ , q_ν , r_β and colors a , b , c and d respectively. For simplicity, we will restrict in what follows to the color gauge group SU(2).

We begin by considering the 4th order reducible graphs contributing to this process which are shown in Figure III, where all external momenta are incoming, with $k+p+q+r=0$.

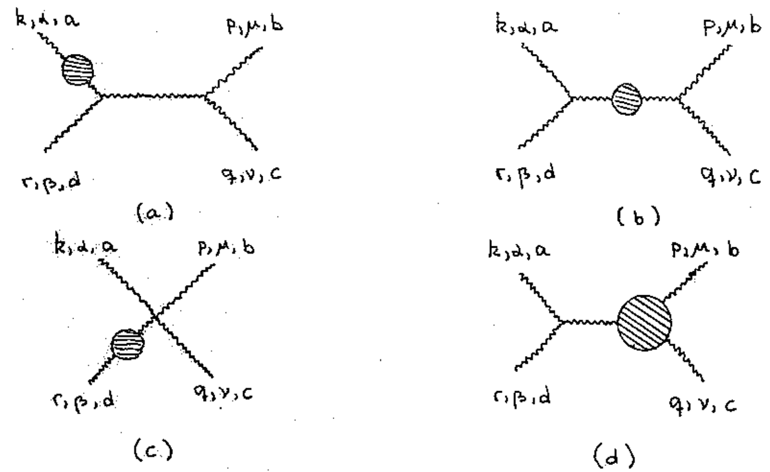

Figure III - Self-energy and vertex corrections contributing to the reducible 4-point vertex. All distinct permutations are to be understood.

Using the well known expression for the gluon self energy [14], we obtain, from graphs containing self-energy insertions, the contribution:

$$M^{abcd}_{\alpha\mu\nu\beta}(k,p,q) = \frac{ig^4}{12\pi^2}\frac{5}{\varepsilon}\left\{\delta_{ab}\delta_{cd}\,M^0_{\alpha\mu\nu\beta}+(b\leftrightarrow c;\mu\leftrightarrow\nu;p\leftrightarrow q)\right\} \\ + \text{c.p.} \tag{32}$$

where c.p. denotes cyclic permutations over the external legs with corresponding momenta k , p , and q , keeping the r leg fixed. This procedure yields, as expected, a Bose invariant expression.

The Lorentz tensor $M^{0}_{\alpha\mu\nu\beta}$ is given by:

$$M^{0}_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\Big[(k\cdot p\,\delta_{\alpha\beta} + 2\,p_\alpha q_\beta)\,\delta_{\mu\nu} + (p\cdot q\,\delta_{\nu\beta} + 2\,k_\beta p_\nu)\,\delta_{\alpha\mu} - 2\,(k_\mu p_\nu \delta_{\alpha\beta} + p_\alpha q_\mu \delta_{\nu\beta} - p_\alpha p_\nu \delta_{\mu\beta}) - (\mu \leftrightarrow \nu\,;\, p \leftrightarrow q)\Big] \tag{33}$$

We have exhibited explicity only the infrared divergent contributions, in equation (32), with a gauge invariant result, proportional to the tree amplitude.

On the other hand, the graphs containing vertex corrections yield, with the help of equation (29), infrared singularities containing single and double poles:

$$^{V}M^{abcd}_{\alpha\mu\nu\beta}(k,p,q) = \frac{i g^4}{16\pi^2}\Big\{\delta_{ab}\delta_{cd}\Big[\Big(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{2p\cdot q}{\mu^2}\Big) M^{V1}_{\alpha\mu\nu\beta} + \frac{1}{\varepsilon} M^{V2}_{\alpha\mu\nu\beta}\Big] + (b\leftrightarrow c\,;\,\mu\leftrightarrow\nu\,;\,p\leftrightarrow q)\Big\} + c.p. \tag{34}$$

where the Lorentz tensors $M^{Vi}_{\alpha\mu\nu\beta}$ (i=1,2) are expressed as follows:

$$M^{V1}_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\Big[5\,q_\beta p_\alpha \delta_{\mu\nu} + (3\,p\cdot q\,\delta_{\mu\nu} - 5\,k_\mu p_\nu)\,\delta_{\alpha\beta} + 4\,(k_\beta p_\nu \delta_{\alpha\mu} + p_\alpha p_\nu \delta_{\mu\beta} - p_\alpha q_\mu \delta_{\nu\beta}) - (\mu\leftrightarrow\nu\,;\,p\leftrightarrow q)\Big] \tag{35.a}$$

$$M^{V2}_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\Big[(11\,k_\mu p_\nu - 6\,k\cdot p\,\delta_{\mu\nu}) - 11\,p_\alpha q_\beta \delta_{\mu\nu} + 10\,(p_\alpha q_\mu \delta_{\nu\beta} - p_\alpha p_\nu \delta_{\mu\beta} - p_\nu q_\beta \delta_{\alpha\mu}) - (\mu\leftrightarrow\nu\,;\,p\leftrightarrow q)\Big] \tag{35.b}$$

These contributions are not gauge invariant and must be added to the corresponding ones resulting from the irreducible 4th order graphs, shown in figure IV, in order to obtain a gauge invariant result.

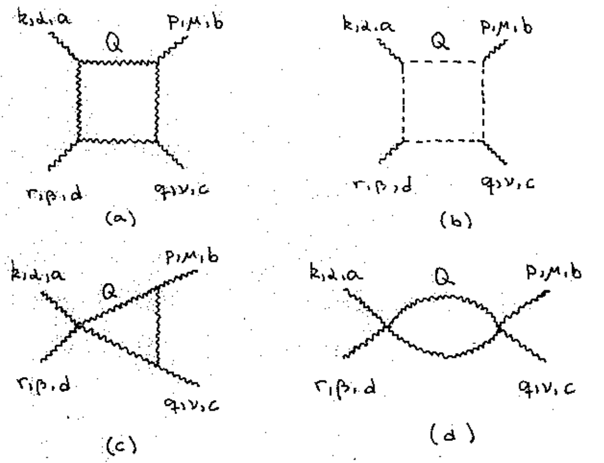

Figure IV - Graphs contributing to the irreducible 4-point function. All other diagrams, obtained by permutations of external gluons, are to be understood.

The contributions associated with the box diagrams (a) and (b), can be conveniently described in terms of the invariants $z_1 = 2k\cdot p\ ;\ z_2 = 2p\cdot q\ ;\ z_3 = 2k\cdot q$, so that $z_1 + z_2 + z_3 = 0$. The integrations over the internal momenta, involve in the numerators polynomials up to 4th power in Q. However, after combining the 4 denominators using the Feynman parametrization, and performing appropriate shifts, the resulting numerator will contain only even powers in Q which we denote by $2m$, so that $m = 0, 1$ or 2. Performing these integrations, one is left with parametric integrals characterized by the expression:

$$H(m,j,k,z_1,z_2,z_3) = \int_0^1 dx \int_0^1 dy\, \frac{x^j\, y^k}{(x z_1 + y z_2 + x y z_3)^{-\varepsilon - m + 2}} \tag{36}$$

Note that these integrals are symmetric under the interchange ($j\leftrightarrow k\ ;\ z_1 \leftrightarrow z_2$) so that it is sufficient to restrict ourselves to the case $j \geq k \geq 0$. It turns out that in our case the indices j, k are related to m in the following manner:

$$0 \leq j+k \leq 4 \quad \text{if} \quad m=0 \tag{37.a}$$

$$0 \leq j+k \leq 2 \quad \text{if} \quad m=1 \tag{37.b}$$

$$j=k=0 \quad \text{if} \quad m=2 \tag{37.c}$$

As shown by Karasinski in reference [9], these integrals can be expressed in terms of hypergeometric functions as follows:

$$H(m,j,k,z_1,z_2,z_3) = -\frac{\Gamma(\varepsilon+m-1)}{\Gamma(\varepsilon+m+j+1)}\frac{1}{z_2}\frac{\partial^{j-k}}{\partial z_1^{j-k}}\frac{1}{z_1}$$
$$\left\{(z_2)^{\varepsilon+m+j}\frac{\partial^k}{\partial z_3^k}\,{}_2F_1\left(1,\varepsilon+m+j,\varepsilon+m+j+1;-\frac{z_3}{z_1}\right)+ \right.$$
$$\left. + (z_1 \leftrightarrow z_2)\right\}\Big|_{z_3=-z_1-z_2} \tag{38}$$

In addition, these integrals appear multiplied by simple functions containing poles in ε, so alltogether the resulting expression contains single and double poles. Physically, the single infrared poles arise when the internal momenta Q is soft or parallel to a given external momenta, whereas the double poles result when both these effects operate together.

It is convenient to simplify these expressions by expanding them in powers of ε. With help of the transformation formulas relating the hypergeometric functions [13] it is straightforward to show that:

$${}_2F_1(1,\varepsilon+p,\varepsilon+p+1;z) = \frac{\varepsilon+p}{z^p}\left[\sum_{l=1}^{\infty} Li_l(z)(-\varepsilon)^{l-1} - \sum_{j=1}^{p-1}\frac{z^j}{\varepsilon+j}\right] \tag{39}$$

where $Li_l(z)$ denote the polylogarithmic functions defined recursively by:

$$Li_1(z) \equiv \ln\frac{1}{1-z} \; ; \; Li_l(z) \equiv \int_0^z Li_{l-1}(x)\,\frac{dx}{x} \tag{40}$$

Actually, for the purpose of studing the infrared structure, it is sufficient to consider, in our case, only terms with $l=1$.

We now turn to contributions associated with diagram (c), which have three denominators, and where the virtual momenta appear at most with a power two in the numerator. This represents a particular case of the general type of integral $I^{a_1a_2a_3}_{\mu_1\cdots\mu_l}$ considered in equation (1), corresponding to $a_1=a_2=a_3=1$ and $l=0$, 1 or 2. For instance, when $l=2$, using equation (17) with $q \rightarrow q+p$, we obtain the result

$$I^{111}_{\mu_1\mu_2} = \frac{i}{(4\pi)^{\varepsilon+2}}(2p\cdot q)^{\varepsilon}\left\{\Gamma(1-\varepsilon)B(\varepsilon,\varepsilon+3)\left[\frac{1}{\varepsilon}\frac{p_{\mu_1}p_{\mu_2}}{2p\cdot q}+\right.\right.$$
$$\left.+\frac{1}{\varepsilon+1}\frac{(p_{\mu_1}q_{\mu_2}+q_{\mu_1}p_{\mu_2})}{2p\cdot q}+\frac{1}{\varepsilon+2}\frac{q_{\mu_1}q_{\mu_2}}{2p\cdot q}\right]$$
$$\left.+\Gamma(-\varepsilon)\frac{B(\varepsilon+1,\varepsilon+2)}{\varepsilon+1}\frac{\delta_{\mu_1\mu_2}}{2}\right\} \tag{41}$$

which contains, in general, single and double pole divergences. As in the previous case, the infrared singularities are associated with configurations where the internal momenta Q is soft and/or colinear to that of an external gluon.

Finaly we consider the contributions resulting from graph (d). These can be expressed in terms of integrals given in equation (2), which contain two denominators, by making in particular $a_1=a_2=1$ and $l=2$. In this way, with help of equation (14), we find the expression:

$$J^{11} = \frac{i}{(4\pi)^{\varepsilon+2}}\Gamma(-\varepsilon)B(\varepsilon+1,\varepsilon+1)(2p\cdot q)^{\varepsilon} \tag{42}$$

which contains only ultraviolet divergent poles.

With the help of the integrals given in equations (36-42), we have performed the lengthy algebra using the computational method SCHOONSHIP. The calculation involves a very large number of terms and, as check, we have isolated the ultraviolet divergent contribution which is given by:

$$D^{abcd}_{\alpha\mu\nu\beta} = \frac{ig^4}{24\pi^2}\frac{1}{\varepsilon}\left\{\delta_{ab}\delta_{cd}\left(\delta_{\alpha\mu}\delta_{\nu\beta}-\delta_{\alpha\nu}\delta_{\mu\beta}\right)+\left(b\leftrightarrow c;\mu\leftrightarrow\nu\right)\right\}+ + \text{c.p.} \tag{43}$$

where the expression in bracket represents the 4-gluon tree vertex. This is in agreement with the result expected for the irreducible 4-point function which arises in consequence of the Slavnov-Taylor identities [14] . Next we perform the renormalization of the ultraviolet singularities according to the scheme described in section II.

The resulting expression for the renormalized irreducible 4-point vertex contains single and double infrared poles. The contributions with double poles, when added to the corresponding ones associated with the reducible 4-point function, given in equation (34), yields the result:

$$\begin{aligned} M^{abcd}_{\alpha\mu\nu\beta}(k,p,q) = \frac{ig^4}{16\pi^2}\frac{1}{\varepsilon^2}\Big\{&\delta_{ab}\delta_{cd}\left(M^1_{\alpha\mu\nu\beta}+M^2_{\alpha\mu\nu\beta}\right)+\\ &+\delta_{ad}\delta_{bc}M^3_{\alpha\mu\nu\beta} +\\ &+\left(\mu\leftrightarrow\nu;\, b\leftrightarrow c;\, p\leftrightarrow q\right)\Big\} + \text{c.p.}\end{aligned} \tag{44}$$

where the Lorentz tensors $M^i_{\alpha\mu\nu\beta}$ (i=1,2,3), are given respectively by:

$$\begin{aligned} M^1_{\alpha\mu\nu\beta} = \frac{1}{k\cdot p}\Big[&\left(12k_\mu k_\nu+10k_\mu p_\nu+2k_\nu q_\mu+2p_\nu q_\mu\right)\delta_{\alpha\beta}-\\ &-\left(11k_\mu k_\beta+10k_\mu p_\beta+q_\mu q_\beta\right)\delta_{\alpha\nu}+\\ &+\left(2p_\alpha k_\mu-\tfrac{21}{2}q_\alpha k_\mu-\tfrac{1}{2}q_\alpha q_\mu+10p_\alpha q_\mu\right)\delta_{\nu\beta}+\\ &+\left(2k_\nu k_\beta+\tfrac{25}{2}k_\nu p_\beta-8p_\nu k_\beta+2p_\nu p_\beta+10\,k\cdot q\,\delta_{\nu\beta}\right)\delta_{\alpha\mu}-\\ &-\left(11p_\alpha k_\nu+q_\alpha k_\nu+12p_\alpha p_\nu+q_\alpha p_\nu\right)\delta_{\mu\beta}+\\ &+\left(8p_\alpha k_\beta+10p_\alpha p_\beta+2q_\alpha q_\beta\right)\delta_{\mu\nu}\Big]+\\ &+\left(\mu\leftrightarrow\nu;\, p\leftrightarrow q\right)\end{aligned} \tag{45.a}$$

$$\begin{aligned} M^2_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\Big[&\left(\tfrac{1}{2}k_\mu k_\nu-\tfrac{11}{2}k_\mu p_\nu+6k_\nu q_\mu-2p_\nu q_\mu+2\,p\cdot q\,\delta_{\mu\nu}\right)\delta_{\alpha\beta}\\ &+\left(2q_\mu p_\beta-2k_\mu k_\beta-6q_\mu k_\beta-4\,p\cdot q\,\delta_{\mu\beta}\right)\delta_{\alpha\nu}\\ &+\left(p_\alpha k_\mu+q_\alpha k_\mu-5p_\alpha q_\mu-4q_\alpha q_\mu\right)\delta_{\nu\beta}+\\ &+\left(k_\nu k_\beta+6p_\nu k_\beta+p_\nu p_\beta+8\,p\cdot q\,\delta_{\nu\beta}\right)\delta_{\alpha\mu}\\ &+\left(6q_\alpha p_\nu-2p_\alpha k_\nu-2q_\alpha k_\nu+4p_\nu p_\alpha\right)\delta_{\mu\beta}\\ &-\left(\tfrac{7}{2}p_\alpha k_\beta+\tfrac{11}{2}p_\alpha p_\beta-2q_\alpha k_\beta+6q_\alpha p_\beta-6\,k\cdot q\,\delta_{\alpha\beta}\right)\delta_{\mu\nu}\Big]\end{aligned} \tag{45.b}$$

$$\begin{aligned} M^3_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\Big[&\left(\tfrac{1}{2}k_\mu k_\nu+\tfrac{1}{2}k_\mu p_\nu-2p_\nu q_\mu+5\,p\cdot q\,\delta_{\mu\nu}\right)\delta_{\alpha\beta}+\\ &+\left(4p_\alpha k_\beta-\tfrac{1}{2}p_\alpha q_\beta\right)\delta_{\mu\nu}-k_\mu k_\beta\delta_{\alpha\nu}+\\ &+\left(p_\nu q_\beta-p_\nu k_\beta-4\,p\cdot q\,\delta_{\nu\beta}\right)\delta_{\alpha\mu}+\\ &+\left(p_\alpha q_\mu-p_\alpha k_\mu-q_\alpha k_\mu\right)\delta_{\nu\beta}\Big]\end{aligned} \tag{45.c}$$

This Bose invariant expression is furthermore gauge independent , satisfying the transversality condition:

$$k_\alpha M^{abcd}_{\alpha\mu\nu\beta}(k,p,q,r) = 0 \quad , \tag{46}$$

in accordance with the gauge invariance required by equation (31).

The infrared single poles contributions associated with the gluon-gluon scattering amplitude involve a large number of terms and are presented in Appendix C. It can be verified that these contributions too, form a gauge independent set as expected for a gauge invariant process.

We hope that these results, concerning the behaviour of three and four gluon vertices, might be useful also in other investigations of higher order QCD corrections.

Acknowledgements

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APPENDIX A

The Feynman rules employed in this paper are more fully described in reference [11] . In particular, the gluon propagator in a general covariant gauge is given by:

$$D^{ab}_{\mu\nu}(Q) = \frac{\delta_{ab}}{Q^2 - i\eta}\left(\delta_{\mu\nu} - \lambda\frac{Q_\mu Q_\nu}{Q^2 - i\eta}\right) \tag{A.1}$$

The Feynman gauge is characterized by the vanishing of the gauge parameter λ . The basic 3-gluon vertex is defined with all momenta inwards as follows:

$$V^{\alpha\mu\nu}_{abc}(k,p,q) = -i\bar{g}\, f_{abc}\left[\delta_{\mu\nu}(q-p)_\alpha + \delta_{\nu\alpha}(k-q)_\mu + \delta_{\mu\alpha}(p-k)_\nu\right] \tag{A.2}$$

where in terms of the dimensionless coupling constant g we have $\bar{g} = \bar{\mu}^{-\varepsilon} g$. For a SU(N) Yang-Mills theory, the structure constants f_{abc} satisfy the relations:

$$f_{aa'b'} f_{bb'c'} f_{cc'a'} = \frac{N}{2} f_{abc} \;;\; f_{aa'b'} f_{ba'b'} = N\delta_{ab} \tag{A.3}$$

Furthermore, in a space-time with dimension $n=4+2\varepsilon$ we have $\delta_{\mu\nu}\delta_{\mu\nu} = 4+2\varepsilon$. The Feynman denominators are combined with the help of the expression:

$$\frac{1}{D_1^{a_1}\cdots D_k^{a_k} D_{k+1}^{a_{k+1}}} = \frac{\Gamma(a_1+\dots+a_k+a_{k+1})}{\Gamma(a_1)\cdots\Gamma(a_k)\Gamma(a_{k+1})}\,\cdot$$

$$\int_0^1\cdots\int_0^1 dx_1\cdots dx_k\,\frac{\theta(1-x_1-\dots-x_k)\,x_1^{a_1-1}\cdots x_k^{a_k-1}\,(1-x_1-\dots x_k)^{a_{k+1}-1}}{\left[x_1(D_1-D_k)+x_2(D_2-D_k)+\cdots+D_{k+1}\right]^{a_1+\dots+a_{k+1}}} \tag{A.4}$$

Finally we perform the n-dimensional integrations over the internal momenta using the relation:

$$\int \frac{d^mQ}{(2\pi)^m} \frac{Q_{\mu_1}\cdots Q_{\mu_{2r}}}{(Q^2+m^2)^j} = \frac{i}{(4\pi)^{\frac{m}{2}}} \sum_{P(\mu_1\cdots\mu_{2r})} (m^2)^{\frac{m}{2}+r-j} \frac{\Gamma(-\frac{m}{2}-r+j)}{\Gamma(j)} \times$$

$$\frac{\delta_{\mu_1\mu_2}\cdots\delta_{\mu_{2r-1}\mu_{2r}}}{2^{2r}\ r!} \tag{A.5}$$

where P denotes permutations over the indicated indices.

APPENDIX B

We list in this appendix the expressions for the Lorentz factors $K^{\alpha\mu\nu;\mu_1\cdots\mu_\ell}_{a_1 a_2 a_3;j}$ associated with the contributions of diagram 2. We obtain in a general covariant gauge the following explicit results:

$$K^{\alpha\mu\nu}_{111;0} = \left[(3-2\varepsilon)p_\mu q_\nu + 3q_\mu q_\nu + \tfrac{1}{2}(5q^2+3p^2)\delta_{\mu\nu}\right]p_\alpha - \\ -(2p^2p_\nu + 4q^2p_\nu + 5p^2q_\nu)\delta_{\alpha\mu} \tag{B.1}$$

$$K^{\alpha\mu\nu;\beta}_{111;0} = \left[3p_\mu p_\nu - (2\varepsilon+2)p_\mu q_\nu + 8q_\mu p_\nu\right]\delta_{\alpha\beta} + \\ + \left[(4\varepsilon-6)q_\alpha q_\nu - (4\varepsilon+2)p_\alpha p_\nu + 3p_\alpha q_\nu - 13q_\alpha p_\nu\right]\delta_{\mu\beta} + \\ + \left[(5p^2-6q^2)\delta_{\nu\beta} - 8p_\nu p_\beta - 4q_\nu p_\beta + 3q_\nu q_\beta\right]\delta_{\alpha\mu} + \\ + \left[6p^2\delta_{\alpha\beta} + 3p_\alpha p_\beta + \tfrac{1}{2}p_\alpha q_\beta\right]\delta_{\mu\nu} \tag{B.2}$$

$$K^{\alpha\mu\nu;\beta\gamma}_{111;0} = -(8\varepsilon+9)(\delta_{\alpha\gamma}\delta_{\mu\beta}q_\nu - \delta_{\mu\gamma}\delta_{\nu\beta}p_\alpha) - \\ - \left[2\delta_{\nu\gamma}q_\beta + 4\delta_{\beta\gamma}p_\nu + 3\delta_{\beta\gamma}q_\nu\right]\delta_{\alpha\mu} + \\ + \left[2\delta_{\alpha\gamma}p_\beta + 3\delta_{\beta\gamma}p_\alpha\right]\delta_{\mu\nu} \tag{B.3}$$

$$K^{\alpha\mu\nu}_{211;1} = -p^2q^2\delta_{\alpha\mu}q_\nu + \left[(p^2+q^2)q^2p_\mu + 2p^2q^2q_\mu\right]\delta_{\alpha\nu} + \\ + \left[\tfrac{1}{2}(p^2-q^2)p^2\delta_{\mu\nu} + 2p^2q_\mu p_\nu\right](p_\alpha+q_\alpha) + \\ + \left[\tfrac{1}{2}(p^2+3q^2)(p_\nu+q_\nu)p_\mu + p^2q_\mu q_\nu\right]p_\alpha + \\ + \tfrac{1}{2}(p^2+3q^2)q_\alpha p_\mu p_\nu \tag{B.4}$$

$$K^{\alpha\mu\nu;\beta}_{211;1} = 2p^2(q^2\delta_{\nu\beta} + q_\nu q_\beta)\delta_{\alpha\mu} + \\ + \left[-p^2q^2\delta_{\mu\beta} + q^2p_\mu(p_\beta - 2q_\beta) - 2p^2(p_\mu p_\beta + 2q_\mu q_\beta)\right]\delta_{\alpha\nu} + \\ + \left[-2p^2q^2\delta_{\alpha\beta} - p^2(p_\alpha(p_\beta+q_\beta) + q_\alpha(p_\beta-q_\beta))\right]\delta_{\mu\nu} + \\ + \left[q^2p_\mu(2p_\nu + \tfrac{1}{2}q_\nu) + p^2q_\nu(\tfrac{1}{2}p_\mu + q_\mu)\right]\delta_{\alpha\beta} - \\ - p^2\left[p_\alpha(p_\nu+q_\nu) + q_\alpha p_\nu\right]\delta_{\mu\beta} + \\ + \left[p^2(p_\mu+2q_\mu)(p_\alpha+q_\alpha) + q^2(q_\alpha-p_\alpha)p_\mu\right]\delta_{\nu\beta} + \\ + \left[2p_\nu p_\beta + p_\nu q_\beta + q_\nu p_\beta - 2q_\nu q_\beta\right]p_\alpha p_\mu + \\ + \left[2p_\nu p_\beta - p_\nu q_\beta + q_\nu p_\beta\right]q_\alpha p_\mu \tag{B.5}$$

$$K^{\alpha\mu\nu;\beta\gamma}_{211;1} = -\left[4p^2\delta_{\nu\gamma}q_\beta + (p^2-q^2)\delta_{\beta\gamma}q_\nu\right]\delta_{\alpha\mu} + \\ + \left[(p^2+q^2)(2\delta_{\mu\gamma}q_\beta + \delta_{\beta\gamma}p_\mu) - 2p_\mu p_\beta q_\gamma + 2(p^2-q^2)\delta_{\beta\gamma}q_\mu\right]\delta_{\alpha\nu} + \\ + \left[2p^2\delta_{\alpha\gamma}q_\beta - \tfrac{1}{2}(p^2-q^2)\delta_{\beta\gamma}p_\alpha - \tfrac{1}{2}(5p^2-q^2)\delta_{\beta\gamma}q_\alpha\right]\delta_{\mu\nu} + \\ + \left[-(p^2+q^2)\delta_{\mu\beta}q_\nu - 2p_\mu p_\nu q_\beta\right]\delta_{\alpha\gamma} + \\ + \left[\tfrac{1}{2}(p^2-q^2)\delta_{\nu\beta}p_\alpha - \tfrac{1}{2}(p^2+q^2)\delta_{\nu\beta}q_\alpha + p_\alpha(2p_\nu q_\beta - q_\nu p_\beta + \\ + q_\nu q_\beta) + q_\alpha(2p_\nu q_\beta + q_\nu p_\beta)\right]\delta_{\mu\gamma} + \\ p_\mu(4p_\alpha p_\beta - q_\alpha p_\beta)\delta_{\nu\gamma} + \left[p_\alpha(2p_\mu q_\nu - 2q_\mu p_\nu - q_\mu q_\nu) + \\ q_\alpha(2p_\mu p_\nu + p_\mu q_\nu - 2q_\mu p_\nu\right]\delta_{\beta\gamma} \tag{B.6}$$

$$K^{\alpha\mu\nu;\beta\gamma\rho}_{211;1} = -\left[4\delta_{\mu\rho} q_\beta q_\gamma + (p^2\delta_{\mu\rho} - p_\mu p_\rho - 4 q_\mu q_\rho)\delta_{\beta\gamma}\right]\delta_{\alpha\nu} + \\ +\left[p^2\delta_{\alpha\beta}\delta_{\nu\rho} + (-2q_\nu p_\beta + 3 q_\nu q_\beta)\delta_{\alpha\rho}\right]\delta_{\mu\gamma} + \\ +\left[\delta_{\alpha\rho} p_\mu p_\beta + \delta_{\mu\rho}(p_\alpha p_\beta + p_\alpha q_\beta - 3 q_\alpha p_\beta + 3 q_\alpha q_\beta)\right]\delta_{\nu\gamma} + \\ +\left[2(p^2-q^2)\delta_{\nu\rho} - 2q_\nu q_\rho\right]\delta_{\beta\gamma}\delta_{\alpha\mu} + \\ +\left[2q^2\delta_{\alpha\rho} + p_\alpha p_\rho + p_\alpha q_\rho + q_\alpha p_\rho - q_\alpha q_\rho\right]\delta_{\beta\gamma}\delta_{\mu\nu} + \\ +\left[-\delta_{\alpha\rho} q_\mu q_\nu + \delta_{\mu\rho} q_\alpha q_\nu - \delta_{\nu\rho}(4p_\alpha p_\mu + 2p_\alpha q_\mu + 2q_\alpha p_\mu + \\ + 2q_\alpha q_\mu)\right]\delta_{\beta\gamma} \qquad (B.7)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho\sigma}_{211;1} = 2\delta_{\alpha\gamma}\delta_{\mu\rho}\delta_{\nu\sigma}(2p_\beta - q_\beta) + 2\delta_{\beta\gamma}\left[(2\delta_{\alpha\mu}\delta_{\nu\sigma} + \\ + \delta_{\alpha\nu}\delta_{\mu\sigma} - \delta_{\mu\nu}\delta_{\alpha\sigma})q_\rho - \delta_{\alpha\rho}\delta_{\mu\sigma} q_\nu - \\ - 2\delta_{\mu\rho}\delta_{\nu\sigma} q_\alpha\right] + \delta_{\beta\gamma}\delta_{\rho\sigma}\left[\delta_{\alpha\mu} q_\nu - 2\delta_{\alpha\nu} q_\mu + 2\delta_{\mu\nu} q_\alpha\right] \qquad (B.8)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho\sigma\tau}_{211;1} = 2\delta_{\beta\gamma}(2\delta_{\alpha\rho}\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\rho\sigma}\delta_{\alpha\mu}\delta_{\nu\tau}) \qquad (B.9)$$

$$K^{\alpha\mu\nu;\beta\gamma}_{112;1} = \left[-2p^2\delta_{\nu\gamma} p_\beta - 8p_\nu p_\beta q_\gamma + 4q_\nu p_\beta p_\gamma - 2q_\nu p_\beta q_\gamma\right]\delta_{\alpha\mu} + \\ + 4\delta_{\mu\nu} p_\alpha p_\beta p_\gamma + \left[-q^2\delta_{\mu\beta} q_\nu + 4p_\mu p_\nu q_\beta - 3p_\mu q_\nu p_\beta\right]\delta_{\alpha\gamma} + \\ + \left[\tfrac{1}{2}(p^2-q^2)\delta_{\nu\beta} p_\alpha + p_\alpha(2p_\nu p_\beta - 3q_\nu p_\beta + q_\nu q_\beta) + \\ + q_\alpha(2p_\nu q_\beta + q_\nu p_\beta)\right]\delta_{\mu\gamma} \qquad (B.10)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho}_{112;1} = -4\delta_{\alpha\mu}\delta_{\nu\rho} p_\beta(p_\gamma + q_\gamma) + 4\delta_{\mu\nu}\delta_{\alpha\rho} p_\beta q_\gamma + \\ + \delta_{\alpha\rho}\delta_{\mu\gamma} q_\nu(-2p_\beta + 3q_\beta) + \delta_{\mu\rho}\delta_{\nu\gamma} p_\alpha(3p_\beta + q_\beta) \qquad (B.11)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho\sigma}_{112;1} = 2\delta_{\alpha\gamma}\delta_{\mu\rho}\delta_{\nu\sigma} p_\beta \qquad (B.12)$$

$$K^{\alpha\mu\nu}_{111;1} = p_\alpha p_\mu q_\nu \qquad (B.13)$$

$$K^{\alpha\mu\nu;\beta}_{111;1} = \left[-p^2\delta_{\nu\beta} + 4p_\nu p_\beta - 4p_\nu q_\beta + 6q_\nu p_\beta - q_\nu q_\beta\right]\delta_{\alpha\mu} + \\ + \left[-2p_\alpha p_\beta + 2p_\alpha q_\beta\right]\delta_{\mu\nu} - (2p_\nu + q_\nu)p_\mu\delta_{\alpha\beta} - \\ - (p_\alpha p_\nu + 2p_\alpha q_\nu + q_\alpha p_\nu + q_\alpha q_\nu)\delta_{\mu\beta} \qquad (B.14)$$

$$K^{\alpha\mu\nu;\beta\gamma}_{111;1} = -2(p_\beta + q_\beta)\delta_{\alpha\mu}\delta_{\nu\gamma} - 4\delta_{\mu\nu}\delta_{\alpha\gamma} p_\beta - \delta_{\alpha\gamma}\delta_{\mu\beta} q_\nu - \\ - \delta_{\mu\gamma}\delta_{\nu\beta} p_\alpha + \delta_{\beta\gamma}\left[2\delta_{\alpha\mu}(p_\nu + q_\nu) - \delta_{\mu\nu} p_\alpha\right] \qquad (B.15)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho}_{111;1} = -\delta_{\alpha\beta}(\delta_{\mu\gamma}\delta_{\nu\rho} + \delta_{\mu\nu}\delta_{\gamma\rho}) \qquad (B.16)$$

$$K^{\alpha\mu\nu}_{221;2} = \tfrac{1}{2}(p^2-q^2)\left[-p^2q^2\delta_{\mu\nu} p_\alpha + q^2 p_\alpha p_\mu p_\nu + \tfrac{1}{2}(p^2+q^2)p_\alpha p_\mu q_\nu + \\ + p^2 p_\alpha q_\mu q_\nu\right] \qquad (B.17)$$

$$K^{\alpha\mu\nu;\beta}_{221;2} = \left[-p^2q^2\delta_{\mu\nu} p_\alpha + q^2 p_\alpha p_\mu p_\nu + p^2 p_\alpha q_\mu q_\nu\right](p_\beta + q_\beta) + \\ + \tfrac{1}{2}(p^2+q^2)p_\alpha p_\mu p_\nu(p_\beta + q_\beta) \qquad (B.18)$$

$$K^{\alpha\mu\nu;\beta\gamma}_{221;2} = (p^2-q^2)\left[\tfrac{1}{2}(p^2+q^2)(\delta_{\mu\nu}\delta_{\beta\gamma} p_\alpha - \delta_{\mu\gamma}\delta_{\nu\beta} p_\alpha) + \\ + \delta_{\mu\gamma} q_\nu q_\beta(p_\alpha + q_\alpha) - \delta_{\beta\gamma} p_\alpha(p_\mu p_\nu + q_\mu q_\nu)\right] \qquad (B.19)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho}_{221;2} = \left[(p^2+q^2)(\delta_{\rho\gamma}\delta_{\mu\nu} - \delta_{\mu\rho}\delta_{\nu\gamma})p_\alpha + \delta_{\mu\rho} q_\nu q_\gamma(p_\alpha + q_\alpha) - \\ - \delta_{\rho\gamma} p_\alpha(p_\mu p_\nu + q_\mu q_\nu)\right](p_\beta + q_\beta) \qquad (B.20)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho\sigma}_{221;2} = \tfrac{1}{2}(p^2-q^2)p_\alpha\delta_{\beta\gamma}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\rho\sigma}\delta_{\mu\nu}) \qquad (B.21)$$

$$K^{\alpha\mu\nu;\beta\gamma\rho\sigma\tau}_{221;2} = \left[\delta_{\beta\gamma}\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\beta\gamma}\delta_{\tau\sigma}\delta_{\mu\nu}\right](p_\rho + q_\rho)p_\alpha \qquad (B.22)$$

$$\begin{aligned} K^{\alpha\mu\nu;\beta\gamma}_{212;2} = {} & q^2\left(-p^2\delta_{\mu\gamma} + 2p_\mu p_\gamma\right)q_\beta\delta_{\alpha\nu} + q^2\left(p^2\delta_{\mu\beta} - p_\mu p_\beta\right)q_\nu\delta_{\alpha\gamma} + \\ & + \left[\tfrac{1}{2}(p^2-q^2)\left(-p^2\delta_{\mu\gamma}\delta_{\nu\beta} + \delta_{\nu\gamma}p_\mu p_\beta\right) + \right. \\ & \left. + p^2\delta_{\mu\gamma}\left(q_\nu p_\beta - 2p_\nu q_\beta\right)\right]\left(p_\alpha + q_\alpha\right) - \\ & - p^2\delta_{\mu\gamma}p_\alpha q_\nu q_\beta + p_\alpha p_\mu\left(2p_\nu p_\beta q_\gamma - q_\nu p_\beta p_\gamma + q_\nu p_\beta q_\gamma\right) + \\ & + q_\alpha p_\mu\left(2p_\nu p_\beta q_\gamma - q_\nu p_\beta p_\gamma\right) \end{aligned} \tag{B.23}$$

$$\begin{aligned} K^{\alpha\mu\nu;\beta\gamma\rho}_{212;2} = {} & 4\left(p^2\delta_{\mu\rho}q_\beta q_\gamma - p_\mu p_\beta q_\gamma p_\rho\right)\delta_{\alpha\nu} + \\ & + 3\left(p_\mu q_\nu p_\beta q_\gamma - p^2\delta_{\mu\gamma}q_\nu q_\beta\right)\delta_{\alpha\rho} + \\ & + \left[-p^2\delta_{\mu\rho}\delta_{\nu\gamma}p_\alpha + \delta_{\nu\rho}p_\alpha p_\mu p_\gamma\right]\left(p_\beta + q_\beta\right) + \\ & + \left[-p^2\delta_{\mu\rho}\delta_{\nu\gamma}q_\alpha + \delta_{\nu\rho}q_\alpha p_\mu p_\gamma\right]\left(p_\beta + 3q_\beta\right) \end{aligned} \tag{B.24}$$

$$K^{\alpha\mu\nu;\beta\gamma\rho\sigma}_{212;2} = 2\left(p^2\delta_{\alpha\gamma}\delta_{\mu\rho}\delta_{\nu\sigma}q_\beta - \delta_{\alpha\rho}\delta_{\nu\sigma}p_\mu p_\beta q_\gamma\right) \tag{B.25}$$

$$\begin{aligned} K^{\alpha\mu\nu;\beta}_{211;2} = {} & q^2\left(p^2\delta_{\mu\beta} - p_\mu p_\beta\right)\delta_{\alpha\nu} + \left[p^2\delta_{\mu\beta}\left(p_\alpha + q_\alpha\right) - p_\alpha p_\mu p_\beta - \right. \\ & \left. - q_\alpha p_\mu p_\beta\right]\left(p_\nu + q_\nu\right) \end{aligned} \tag{B.26}$$

$$\begin{aligned} K^{\alpha\mu\nu;\beta\gamma}_{211;2} = {} & 4\left(-p^2\delta_{\mu\gamma}q_\beta + p_\mu p_\beta q_\gamma\right)\delta_{\alpha\nu} + \\ & + \left(p^2\delta_{\mu\beta}q_\nu - p_\mu q_\nu p_\beta\right)\delta_{\alpha\gamma} + \\ & + \left(p^2\delta_{\mu\beta}q_\alpha - q_\alpha p_\mu p_\beta\right)\delta_{\nu\gamma} \end{aligned} \tag{B.27}$$

$$K^{\alpha\mu\nu;\beta\gamma\rho}_{211;2} = \left(p^2\delta_{\mu\rho} - p_\mu p_\rho\right)\left(\delta_{\beta\gamma}\delta_{\alpha\nu} - \delta_{\alpha\beta}\delta_{\nu\gamma}\right) \tag{B.28}$$

$$\begin{aligned} K^{\alpha\mu\nu;\beta\gamma}_{222;3} = {} & \tfrac{1}{2}(p^2-q^2)\left[p^2\delta_{\mu\gamma}\left(q^2\delta_{\nu\beta}p_\alpha - q_\nu q_\beta\left(p_\alpha + q_\alpha\right)\right) + \right. \\ & \left. + p_\alpha p_\mu q_\nu p_\beta q_\gamma\right] \end{aligned} \tag{B.29}$$

$$\begin{aligned} K^{\alpha\mu\nu;\beta\gamma\rho}_{222;3} = {} & p^2\left[q^2 p_\alpha\delta_{\nu\gamma} - \left(p_\alpha + q_\alpha\right)q_\nu q_\gamma\right]\delta_{\mu\rho}\left(p_\beta + q_\beta\right) + \\ & + p_\alpha p_\mu q_\nu\left(p_\beta p_\gamma q_\rho + q_\beta q_\gamma p_\rho\right) \end{aligned} \tag{B.30}$$

When $a_1 = a_2$, the permutations $\mu \leftrightarrow \nu$ and $p \leftrightarrow q$ are always to be understood. All other non-vanishing functions are obtained from the above expressions, by the symultaneous interchange of Lorentz factors $\mu \leftrightarrow \nu$, gluon momenta $p \leftrightarrow q$ and indices $a_1 \leftrightarrow a_2$.

APPENDIX C

We present here the contributions with single infrared poles resulting from the one particle irreducible 4 point function, together with the corresponding ones associated with the 4 point vertex given in eq. (34). Using momenta conservation $k+p+q+r=0$, we can express then in terms of the folowing Bose invariant function:

$$\begin{aligned} M^{abcd}_{\alpha\mu\nu\beta}(k,p,q) = \frac{i g^4}{16\pi^2}\frac{1}{\varepsilon}\Big\{ & \delta_{ab}\delta_{cd}\left(M^4_{\alpha\mu\nu\beta} + M^5_{\alpha\mu\nu\beta} + M^6_{\alpha\mu\nu\beta} + \right. \\ & \left. + M^7_{\alpha\mu\nu\beta} + M^8_{\alpha\mu\nu\beta}\right) + \\ & + \delta_{ad}\delta_{bc}\left(M^9_{\alpha\mu\nu\beta} + M^{10}_{\alpha\mu\nu\beta}\right) + \\ & \left(\mu\leftrightarrow\nu;\ b\leftrightarrow c;\ p\leftrightarrow q\right)\Big\} + \text{c.p.} \end{aligned} \tag{C.1}$$

where the Lorentz tensors $M^i_{\alpha\mu\nu\beta}$ are given explicitly as follows:

$$
\begin{aligned}
M^{4}_{\alpha\mu\nu\beta} = \frac{1}{k\cdot p}\ln\frac{2k\cdot p}{\mu^{2}}\Big[& \left(4k_\mu k_\nu + 2k_\mu p_\nu + 2k_\nu q_\mu + 2p_\nu q_\mu\right)\delta_{\alpha\beta} + \\
& + \left(2k_\mu p_\alpha - \tfrac{5}{2}k_\mu q_\alpha + 2k\cdot q\,\delta_{\alpha\mu} + 2p_\alpha q_\mu - \tfrac{1}{2}q_\alpha q_\mu\right)\delta_{\nu\beta} + \\
& + \left(-3k_\nu p_\alpha - k_\nu q_\alpha - 4p_\alpha p_\nu - p_\alpha q_\nu\right)\delta_{\mu\beta} + \\
& + \left(-3k_\mu k_\beta - 2k_\mu p_\beta - q_\beta q_\mu\right)\delta_{\alpha\nu} + \\
& + \left(2p_\nu p_\beta - \tfrac{5}{2}k_\nu k_\beta - \tfrac{9}{2}k_\nu q_\beta\right)\delta_{\alpha\mu} + \\
& + \left(2q_\alpha q_\beta + 2p_\alpha p_\beta\right)\delta_{\mu\nu}\Big] + (\mu \leftrightarrow \nu;\ p \leftrightarrow q)
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
M^{5}_{\alpha\mu\nu\beta} = \frac{1}{k\cdot q}\ln\frac{2k\cdot p}{\mu^{2}}\Big[& \left(8k_\mu k_\nu + 8k_\nu q_\mu\right)\delta_{\alpha\beta} - \\
& - \left(8k_\mu q_\alpha + 8q_\alpha q_\mu\right)\delta_{\nu\beta} + \\
& + \left(8q_\alpha p_\nu - 8p_\alpha k_\nu + 4k\cdot q\,\delta_{\alpha\nu}\right)\delta_{\mu\beta} + \\
& + \left(8k_\mu q_\beta - 8q_\mu k_\beta + 8k\cdot p\,\delta_{\mu\beta}\right)\delta_{\alpha\nu} - \\
& - 8q_\alpha p_\beta\delta_{\mu\nu} + 8k_\nu p_\beta\,\delta_{\alpha\mu}\Big] + \\
& + (\mu \leftrightarrow \nu;\ p \leftrightarrow q)
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
M^{6}_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\ln\frac{2p\cdot q}{\mu^{2}}\Big[& \left(2p\cdot q\,\delta_{\mu\nu} - \tfrac{11}{2}k_\mu p_\nu + 6k_\nu q_\mu + \tfrac{1}{2}k_\mu k_\nu - 2q_\mu p_\nu\right)\delta_{\alpha\beta} + \\
& + \left(k_\mu p_\alpha + k_\mu q_\alpha - 5q_\mu p_\alpha - 4q_\mu q_\alpha\right)\delta_{\nu\beta} + \\
& + \left(4p_\alpha p_\nu + 6q_\alpha p_\nu - 2p_\alpha k_\nu - 2q_\alpha k_\nu\right)\delta_{\mu\beta} - \\
& - \left(2k_\mu k_\beta + 8q_\mu k_\beta + 2q_\mu q_\beta - 8p\cdot q\,\delta_{\mu\beta}\right)\delta_{\alpha\nu} + \\
& + \left(k_\nu k_\beta + 6p_\nu k_\beta + p_\nu p_\beta + 4p\cdot q\,\delta_{\nu\beta}\right)\delta_{\alpha\mu} + \\
& + \left(2p\cdot q\,\delta_{\alpha\beta} - 6k\cdot q\,\delta_{\alpha\beta} + \tfrac{7}{2}p_\alpha q_\beta - 2p_\alpha p_\beta - \right. \\
& \left. - 6q_\alpha p_\beta + 2q_\alpha k_\beta\right)\delta_{\mu\nu}\Big]
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
M^{7}_{\alpha\mu\nu\beta} = \frac{1}{k\cdot p}\Big[& -\left(12k_\mu k_\nu + 10k_\mu p_\nu + 2k_\nu q_\mu + 2p_\nu q_\mu\right)\delta_{\alpha\beta} + \\
& + \left(\tfrac{1}{4}p_\alpha k_\mu + \tfrac{21}{4}q_\alpha k_\mu - 4k\cdot q\,\delta_{\alpha\mu} - \tfrac{31}{4}p_\alpha q_\mu - \tfrac{5}{2}q_\alpha q_\mu\right)\delta_{\nu\beta} + \\
& + \left(\tfrac{41}{4}p_\alpha k_\nu + \tfrac{13}{4}q_\alpha k_\nu + \tfrac{27}{2}p_\alpha p_\nu + \tfrac{13}{4}q_\alpha p_\nu\right)\delta_{\mu\beta} + \\
& + \left(\tfrac{41}{4}k_\mu k_\beta + 7k_\mu p_\beta + \tfrac{13}{4}q_\mu q_\beta\right)\delta_{\alpha\nu} + \\
& + \left(\tfrac{21}{4}k_\nu k_\beta + 5k_\nu q_\beta + \tfrac{31}{4}p_\nu k_\beta - \tfrac{1}{4}p_\nu q_\beta\right)\delta_{\alpha\mu} + \\
& + \left(2p_\alpha p_\beta + 8p_\alpha q_\beta - 2q_\alpha q_\beta\right)\delta_{\mu\nu}\Big] + (\mu \leftrightarrow \nu;\ p \leftrightarrow q)
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
M^{8}_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\Big[& \left(\tfrac{5}{2}k_\mu k_\nu - 8p\cdot q\,\delta_{\mu\nu} - 12k\cdot p\,\delta_{\mu\nu} + \tfrac{43}{4}k_\mu p_\nu - \tfrac{33}{4}q_\mu k_\nu - \tfrac{1}{4}q_\mu p_\nu\right)\delta_{\alpha\beta} + \\
& + \left(\tfrac{23}{4}p_\alpha q_\mu + \tfrac{5}{2}q_\alpha q_\mu - \tfrac{13}{4}p_\alpha k_\mu - \tfrac{13}{4}q_\alpha k_\mu\right)\delta_{\nu\beta} + \\
& + \left(2p_\alpha k_\nu + 2q_\alpha k_\nu - 4p_\alpha p_\nu - 6q_\alpha p_\nu\right)\delta_{\mu\beta} + \\
& + \left(2k_\mu k_\beta + 8q_\mu k_\beta + 2q_\mu q_\beta\right)\delta_{\alpha\nu} - \\
& - \left(\tfrac{13}{4}k_\nu k_\beta + 9p_\nu k_\beta + \tfrac{13}{4}p_\nu p_\beta\right)\delta_{\alpha\mu} - \\
& - \left(11p_\alpha q_\beta + \tfrac{1}{4}p_\alpha p_\beta + 8q_\alpha k_\beta + \tfrac{33}{4}q_\alpha q_\beta\right)\delta_{\mu\nu}\Big]
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
M^{9}_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\ln\frac{2p\cdot q}{\mu^{2}}\Big[& \left(5p\cdot q\,\delta_{\mu\nu} + \tfrac{1}{2}k_\mu k_\nu + \tfrac{1}{2}k_\mu p_\nu - 2q_\mu p_\nu\right)\delta_{\alpha\beta} + \\
& + \left(p_\alpha q_\mu - 4p\cdot q\,\delta_{\alpha\mu} - p_\alpha k_\mu - q_\alpha k_\mu\right)\delta_{\nu\beta} - \\
& - k_\mu k_\beta\delta_{\alpha\nu} + \left(p_\nu q_\beta - p_\nu k_\beta\right)\delta_{\alpha\mu} + \\
& + \left(4p_\alpha k_\beta - \tfrac{1}{2}p_\alpha q_\beta\right)\delta_{\mu\nu}\Big]
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
M^{10}_{\alpha\mu\nu\beta} = \frac{1}{p\cdot q}\Big[& \left(\tfrac{5}{2}k_\mu k_\nu - 2p\cdot q\,\delta_{\mu\nu} + \tfrac{5}{2}k_\mu p_\nu - \tfrac{1}{4}q_\mu p_\nu\right)\delta_{\alpha\beta} + \\
& + \left(4p\cdot q\,\delta_{\alpha\mu} - \tfrac{5}{4}p_\alpha k_\mu - \tfrac{5}{4}q_\alpha k_\mu - \tfrac{1}{4}p_\alpha q_\mu\right)\delta_{\nu\beta} - \\
& - \left(\tfrac{5}{4}k_\mu k_\beta - \tfrac{5}{8}q_\mu p_\beta\right)\delta_{\alpha\nu} - \tfrac{3}{2}p_\alpha p_\nu\delta_{\mu\beta} + \\
& + \left(\tfrac{1}{4}p_\nu k_\beta + \tfrac{5}{8}p_\nu q_\beta\right)\delta_{\alpha\mu} + \\
& + \left(3p_\alpha k_\beta + \tfrac{5}{2}p_\alpha p_\beta\right)\delta_{\mu\nu}\Big]
\end{aligned}
\tag{C.8}
$$

These terms yield , as expected , a gauge invariant contribution satisfying the transversality condition (46) .

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