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LOGARITHMIC CORRECTIONS TO THE UNCERTAINTY  
PRINCIPLE AND INFINITUDE OF THE NUMBER OF  
BOUND STATES OF N-PARTICLE SYSTEMS

by

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LOGARITHMIC CORRECTIONS TO THE UNCERTAINTY PRINCIPLE AND  
INFINITUDE OF THE NUMBER OF BOUND STATES OF N-PARTICLE SYSTEMS

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ABSTRACT

We show that critical long distance behaviour for a two-body potential, defining the finiteness or infinitude of the number of negative eigenvalues of Schrödinger operators in  $v$ -dimensions, are given by  $v_k(r) = -\left(\frac{v-2}{2r}\right)^2 - \frac{1}{(2r \ln r)^2} + \dots - \frac{1}{(2r \ln r \ln \ln r \dots \ln^{(k)} r)^2}$  where  $k=0,1,\dots$  for  $v \neq 2$  and  $k=1,2,\dots$  if  $v=2$ . This result is a consequence of logarithmic corrections to an inequality known as Uncertainty Principle.

If the continuum threshold in the N-body problem is defined by a two-cluster break up our results generate corrections to the existing sufficient conditions for the existence of infinitely many bound states.

I. INTRODUCTION

It is well known that the finiteness or infinitude of the number of bound-states of negative energy of a Schrödinger operator  $[-\Delta+V]$  is controlled by the long-distance behaviour of the potential [1,2,4,11]. For dimension  $v \neq 2$  a finiteness-infinitude borderline is set by a fall-off  $\sim \left(\frac{v-2}{2r}\right)^2$  as  $r \rightarrow \infty$ . Not coincidentally, for the quadratic form  $(\psi, [-\Delta+V]\psi)$ ,  $\psi \in C_0^\infty(\mathbb{R}^v \setminus \{0\})$  and  $V$  being a Kato potential<sup>†</sup>, the following results hold:

A) "Uncertainty Principle Lemma" [2,3,8,12]

If  $V(x) \geq -\left(\frac{v-2}{2r}\right)^2$  then

$$(\psi, [-\Delta+V]\psi) \geq 0 \quad (1.1)$$

and

B) If, for  $\alpha > 1$ ,  $r \geq R_0 > 0$ ,  $V(x) \leq -\alpha \left(\frac{v-2}{2r}\right)^2$  then there exists an infinite sequence  $\{\psi_n \in C_0^\infty(\mathbb{R}^v \setminus \{0\})\}_{n \geq 1}$  with disjoint supports, such that

$$(\psi_n, [-\Delta+V]\psi_n) < 0 \quad (1.2)$$

<sup>†</sup>Through this paper we will assume that the potential functions satisfy the Kato condition,  $V \in L^2(\mathbb{R}^v) + L_\epsilon^\infty(\mathbb{R}^v)$  i.e., for any  $\epsilon > 0$ , there exists a decomposition

$$V = V_{1,\epsilon} + V_{2,\epsilon} \quad \text{with} \quad V_{1,\epsilon} \in L^2(\mathbb{R}^v), V_{2,\epsilon} \in L^\infty(\mathbb{R}^v)$$

and  $\|V_{2,\epsilon}\|_\infty < \epsilon$ .

This condition will ensure self-adjointness of the relevant Hamiltonians, both for the two-body and the N-body case. [2].

<sup>(\*)</sup>Partially supported by the CNPq.

From A) (as proved by Simon [1] for  $v=3$ ) it follows that if  $V(x) \geq -\left(\frac{v-2}{2r}\right)^2$  for  $r \geq R_0 > 0$ , then  $[-\Delta + V]$  has at most finitely many negative eigenvalues. Under the assumptions of B), the "min-max principle" implies the existence of infinitely many eigenstates of negative energy.

For  $v=2$ , however, property A is trivial and property B is false!

The original purpose of our investigation was to determine the critical asymptotic behaviour of the potential for  $v=2$ . The answer is that for  $v=2$  the critical (in the same sense as above) long distance fall-off is  $\sim -\frac{1}{(2r \ln r)^2}$ . This follows from appropriately modified versions of A and B above.

Nevertheless, it turns out that the  $v=2$  result is only the first term of an infinite series of logarithmic corrections for  $v=1$  and 3 results! This is a consequence of the following chain of facts:

i) Under suitable domain restrictions, the unitary operator  $T_v: L^2(\mathbb{R}_+, r^{v-1} dr) \rightarrow L^2(\mathbb{R}_+, dr)$ ,  $(T_v \psi)(r) = r^{\frac{v-1}{2}} \psi(r)$  establishes a unitary equivalence between the radial part of the 2-dimensional Laplacean and the critically perturbed radial part of the  $v$ -dimensional Laplacean:

$$T_v \left( -\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \right) T_v^{-1} = T_v \left[ -\frac{1}{r^{v-1}} \frac{d}{dr} r^{v-1} \frac{d}{dr} - \left( \frac{v-2}{2r} \right)^2 \right] T_v^{-1} = -\frac{d^2}{dr^2} - \frac{1}{4r^2} \quad (1.3)$$

More generally, if  $a: \mathbb{R}_+ \setminus N_a \rightarrow \mathbb{R}_+$  is  $C^\infty$  and  $a(r) > 0$  for all  $r \in \mathbb{R}_+ \setminus N_a$ , where  $N_a$  is a finite set, then the unitary operator  $U_a: L^2(\mathbb{R}_+, a dr) \rightarrow L^2(\mathbb{R}_+, dr)$ , given by  $(U_a \psi)(r) = a^{1/2} \psi(r)$  transforms the "radial  $a$ -Laplacean"

as:

$$U_a \left( -\frac{1}{a} \frac{d}{dr} a \frac{d}{dr} \right) U_a^{-1} = \left[ -\frac{d^2}{dr^2} - \frac{1}{4} \left( \frac{a'}{a} \right)^2 + \frac{1}{2} \left( \frac{a''}{a} \right) \right] \quad (1.4)$$

when restricted, for instance, to  $C_0^\infty(\mathbb{R}_+ \setminus N_a)$ . (From now on we shall use a prime to denote derivatives with respect to  $r$ .)

Remark. Since  $\left( -\frac{1}{a} \frac{d}{dr} a \frac{d}{dr} \right)$  is a positive operator, when restricted to  $C_0^\infty(\mathbb{R}_+ \setminus N_a)$ , (1.3) provides a trivial proof of the "Uncertainty Principle Lemma".

ii) For a class of functions  $a(r)$  as above, it is possible to find a critical potential  $v_a$  for the  $a$ -Laplacean. It is given by

$$v_a(r) = -\frac{1}{(2a(r)h(r))^2} \quad (1.5)$$

where  $h$  is a monotonic function satisfying

$$h'(r) = \frac{1}{a(r)} \quad (1.6)$$

In fact, denoting by  $S_a$  the finite set where  $a$  or  $h$  are zero, we prove

Lemma 1. If  $\psi \in C_0^\infty(\mathbb{R}_+ \setminus S_a)$  then

$$\int (\psi')^2 a dr \geq \int v_a \psi^2 a dr \quad (1.7)$$

Lemma 2. If  $\lim_{r \rightarrow \infty} h(r) = \infty$  then, given  $\epsilon > 0$  arbitrary, there exists an infinite family of non-zero functions, with disjoint

supports,  $\{\psi_n \in C_0^\infty(\mathbb{R}_+ \setminus S_a)\}_{n \geq 1}$ , such that

$$\int (\psi'_n)^2 a dr < (1 + \epsilon) \int v_a \psi_n^2 a dr. \quad (1.8)$$

Remarks. (1.7) is a version of an inequality of Hardy [2,3,8,12] known as the "Uncertainty Principle Lemma". Lemma 2 says that the constants appearing in the definition  $v_a$  are best possible.

iii) Finally, the whole procedure may be iterated provided we can find  $b: \mathbb{R}_+ \setminus N_b \rightarrow \mathbb{R}_+$ , with the same assumed properties of  $a(r)$ , such that

$$U_b \left( -\frac{1}{b} \frac{d}{dr} b \frac{d}{dr} \right) U_b^{-1} = U_b \left[ -\frac{1}{a} \frac{d}{dr} a \frac{d}{dr} + v_a(r) \right] U_a^{-1}. \quad (1.9)$$

Starting with  $a=r$  and iterating the whole procedure we obtain the result that the potentials

$$v_k(r) = -\left(\frac{\nu-2}{2r}\right)^2 - \left(\frac{1}{2r \ln r}\right)^2 - \dots - \left(\frac{1}{2r \ln r \ln_{k_2} r \dots \ln_{k_{k-1}} r}\right)^2 \quad (1.10)$$

for  $k \geq 0$  if  $\nu \neq 2$  and  $k \geq 1$  if  $\nu = 2$  are critical, i.e., for some  $r \geq R_0 > 0$ ,

(a) if  $V(x) > (1+\epsilon)v_k(r)$  then  $[-\Delta+V]$  has finitely many negative eigenvalues or

(b) if  $V(x) \leq v_{k-1}(r) - \frac{1+\epsilon}{(2r \ln r \dots \ln_{k-1} r)^2}$ , for some  $\epsilon > 0$ , then  $[-\Delta+V]$  has infinitely many negative eigenvalues.

Notation: For  $k \geq 2$ ,  $\ln_{(k_2)} r = \ln \ln_{(k_2-1)} r$   
and  $\ln_{(1)} r = \ln r$ .

Our results amount, in fact, to logarithmic corrections to the "Uncertainty Principle", a widely used tool in the proofs of self-adjointness of strongly singular potentials (see, for instance, [8], [12] and [2]). In a separate paper [13] we discuss the implications of our results to this problem.

Relative to the two-body problem, the N-body problem presents the extra difficulty of locating the threshold (the infimum of the essential spectrum of the N-body Hamiltonian with center of mass motion removed). However, if the threshold as given by Hunziker's theorem [5], is defined by a two-cluster break up we can extend the results of Simon [1] concerning sufficient conditions for the existence of infinitely many bound-states.

This paper is organized as follows. In section II we prove lemmas 1 and 2 and discuss the 2-body problem. In section III the N-body problem is briefly discussed.

## II. THE TWO-BODY PROBLEM: FINITENESS AND INFINITUDE

A general proof of inequalities of type (1.7) can be found in [8]. For completeness we present the following simple

Proof of Lemma 1:-

Let  $\psi(r) = g(r)\varphi(r)$ , where  $g^2 = h$ . Then

$$\begin{aligned} \int (\psi')^2 a dr &\geq \int \varphi^2 (g')^2 a dr + 2 \int g g' \varphi \varphi' a dr = \\ &= \int \varphi^2 \left(\frac{g'}{g}\right)^2 a dr + \frac{1}{2} \int (\varphi^2)' (g^2)' a dr = \\ &= \int \varphi^2 v_a a dr. \end{aligned} \quad \text{q.e.d.}$$

.7.

Proof of Lemma 2:-

1) Let us first consider the case  $a(r) = 1$  and  $h(r) = r$ .

Since for  $\psi = r^{1/2} \varphi$

$$\int (\psi')^2 dr = \left\{ 1 + \frac{\int (\varphi')^2 r dr}{\int \frac{\varphi^2}{r} dr} \right\} \int \psi^2 v_a dr$$

it is enough to show the existence of an infinite sequence  $\{\varphi_n \in C_0^\infty(\mathbb{R}_+ \setminus S_a)\}_{n \geq 1}$  such that

$$\frac{\int (\varphi_n')^2 r dr}{\int \frac{\varphi_n^2}{r} dr} < \varepsilon \quad (2.1)$$

The l.h.s. of (2.1) is scale invariant, i.e.,

$$\frac{\int (\varphi_n')^2 r dr}{\int \frac{\varphi_n^2}{r} dr} = \frac{\int (\varphi')^2 r dr}{\int \frac{\varphi^2}{r} dr}$$

where  $\varphi_n(r) = \varphi(\alpha_n r)$ . It is, therefore, sufficient to find just one  $\varphi \in C_0^\infty(\mathbb{R}_+ \setminus S_a)$  satisfying (2.1) and the infinite sequence  $\varphi_n = \varphi(\alpha_n r)$  will be generated by suitably choosing  $\alpha_n$  to make the supports disjoint. A possible choice of  $\varphi$  is, as given in [9],

.8.

$$\varphi(r) = \begin{cases} 0 & , r \leq R_0 \\ \rho(r-R_0) & , R_0 \leq r \leq R_0+1 \\ 1 & , R_0+1 \leq r \leq R_0+N \\ \rho(1 - \frac{r-R_0}{N}) & , R_0+N \leq r \leq R_0+2N \\ 0 & , r \geq R_0+2N \end{cases}$$

with  $R_0 > \max_{r \in S_a} r$  and  $\rho \in C^\infty(\mathbb{R}_+)$  with  $\rho(r) = 0$  if  $0 \leq r \leq \frac{1}{4}$ ,  $\rho(r) = 1$  if  $r \geq \frac{3}{4}$ . Since  $\lim_{N \rightarrow \infty} \left( \frac{\int (\varphi')^2 r dr}{\int \frac{\varphi^2}{r} dr} \right) = 0$ ,

it is enough to take  $N$  sufficiently large to verify (2.1).

2) Let now  $\tilde{\psi}(r) = \psi(h(r))$ . Then

$$\int (\tilde{\psi}')^2 a dr = \int (\psi')^2 dr$$

and

$$\int (\tilde{\psi})^2 v_a a dr = \frac{1}{4} \int \frac{\psi^2}{r^2} dr$$

Taking then  $\tilde{\varphi}_n = \varphi_n \circ f$  with  $\varphi_n$  as given in part 1) makes the proof complete.

q.e.d.

Remarks. The assumption  $\lim_{r \rightarrow \infty} h(r) = \infty$  is used to guarantee that the functions  $\tilde{\varphi}_n(r) = \varphi_n(h(r))$  are not identically zero. It is not the best possible assumption for the result is still true if  $a(r) = r^n$ ,  $n \geq 1$ . However, some assumption on  $a(r)$  is required as the result is false if  $a(r)h(r) = r^n$ ,  $n \geq 1$ .

We now describe how, starting with  $a_0 = r$ , it is

possible to generate an infinite chain of logarithmic corrections to the "Uncertainty Principle" as described by Lemmas 1 and 2.

Let  $a_n(r) = a_{n-1}(r) \ln_{(n)} r$ ,  $n = 1, 2, \dots$

A straightforward computation gives, for all  $\psi \in C_0^\infty(\mathbb{R}_+ \setminus S_{a_n})$ ,

$$U_{a_n} \left( -\frac{1}{a_n} \frac{d}{dr} a_n \frac{d}{dr} \right) U_{a_n}^{-1} \psi = U_{a_{n-1}} \left( -\frac{1}{a_{n-1}} \frac{d}{dr} a_{n-1} \frac{d}{dr} + v_{a_{n-1}} \right) U_{a_{n-1}}^{-1} \psi, \quad (2.2)$$

with  $U_a$  as given in the introduction. Therefore, applying Lemmas 1 and 2 to  $a_n$  obtains

Lemma 3. Let  $v_k(r)$  be given by

$$v_0(r) = -\frac{(v-2)^2}{4r^2} \quad (2.3a)$$

$$v_k(r) = v_{k-1}(r) - \frac{1}{(2r \prod_{n=1}^k \ln_{(n)} r)^2}, \quad k=1, 2, \dots \quad (2.3b)$$

Then:

(a) For  $\psi \in C_0^\infty(\mathbb{R}_+ \setminus S_{a_k})$

$$\int (\psi')^2 dr \geq \int \psi^2 v_k dr \quad (2.4)$$

and

(b) For  $\epsilon > 0$ , there exists an infinite sequence of non-zero functions, with disjoint supports,  $\{\psi_n \in C_0^\infty(\mathbb{R}_+ \setminus S_{a_k})\}$  such that

$$\int (\psi_n')^2 dr \leq \int (\psi_n)^2 \left[ v_k - \frac{1+\epsilon}{(2r \prod_{n=1}^k \ln_{(n)} r)^2} \right] dr$$

One of the main ingredients in our discussion below is the so-called "Min-Max Principle": Let  $H$  be a self-adjoint operator in Hilbert space with quadratic form domain  $Q(H)$ , and for  $n = 1, 2, \dots$  let

$$\mu_n(H) = \sup_{\varphi_1, \dots, \varphi_{n-1}} \inf_{\psi \in [\varphi_1, \dots, \varphi_{n-1}]^\perp} (\psi, H\psi), \quad \|\psi\| = 1, \psi \in Q(H) \quad (2.6)$$

where  $[\varphi_1, \dots, \varphi_{n-1}]^\perp$  indicates the orthogonal complement of the subspace generated by  $\varphi_1, \dots, \varphi_{n-1}$ . Then, for each  $n$ , either

- (a) there are  $n$  eigenvalues (counting multiplicities) below the bottom of the essential spectrum, and  $\mu_n(H)$  is the  $n$ -th eigenvalue counting multiplicity in increasing order
- or
- (b)  $\mu_n$  is the bottom of the essential spectrum, and in this case,  $\mu_n = \mu_{n+1} = \dots$  and there are at most  $(n-1)$  eigenvalues (counting multiplicity) below  $\mu_n$ .

We are now prepared to state and prove our main results.

Theorem 1. Let  $V$  be a Kato potential in  $L^2(\mathbb{R}^v)$ ,  $v = 1, 2, 3$ , such that for some  $R_0 > 1$  and  $\epsilon > 0$ ,

$$V(x) \leq v_k(r) - \frac{1+\epsilon}{(2r \prod_{n=1}^k \ln_{(n)} r)^2}, \quad \begin{matrix} k=0, \dots & \text{if } v \neq 2 \\ k=1, 2, \dots & \text{if } v = 2 \end{matrix}$$

Then, the operator  $[-\Delta+V]$  has infinitely many negative eigenvalues.

Proof. By the min-max principle, it is sufficient to exhibit an infinite sequence  $\{\psi_n \in Q(-\Delta+V)\}_{n \geq 1}$ , with disjoint supports, such that  $(\psi_n, [-\Delta+V]\psi_n) < 0$ . The existence of such a sequence follows from Lemma 3.

q.e.d

Theorem 2. Let  $V$  be a Kato potential in  $L^2(\mathbb{R}^v)$ ,  $v=1,2,3$ , such that, for  $R_0 > 1$ ,  $c < 1$  and  $k$ ,

$$V(x) \geq c \chi_k(r) \quad \text{if} \quad r > R_0$$

where  $k=0,1,\dots$  if  $v \neq 2$  and  $k=1,2,\dots$  if  $v=2$ .

Then  $[-\Delta+V]$  has at most finitely many negative eigenvalues.

Proof. We first decompose our operator into

$$-\Delta + V = (-c\Delta + V\chi_2) + (-(1-c)\Delta + V\chi_1),$$

where  $\chi_1 \in C_0^\infty$ ,  $\chi_1(x) = 1$  if  $r \leq R_0$ ,  $0 \leq \chi_1 \leq 1$  and  $\chi_2(x) = 1 - \chi_1(x)$ .

From a simple application of the min-max principle, it follows that if both operators  $A = -(1-c)\Delta + V\chi_1$  and  $B = -c\Delta + V\chi_2$  (which are essentially self-adjoint in the same domain and have the same essential spectrum) have finitely many negative eigenvalues then the same holds for  $-\Delta+V = A+B$  (see for instance [2], vol. IV exercise 129 pg 379).

That the operator  $A$  has finitely many negative eigenvalues is a standard result since the potential  $V\chi_1$  has compact support (see for instance [2], vol. IV exercise 20 pg 366). On the other hand, by assumption,  $B > c(-\Delta + \chi_2 v_k)$  and

it is therefore sufficient to show that the operator  $-\Delta + \chi_2 v_k$  has finitely many negative eigenvalues. If  $v \geq 2$  it is sufficient to consider the operator  $-\Delta + \chi_2 v_k$  restricted to the subspace  $\mathcal{H}_0$  of spherically symmetric functions since in  $\mathcal{H}_0^\perp$  the operator is positive! The restriction to  $\mathcal{H}_0$  is given by the operator

$$H_k = \left\{ -\frac{1}{r^{v-1}} \frac{d}{dr} r^{v-1} \frac{d}{dr} + \chi_2 v_k(r) \right\}.$$

For  $v=1$  we consider the operator  $(-\frac{d^2}{dx^2})_D + \chi_2 v_k$ , with Dirichlet boundary conditions on  $\pm R_0$ .

For  $v=2,3$  a similar argument applies for the operator  $H_k$  restricted to  $\mathcal{H}_0$ , thus concluding the proof.  
q.e.d.

Remarks. From the proofs it is clear that the finiteness or infinitude is controlled by the following limits:

$$\mu_0 = \lim_{r \rightarrow \infty} (2r)^2 V(x), \dots, \mu_k = \lim_{r \rightarrow \infty} \left\{ (2r)^2 \frac{1}{\omega_{v-1}} \int_{\omega_{v-1}}^r [V(x) - \chi_k(r)] \right\}.$$

Indeed finiteness is implied by

$$\begin{aligned} \mu_0 = \dots = \mu_{k-1} &= -1, \chi_k > -1 && \text{for some } k \geq 0 \text{ if } v \neq 2 \\ \mu_1 = \dots = \mu_{k-1} &= -1, \chi_k > -1 && \text{for some } k \geq 1 \text{ if } v = 2, \end{aligned}$$

whereas infinitude is guaranteed by

$$u_0 = \dots = u_{k-1} = -1, \mu_k < -1 \quad \text{for some } k \geq 0 \quad \text{if } \nu \neq 2$$

$$u_1 = \dots = u_{k-1} = -1, \mu_k < -1 \quad \text{for some } k \geq 1 \quad \text{if } \nu = 2$$

III. THE N-BODY PROBLEM: INFINITUDE

This section constitutes a sort of appendix of section 3 of Barry Simon's work [1]. Therefore we shall not give all the details and instead we shall be rather sketchy.

Let us consider a system of N-particles, with masses  $m_i, i=1, \dots, N$ , in  $\nu=1, 2$  or 3 dimensions, interacting via two-body Kato potentials  $V_{ij}(\vec{r}_i - \vec{r}_j)$ . The Hamiltonian  $H_N$ , after removal of the center of mass motion,

$$H_N = \sum_{i=1}^N \frac{p_i^2}{2m_i} + \sum_{i < j} V_{ij}(\vec{r}_i - \vec{r}_j) - \frac{(\sum p_i)^2}{2(\sum m_i)}$$

has the infimum  $\sigma$  of its essential spectrum given by Hunziker's theorem [5]:

$$\sigma = \min_{\substack{D, D_2 = \emptyset \\ D \cup D_2 = \{1, \dots, N\}}} [\sigma_D + \sigma_{D_2}]$$

where  $\sigma_D = \text{infimum spectrum } H_D$ ; here  $H_D$  denotes the Hamiltonian of the cluster  $D \subset \{1, \dots, N\}$ , with center of mass kinetic energy removed. If  $\sigma = \sigma_{D_1} + \sigma_{D_2}$  and  $H_{D_1}$  and  $H_{D_2}$  have discrete ground states at the bottom of their spectra we say, after [1], that the system has a "two-cluster continuum limit".

It should be remarked that there is a number of situations for which it can be proved that the system has a "two-cluster continuum limit", namely: a) for  $\nu=1, 2$  a sufficient condition is that  $\int V_{ij}(\vec{r}) d^{\nu}x < 0$  [6]; b) for  $\nu=3$  a sufficient condition is that  $V_{ij}$ 's are purely attractive and hold a bound state [7].

As in [1], if we are in the two-cluster limit case, sufficient conditions for infinitude can be obtained by reducing the analysis to that of an effective two-body problem.

Theorem 3. Let  $V_{ij}$  be Kato potentials that are  $C^\infty$  functions on an open set of  $\mathbb{R}^\nu$  whose complement has zero measure and let  $\{ \}$  be given by a two cluster break up  $(D_1, D_2)$ , with

reduced mass  $\mu_{D_1, D_2} = \left( \frac{1}{\sum_{i \in D_1} m_i} + \frac{1}{\sum_{j \in D_2} m_j} \right)^{-1}$ . Denoting by  $\vec{R}$  the relative coordinate of the center of masses of clusters  $D_1$  and  $D_2$ , if the potential

$$\tilde{V}_{D_1, D_2}(\vec{R}) = 2\mu_{D_1, D_2} \sum_{\substack{i \in D_1 \\ j \in D_2}} V_{ij}(\vec{R})$$

satisfies the assumptions of theorem 1, then  $H_N$  has infinitely many eigenvalues below  $\{ \}$ .

Remark. We believe that this theorem holds for Kato potentials without that extra smoothness assumption.

Proof. Since  $H_N = H_{D_1} + H_{D_2} + V_{D_1, D_2} - \frac{1}{2\mu_{D_1, D_2}} \Delta_{\vec{R}}$ , where

$$V_{D_1, D_2} = \sum_{\substack{i \in D_1 \\ j \in D_2}} V_{ij}(\vec{x}_i - \vec{x}_j)$$

is the intercluster potential, for

$$\psi = \psi_{D_1} \psi_{D_2} \phi \quad \text{we have}$$



$$(\psi, H_N \psi) = E_{D_1} + E_{D_2} + (\phi, [-\frac{1}{2\mu_{D_1 D_2}} \Delta_{\vec{R}} + \bar{V}] \phi)$$

where 
$$\bar{V}(\vec{R}) = \sum_{\substack{i \in D_1 \\ j \in D_2}} (\psi_{D_1} \psi_{D_2}, V_{ij}(\vec{x}_i - \vec{x}_j) \psi_{D_1} \psi_{D_2})$$

is the effective intercluster potential when the clusters  $D_1$  and  $D_2$  are in their bound states  $\psi_{D_1}$  and  $\psi_{D_2}$ , respectively, with corresponding energies  $E_{D_1}$  and  $E_{D_2}$ .

The proof of the theorem is completed by the following generalization of Proposition 5 in [1]:

Lemma 4. Let  $\psi_{D_1}$  be a bound state of  $H_{D_1}$ , a  $k_1$ -body system with Kato potentials that are  $C^\infty$  functions on an open set of  $\mathbb{R}^{v_1}$  whose complement has zero measure. Let  $V_{ij}$  be Kato potentials such that for some  $\gamma \leq 2$  and  $\ell \geq 1$

$$\lim_{r \rightarrow \infty} (2r \prod_{n=1}^{\ell} \ln_{n+1} r)^{\gamma} [V_{ij}(\vec{x}) - v_{\ell-1}(r)] \leq C_{\ell}$$

$v_{\ell}$  given by (2.3). Let

$$\bar{V}_{ij}(\vec{R}) = \int |\psi_{D_1}(\vec{r}_1)|^2 |\psi_{D_2}(\vec{r}_2)|^2 V_{ij}(\vec{r}_j(\vec{R}, \vec{r}_1, \vec{r}_2)) d_{\vec{r}_1}^{v_1} d_{\vec{r}_2}^{v_2}$$

where  $\vec{r}_j(\vec{R}, \vec{r}_1, \vec{r}_2)$  is the distance between particles  $i \in D_1$  and  $j \in D_2$ , in terms of the internal coordinates  $\vec{r}_i(\vec{r}_i)$  of  $D_1$  and the distance  $\vec{R}$  between the centers of mass of  $D_1$  and  $D_2$ . Then

$$\lim_{R \rightarrow \infty} (2R \prod_{n=1}^{\ell} \ln_{n+1} R)^{\gamma} [\bar{V}_{ij}(\vec{R}) - v_{\ell-1}] \leq C_{\ell}$$

Proof. The proof follows by repetition of the steps in [1, Proposition 5] having in mind that the extra smoothness assumption on the potentials ensures that the function

$$f(\vec{r}_0) = \int d^{v_1} d^{v_2} \frac{1}{r^{v_1+v_2-\nu}} |\psi_{D_1}(\vec{r}_1)|^2 |\psi_{D_2}(\vec{r}_2)|^2$$

(with integration over all coordinates but  $\vec{r}_0 : \vec{r}_j(\vec{R}, \vec{r}_1, \vec{r}_2), \vec{R} - \vec{r}_0$ ) decays faster than any power:

$$\sup_{\vec{r}_0} |f(\vec{r}_0) (1+r_0^n)| < \infty$$

for all  $n$ . This is a result by Hunziker [5].

q.e.d.

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