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A HIERARCHICAL MODEL EXHIBITING THE
KOSTERLITZ-THOULESS FIXED POINT

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ABSTRACT

We construct a hierarchical model for 2-d Coulomb gases displaying a line stable of fixed points describing the Kosterlitz-Thouless phase transition. For Coulomb gases corresponding to \mathbb{Z}_N -models these fixed points are stable for an intermediate temperature interval.

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The Kosterlitz-Thouless phase transition [1,2] occurs in a class of 2-dimensional systems like the plane rotator, Coulomb gases and \mathbb{Z}_N models, $N \gg 1$ [3,4,12]. For the plane rotator it is characterized by a change of exponential to power law decay of correlation function as the temperature is lowered. The physics of this transition is explained by the competition of the self-energy and entropy of the defects (vortices) occurring in the system [1,2,3].

Renormalization group (R.G.) methods have been employed to discuss the phenomenon. Usually, calculations with R.G. make an approximation of disregarding non-local contributions to the transformed Hamiltonian. In the so called hierarchical models no non-local terms appear and therefore the above approximation scheme is exact.

For the Kosterlitz-Thouless phase transition however, the only existing hierarchical model is the one for which the so-called Migdal-Kadanoff R.G. formula is exact. The trouble with this approach is that the Migdal-Kadanoff recursion formulae, as seen numerically by José et al. [3] and rigorously proved by Ito [7], have no stable fixed point other than the $T = \infty$ one.

In this letter we describe a 2-d hierarchical model such that the line of fixed points corresponding to massless gaussian theories is, for $0 < T \leq T_c < \infty$, (globally) stable against a class of perturbations that include Coulomb gas-type of interactions. Therefore those Coulomb gases have, for $T < T_c$, an asymptotic behavior of massless gaussian field. This is the Kosterlitz-Thouless phase transition.

Our model incorporates the ideas of Wilson [8], as formulated by Gawedzki and Kupiainen [9,10] of decomposing the field operator ϕ into a sum of two fields ψ and ξ describing

the block-spin and fluctuation variables respectively. It is described as follows.

The starting point is the hierarchical 2-dimensional massless gaussian field $\phi(x)$ in a cubic lattice Z^2 , defined by the two-point function:

$$\langle e^{i\phi(x)} e^{-i\phi(y)} \rangle = \int d\mu_{\beta G_H} e^{i(\phi(x) - \phi(y))} = \left[-\frac{\beta}{2\pi} [N_L(x,y) - N_L(0,0)] \right] \quad (1)$$

The measure $\int d\mu_{\beta G_H}(\phi)$ is formally given by $\exp\{-\frac{1}{2}\beta \sum_{x,y} \phi(x) G_H^{-1}(x,y) \phi(y)\} \prod_x d\phi(x)$ where $G_H(x,y) = -\frac{1}{2\pi} N_L(x,y) \ln L$ plays the role of a Green's function for the "hierarchical laplacean".

Here $L > 1$ is an integer representing a scale parameter in the model and $N_L(x,y)$, the "hierarchical distance" between x and y , is the smallest positive integer N such that $[L^{-N}x] = [L^{-N}y]$, ($[Z]$ denotes the vector formed with the integer part of the components of $Z \in R^2$), and so $N_L(0,0) = 1$. Our choice of the free hierarchical covariance is made as to guarantee that the asymptotic behavior of correlation function of exponentials of the field $\phi(x)$, are given by:

$$\langle e^{i(\phi(x) - \phi(y))} \rangle \underset{|x-y| \rightarrow \infty}{\sim} C |x-y|^{-\frac{\beta}{2\pi}} \quad (2)$$

Notice that this differs from the usual formulation [10,11] of the hierarchical models (for $d > 2$) where the correlation function of the fields themselves are asymptotically equal to that of a usual free massless theory.

Following Gawedzki and Kupiainen [11] we introduce the orthogonal decomposition:

$$\phi(x) = \psi\left(\left[\frac{x}{L}\right]\right) + \xi\left(L\left[\frac{x}{L}\right]\right) \quad (3)$$

where ψ is a gaussian field with two-point function:

$$\langle e^{i(\psi(x) - \psi(y))} \rangle = \langle e^{i(\phi(x) - \phi(y))} \rangle \quad (4)$$

and ξ is the gaussian "fluctuation field" determined by

$$\langle e^{i(\xi(x) - \xi(y))} \rangle = \int e^{i(\xi(x) - \xi(y))} d\nu(\xi) = \begin{cases} 1 & \text{if } x=y \\ \left[-\frac{\beta}{2\pi} \right] & \text{if } x \neq y \end{cases} \quad (5)$$

$$\langle e^{it\xi(x)} \rangle = \left[-\frac{\beta}{4\pi} t^2 \right]$$

Notice that $\xi(x)$ is independent of $\xi(y)$ if $x \neq y$ and that the contribution of the $\xi(L\left[\frac{x}{L}\right])$ is constant when x varies in a given block of side L .

The class of models we are going to consider is obtained by a local perturbation Λ of the gaussian-measure $d\mu_{\beta G_H}$:

$$\langle F(\phi) \rangle_{\Lambda} = Z^{-1} \int F(\phi) \Lambda(\phi) d\mu_{\beta G_H}(\phi) = Z^{-1} \int F(\phi) \prod_{x \in Z^2} \lambda(\phi(x)) d\mu_{\beta G_H}(\phi) \quad (6)$$

The renormalization group transformation $\Lambda \rightarrow R_L \Lambda$ is defined by integration over the "fluctuation variables" ξ :

$$(R_L \Lambda)(\psi) = \int d\nu(\xi) \Lambda(\phi) \quad (7)$$

and it corresponds to the usual block spin transformation. This transformation is in fact of a local nature. It follows from (3)

and (5) that

$$(R_L \Lambda)(\psi) = \prod_{x \in \mathbb{Z}^2} (r_L \lambda)(\psi(x)) \quad (8)$$

where

$$(r_L \lambda)(\psi) = \int d\nu(\xi) [\lambda(\psi + \xi)]^2 \quad (9)$$

By construction the free theory, $\lambda=1$, is a fixed point of the transformation r_L . We are interested in analysing the stability of this fixed point with respect to a special class τ of local perturbation λ . The choice of this class has to meet two requirements: 1) it must contain $\lambda(\phi) = \exp\{Z \cos \phi\}$ as this represents the "standard" Coulomb gas and 2) it must be closed under the renormalization group transformation (9), i.e., $r_L \lambda \in \tau$ if $\lambda \in \tau$.

A minimal choice of τ fulfilling 1) and 2) is:

$$\lambda(\phi) = \sum_{q \in \mathbb{Z}} Z_q e^{iq\phi} \quad (10)$$

where $Z \in \ell^1$ i.e. $\sum_{q \in \mathbb{Z}} |Z_q| < \infty$, and $Z_q = Z_{-q}$.

If we write:

$$(r_L \lambda)(\phi) = \sum_{q \in \mathbb{Z}} Z'_q e^{iq\phi} \quad (11)$$

An explicit computation shows:

$$Z'_q = \int \prod_{i=1}^k \delta(q_i - q) \sum_{q_1, \dots, q_k \in \mathbb{Z}} Z_{q_1} \dots Z_{q_k} \quad (12)$$

in particular $Z' \in \ell^1$.

The stability of the fixed point $\lambda_0(\phi) = 1$ corresponding $Z_q^{(0)} = \delta_{q,0}$ can be analyzed by linearizing the R.G. transformation (12) around $Z_q^{(0)}$. The linearized transformation $Z' = AZ$ is given by the diagonal matrix:

$$A_{qq'} = \begin{cases} \delta_{qq'} L^{2 - \frac{\beta}{4\pi} q^2} & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases} \quad (13)$$

Therefore if $\beta > \beta_c = 8\pi$ the eigenvalues γ_n of A satisfy $|\gamma_n| < 1$ and the fixed point is stable.

In fact it is not difficult to show that the fixed point is globally attractive, i.e., for any $\lambda \in \tau$, $\lim_{N \rightarrow \infty} (r_L^N \lambda) = \lambda_0$ if $\beta > \beta_c$. Full mathematical detail will be presented elsewhere [14].

For the Z_N -models, $N \gg 1$, the Kosterlitz-Thouless phenomenon is characterized [3,12] by the existence of an intermediate temperature interval $I_N = [\beta_N, \bar{\beta}_N]$ such that for $\beta_N < \beta < \bar{\beta}_N$ the correlation function decay polynomially. These models can be shown [12] to be equivalent to two interacting Coulomb gases with integer charges m and n at temperatures β and $\beta' = \frac{4\pi^2 N^2}{\beta}$ respectively. A simplified hierarchical version of this system is given by probability distribution:

$$d\mu_{\beta, \beta'}(\phi) = \prod_{x \in \mathbb{Z}^2} \lambda^{(m)}(\phi(x)) \quad (14)$$

where

$$\lambda^{(m)}(\phi) = \sum_{m, n \in \mathbb{Z}} g_{mn} e^{i(m + \frac{2\pi N}{\beta} n)\phi} \quad (15)$$

Now if $g_{mn} = g_{nm}$, $d\mu_{\beta G_H}^\lambda(\phi)$ is invariant under $\beta \rightarrow \frac{(2\pi)^2 N^2}{\beta}$, and this expresses the self-duality typical of Z_N -symmetric models [12].

A renormalization group transformations (9) acts on a model given by $g = \{g_{mn}, m, n \in \mathbb{Z}\}$ through (15) transforming it on another model of the same class given by a different set $g' = \{g'_{mn}, m, n \in \mathbb{Z}\} = r_L g$:

$$g'_{mn} = \left[\frac{\beta}{4\pi} \left(m + \frac{2\pi N}{\beta} n \right)^2 \sum_{\substack{m_1, \dots, m_L \\ \sum_{i=1}^L m_i = m}} \sum_{\substack{n_1, \dots, n_L \\ \sum_{i=1}^L n_i = n}} \left(\prod_{i=1}^L g_{m_i n_i} \right) \right] \quad (16)$$

The linearized transformation is given by the matrix:

$$B_{m'n'; mn} = \begin{cases} \left[2 - \frac{\beta}{4\pi} \left(m + \frac{2\pi N}{\beta} n \right)^2 \right] \delta_{mm'} \delta_{nn'} & , m \neq 0 \text{ and (or) } n \neq 0 \\ 0 & , m = 0 \text{ and } n = 0 \end{cases} \quad (17)$$

Therefore we have stability if and only if $\beta > 8\pi \equiv \underline{\beta}_N$ and $\beta < \frac{\pi N^2}{2} \equiv \overline{\beta}_N$. The two conditions are incompatible if $N < 4$ and for $N > 4$ $\overline{\beta}_N > \underline{\beta}_N$ and there will be a soft intermediate phase for $\beta \in (\underline{\beta}_N, \overline{\beta}_N)$.

Finally we should remark that our results and techniques admit a natural extension in order to cover $U(1)$ and Z_N -lattice gauge theories in 4-dimensions [14] as the deconfining phase transition of these models are of the same nature as the Kosterlitz-Thouless phenomenon in 2-dimensions [12,13,15,16].

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