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SYMMETRICAL REPRESENTATION OF THE TRANSITION MATRIX I: FORMAL DEVELOPMENT

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SYMMETRICAL REPRESENTATION OF THE TRANSITION MATRIX I:
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ABSTRACT

An expression for the scattering T-matrix which is symmetrical with respect to the outgoing and ingoing wave functions is derived. The elastic matrix element then takes the form

$$\langle \vec{k}' | T | \vec{k} \rangle = \int_0^1 d\lambda \, \langle \psi_{\vec{k}'}^{(-)}(\lambda V) | V | \psi_{\vec{k}}^{(+)}(\lambda V) \rangle$$

with $\psi^{\pm}(\lambda V)$ obtained from the scaled potential λV. Several examples are worked out, and application to neutron-nucleus scattering is made.

I. INTRODUCTION

The basic quantity in the quantum theory of scattering is the T-matrix, whose on-shell matrix elements, taken with respect to plane waves, supplies the scattering amplitude through which all scattering observables, such as the elastic scattering differential cross section, spin polarization and spin rotation, are calculated. The well known expression for the physical matrix element of T is

$$\langle \vec{k}' | T(E_k) | \vec{k} \rangle = \langle \vec{k}' | V | \psi_{\vec{k}}^{(+)}(V) \rangle \tag{1}$$

where V is the interaction potential (generally complex and energy dependent) and $\psi_{\vec{k}}^{(+)}$ is the exact solution of the Schrödinger equation with the appropriate outgoing wave boundary conditions.

An important qualitative property of the matrix element of Eq. (1) is that it is not symmetrical with respect to the bra and ket. Though in most cases, such as in the numerical evaluation of $\langle \vec{k}' | T | \vec{k} \rangle$ by directly solving the Schrödinger equation, this property does not pose any major problem, in several applications, such as in the development of algebraic algorithm, Padé approximant, or the doorway expansion method, a symmetrical representation for the T-matrix is more appropriate.

Schwinger[1] has suggested one possible symmtri - zation of T, through the use of the square root of the potential operator, namely $\sqrt{V}$. It is then easy to show that the T-matrix, Eq. (1), which, in operator form satisfies

$$T(E) = V + V G_0^{(+)}(E) T(E) \tag{2}$$

where $G_0^{(+)}(E)$ is the free propagator given by

$$G_0^{(+)}(E) = (E - (H-V) + i\varepsilon)^{-1} \tag{3}$$

with H being the total, generally complex, Hamiltonian of the scattering system, can be written as[1,2)]

$$T(E) = V^{1/2} \frac{1}{1 - W^{(+)}(E)} V^{1/2} \tag{4}$$

where $W^{(+)}(E)$ is the scattering operator given by

$$W^{(+)}(E) = V^{1/2} \; G_0^{(+)}(E) \, V^{1/2} \tag{5}$$

Equation (4) is extremely useful for developing non-perturbative approximations for T, required for the treatment of strongly interacting systems, such as nucleon-nucleon and nucleon-nucleus scattering. Recently, extensive use of Eq. (4) has been made in connection with pion-nucleus

scattering, in the energy region dominated by the pion-nucleon Δ-resonance[3)]. In all of these applications, a fundamental requirement for the consistency of the method is the well-behavedness of the operator $V^{1/2}$.

In this paper, we suggest <u>another</u> form of the T-matrix, which, though symmetric in V, does not require the use of the possibly ill-defined operator $V^{1/2}$. The vehicle through which this is accomplished, is a parameter, λ, that multiplies the interaction potential, and whose values are contained in the interval [0,1]. This new symmetrical form of T, which will be derived and extensively discussed in section II, is

$$\langle \mathbf{k}' | T | \mathbf{k} \rangle = \int d\lambda \langle \psi_{\mathbf{k}'}^{(-)}(\lambda V) | V | \psi_{\mathbf{k}}^{(+)}(\lambda V) \rangle \tag{6}$$

where

$$|\psi_{\mathbf{k}'}^{(-)}(\lambda V)\rangle = |\psi_{-\mathbf{k}'}^{(+)*}(\lambda V^*)\rangle$$

and these exact wave functions are calculated with the scaled potential λV. The symmetrized expression above could be of great use to extend the angular range of the validity of the Glauber approximation, as well as to obtain a more rapidly convergent doorway expansion method[3)] for the evaluation of $\langle \mathbf{k}' | T | \mathbf{k} \rangle$.

The plan of this first paper of a series under preparation is as follows. In section II, we develop the theory of the symmetrized T-matrix, which leads to Eq. (6). We then solve, in section III, within this theory several one-dimentional scattering problems. In section IV, we present a detailed account of the calculation of the integrand of Eq. (6) in a realistic scattering situation of n + ^{16}O at several neutron energies. The potential V employed for the purpose is a complex Woods-Saxon interaction whose radial distance dependence follows roughly that of the density distribution of the target ^{16}O nucleus. An important question which is discussed in this section is the possibility of having a particular value of λ which would give the dominant contribution to the integral in Eq. (6), namely the existence of a stationary or more generally, a saddle point in the λ-integral.

Finally in section IV concluding remarks, as well as a general discussion of the applicability of Eq. (6) to high energy (Glauber)[4] scattering, to be fully developed in the second paper of this series, are presented.

II. A GENERAL SYMMETRICAL REPRESENTATION OF THE SCATTERING T-MATRIX

In this section, we present the theory of the symmetrical T-matrix, exemplified by Eq. (6). Let us first write the Lippmann-Schwinger equation for $T(\lambda V)$, obtained from the exact solution of the scattering equation with the potential λV.

$$T(\lambda V) = \lambda V + \lambda V G_0^{(+)}(E)\, T(\lambda V) \tag{7}$$

We formally calculate the derivative of $T(\lambda V)$ with respect to λ, obtaining thus,

$$\frac{d\,T(\lambda V)}{d\lambda} = V + V G_0^{(+)}(E)\, T(\lambda V) + \lambda V G_0^{(+)}(E)\, \frac{d\,T(\lambda V)}{d\lambda} \tag{8}$$

the formal solution of Eq. (8) is

$$\frac{d\,T(\lambda V)}{d\lambda} = \left(1 - \lambda V G_0^{(+)}(E)\right)^{-1} \left[V + V G_0^{(+)}(E)\, T(\lambda V)\right] \tag{9}$$

or

$$\frac{d\,T(\lambda V)}{d\lambda} = \left(1 - \lambda V G_0^{(+)}(E)\right)^{-1} T(\lambda V) \tag{10}$$

Since the operator $(1-\lambda V G_0^{(+)}(E))^{-1}$ is nothing but $(1+T^{(-)}(\lambda V^{+})G_0^{(-)})^{\dagger}$, we have thus

$$\frac{dT(\lambda V)}{d\lambda} = \left(1+T^{(-)}(\lambda V) G_0^{(-)}(E)\right)^{\dagger} V \left(1+G_0^{(+)}(E)T^{(+)}(\lambda V)\right) \tag{11}$$

Taking the plane wave matrix element of (11), and using the L-S equation, we have finally, after integrating over λ,

$$\langle \vec{k}' | T(V) | \vec{k} \rangle = \int_0^1 d\lambda \langle \psi_{\vec{k}'}^{(-)}(\lambda V) | V | \psi_{\vec{k}}^{(+)}(\lambda V) \rangle \tag{12}$$

Eq. (12) is valid for a general complex, energy-dependent interaction. For future use, we write below the partial wave expansion form of T. We have first

$$\psi_{\vec{k}}^{(+)}(\lambda V) = \frac{4\pi}{kr} \sum_{\ell m} (i)^{\ell} e^{i\sigma_\ell} f_\ell^{(\lambda)}(k,r) Y_{\ell m}(\Omega_{\hat{k}}) Y_{\ell m}^{*}(\Omega_{\hat{r}}) \tag{13}$$

$$\begin{aligned} \psi_{\vec{k}'}^{(-)*}(\lambda V) &= \psi_{-\vec{k}'}^{(+)}(\lambda V^{*}) \\ &= \frac{4\pi}{k'r} \sum_{\ell m} (i)^{-\ell} e^{i\sigma_\ell} f_\ell^{(\lambda)}(k',r) Y_{\ell m}^{*}(\Omega_{\hat{k}'}) Y_{\ell m}(\Omega_{\hat{r}}) \end{aligned} \tag{14}$$

Thus, we obtain immediately

$$\langle \vec{k}' | T(E_k) | \vec{k} \rangle = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) e^{2i\sigma_\ell} \left(\int_0^1 d\lambda \int_0^\infty dr \left(f_\ell^{(\lambda)}(k,r)\right)^2 V(r) \right) P_\ell(\hat{k}\cdot\hat{k}') \tag{15}$$

In the above equations, σ_ℓ denotes the Coulomb phase shift and $f_\ell^{(\lambda)}(k,r)$ is the exact radial wave function obtained from the solution of the radial Schrödinger equation with the potential λV.

The λ-integral in Eq. (15) should be compared with the conventional radial integral, namely

$$\int_0^\lambda d\lambda \int_0^\infty dr \left(f_\ell^{(\lambda)}(k,r)\right)^2 V(r) = \int_0^\infty dr\, j_\ell(kr) f_\ell(k,r) V(r) \tag{16}$$

where $j_\ell(kr)$ is the usual spherical Bessel function, which arises from the partial wave expansion of the plane wave and $f_\ell(k,r)$ is just $f_\ell^{(\lambda=1)}(k,r)$. Equation (16) suggests the following useful identity

$$\int_0^\lambda d\lambda \left(f_\ell^{(\lambda)}(k,r)\right)^2 = f_\ell^{(\lambda=0)}(k,r)\, f_\ell^{(\lambda=1)}(k,r) \tag{17}$$

valid for any interaction. Clearly $f_\ell^{(\lambda=0)}(k,r) = j_\ell(kr)$.

Instead of using (17) which would then accomplish nothing in so far as the eventual evaluation of the partial wave series, we concentrate our attention on the λ-dependent radial integrals $\int_0^\infty dr (f_\ell^{(\lambda)}(k,r))^2 V(r)$. This integral is restricted by the range of the interaction $V(r)$. For a Woods-Saxon potential, commonly used to describe nuclear scattering, with a strong imaginary part, it is expected that for λ close to one, the integral above will have a narrow distribution as a function of angular momentum. This is so since the small partial waves will be strongly absorbed and the large partial waves will contribute very little due to the short range nature of $V(r)$. The maximum of this ℓ-function will be situated at an ℓ-value corresponding to a grazing collision. For λ smaller than unity, the lower partial waves will contribute more as the corresponding $f_\ell^{(\lambda)}$ is less damped. Accordingly, one expects on general grounds, that the ℓ-dependent radial integral to behave, as a function of λ, as shown schematically in Fig. (1). This behaviour will be corroborated on in section IV through an exact calculation.

Thus in the small ℓ-region (low partial waves) the small λ region in the λ-integral supplies the dominant contribution. The large ℓ-region contributes very little for all values of λ and finally in the grazing ℓ-region, all λ values contribute roughly the same amount. As a function of ℓ, the λ-integral would then behave schematically as shown in Fig. (2). From this discussion, one may wonder whether there exists a particular value of λ, call it $\lambda_s \in [0,1]$, which supplies the dominant contribution to the λ-integral, in the sense

$$\int_0^1 d\lambda \left(f_\ell^{(\lambda)}(k,r)\right)^2 V(r) \propto f_\ell^{(\lambda_s)}(k,r)\, V(r) \tag{18}$$

Such a situation, if it arises, would be extremely interesting, as it would imply shifting some of the strength of the potential from $|\psi_{\vec{k}}^{(+)}\rangle$ in Eq. (1) to the plane wave $\langle \vec{k}'|$ and thus renders the convergence of, say, a doorway expansion treatment of $\langle \vec{k}'|T|\vec{k}\rangle$, much faster. This questions will be fully addressed in section IV.

So far in our discussion, we have employed the stationary time-independent theory of scattering. For more general applications, in particular, in connection with field theoretic description of the collision, it is more advantageous to develop a time-dependent version of our theory. This goal can be achieved[5] by introducing the evolution operator $U_\lambda(t,t')$, defined by

$$i\frac{\partial}{\partial t} U_\lambda(t,t') = \lambda\, H_I(t)\, U_\lambda(t,t') \tag{19}$$

where $H_I(t)$ in (19) can be cast into the usual form, in terms of the time ordered product of the interaction Hamiltonian,

$$U_\lambda(t,t') = T\exp\left[-i\lambda\int_{t'}^{t} d\tau\, H_I(\tau)\right] \tag{20}$$

or, into the following, more convenient, equivalent expression,

$$U_\lambda(t,t') = 1 - i\int_0^\lambda d\lambda' \int_{t'}^{t} d\tau\, U_{\lambda'}(t,\tau)\, H_I(\tau)\, U_{\lambda'}(\tau,t') \tag{21}$$

From (21) it follows that the usual evolution operator

$U(t,t') \equiv U_{\lambda=1}(t,t')$, can be written, for asymptotic times, as

$$U(\infty,-\infty) = 1 - i\int_0^1 d\lambda \int_{-\infty}^{\infty} dt\; U_\lambda(\infty,t) H_I(t) U_\lambda(t,-\infty) \qquad (22)$$

This formula is our basic result. It allows us to write symmetric expressions for the T-matrix for time dependent potentials as well as to symmetrize the T-matrix within the context of Quantum Field Theory. Thus it represents a major generalization of the time independent description presented earlier.

In order to derive, from (22), symmetric expressions analogous to (6) for time dependent potentials ($V(x,t)$), we will work within the second quantized scheme. In this case, if we represent the non-relativistic field operator by $\phi(x,t)$, the interaction Hamiltonian associated with this problem is

$$H_I = \int d^3x\, \varphi^\dagger(x,t)\left[V(x,t)\right] \varphi(x,t) \qquad (23)$$

Using (23) in (22) and after inserting a complete set of states in (22) it is possible to derive the following expression for the R-matrix (which is just S-1, where S is the S-matrix)

$$\langle f|R|i\rangle = -i\int_0^1 d\lambda \int d^3x\, dt\, \psi^*_{f,\lambda}(x,t)\, V(x,t)\, \psi_{i,\lambda}(x,t) \qquad (24)$$

which is obviously symmetric with regard to the "wave functions" $\psi_{\lambda f}(x,t)$, $\psi_{i\lambda}(x,t)$. These wave functions are defined, within the second quantized scheme, as the matrix elements:

$$\psi_{i,\lambda}(x,t) = \langle 0|\varphi(x,t)\, e^{-iH_0 t}\, U_\lambda(t,-\infty)|i\rangle \qquad (25)$$

$$\psi^\dagger_{f,\lambda}(x,t) = \langle f|\, U_\lambda(\infty,t)\, e^{-iH_0 t}\, \varphi^\dagger(x,t)\,|0\rangle \qquad (26)$$

The above wave functions satisfy the wave equations

$$i\frac{\partial}{\partial t}\psi_\lambda(x,t) = \left[-\frac{\vec{\nabla}^2}{2m} + \lambda V(x,t)\right]\psi_\lambda(x,t) \qquad (27)$$

with asymptotic conditions

$$\lim_{t\to-\infty} \psi_{i,\lambda}(x,t) = \lim_{t\to-\infty} \chi_i(x,t) \qquad (28)$$

$$\lim_{t\to+\infty} \psi_{f,\lambda}(x,t) = \lim_{t\to\infty} \chi_f(x,t) \qquad (29)$$

where χ_i (χ_f) is the wave function of the initial (final) state of the system.

For time independent potentials, one gets, from (24), our expression (6) for the T-matrix, after paying due attention of course to the energy conservation delta function implicity contained in R.

We can further extend the time-dependent approach for the treatment of the Many Body Scattering Problem. We shall illustrate how this can be achieved within the context of the two-body problem. By assuming that the Hamiltonian H can be written as

$$H = \int d^3x \, \varphi^{\dagger}(x,t)\left[-\frac{\nabla^2}{2m} + V(x,t)\right]\varphi(x,t) + \frac{1}{2}\int d^3x \int d^3x' \, \varphi^{\dagger}(x,t)\,\varphi^{\dagger}(x',t)\,U(x',x,t)\,\varphi(x',t)\,\varphi(x,t) \tag{30}$$

where we are using the second quantization approach with $\phi(x,t)$ being a field operator and $V(x,t)$ is an external potential to which the particles are subject and $U(x,x',t)$ is the interaction potential between the particles. In this case, if one uses (22) one can write the following expression for the S-matrix elements

$$\langle f|U(\infty,-\infty)|i\rangle = \delta_{fi} - i\int_{-\infty}^{\infty} dt \int d^3x \int d^3x' \, G^{*}(x,x',t)\left[V(x,t) + \frac{1}{2}U(x,x',t)\right] \times f_s(x,x',t) \tag{31}$$

where the "wave functions" $G_s(x,x',t)$ and $f_s(x,x',t)$ are given, respectively, by:

$$g_{\lambda}^{\dagger}(x,x',t) = \langle f|U_{\lambda}(\infty,t)\,e^{iH_0 t}\,\varphi^{\dagger}(x,t)\,\varphi^{\dagger}(x',t)|0\rangle \tag{32}$$

$$f_{\lambda}(x,x',t) = \langle 0|\varphi(x')\,\varphi(x)\,e^{-iH_0 t}\,U_{\lambda}(t,-\infty)|i\rangle \tag{33}$$

and satisfy the Schrödinger equation

$$\left[-\frac{\nabla^2}{2m} - \frac{\nabla'^2}{2m} + \lambda V(x,t) + \lambda V(x',t)\right]\begin{Bmatrix} f_s(x,x',t) \\ g_s(x,x',t)\end{Bmatrix} = i\frac{\partial}{\partial t}\begin{Bmatrix} f_s(x,x',t) \\ g_s(x,x',t)\end{Bmatrix} \tag{34}$$

satisfying asymptotic conditions analogous to (28) and (29).

If one assumes that one of the particles is free and the other one is in a bound state whose wave function is $\psi_B(x)e^{-iE_b t}$ then the appropriate asymptotic condition in this case

$$\lim_{t\to-\infty} f_s(x,x',t) = \lim_{t\to-\infty} e^{iE_p t}\,e^{-iE_B t} \times \left[e^{ipx}\varphi_B(x') - e^{-ipx'}\varphi_B(x)\right] \tag{35}$$

Before ending this section we comment on the related work of Tikochinsky[6]. The derivation of Eq. (12) presented in Ref. 6 is based on the repeated use of the Gell-Mann-Goldberger relation for the two-potential scattering problem namely, Tikochinsky writes

$$V = \frac{n-1}{n} V + \frac{1}{n} V \tag{36}$$

and

$$\begin{aligned} T_{fi} = & \left\langle \chi_f \left| \frac{n-1}{n} V \right| \psi_i^{(+)}\left(\frac{n-1}{n} V\right) \right\rangle \\ & + \left\langle \psi_f^{(-)}(V) \left| \frac{1}{n} V \right| \psi_i^{(+)}\left(\frac{n-1}{n} V\right) \right\rangle \end{aligned} \tag{37}$$

After n-1 transformations similar to the above, one can write

$$T_{fi} = \sum_{k=1}^{\infty} \frac{1}{n} \left\langle \psi_f^{(-)}\left(\frac{k}{n} V\right) \middle| V \middle| \psi_i^{(+)}\left(\frac{k-1}{n} V\right) \right\rangle \tag{38}$$

Letting $n \to \infty$, one obtains the integral form, Eq. (12)

$$T_{fi} = \int_0^1 d\lambda \left\langle \psi_f^{(-)}(\lambda V) \middle| V \middle| \psi_i^{(+)}(\lambda V) \right\rangle \tag{39}$$

Tikochinsky discusses Eq. (39) in the context of high energy scattering, and shows that the condition usually cited in connection with the validity of Glauber approximation, namely $\frac{V}{E} \ll 1$, can be relaxed to $\frac{\lambda V}{E} \ll 1$ with $0<\lambda<1$, thus extending the validity of this approximation to medium energies. In particular, he finds within a simple model calculation of

nucleon-nucleus scattering at E=35 MeV, that most of the contribution to the λ-integral comes from the vicinity of $\lambda \approx 0.2$, the value at which the integrand peaks. Thus even at this low energy the Glauber approximation may work since $\frac{0.2V}{E} < 1$ for V=50 MeV.

In the next sections we investigate this point further, through schematic and realistic model calculations.

III. EXAMPLES: HIGH ENERGY SEMICLASSICAL APPROXIMATIONS, AND SOLVABLE ONE-DIMENSIONAL PROBLEMS

In this section we evaluate the T-matrix element, as given by Eq. (12) for several analytically solvable problems. We also present the case of the high energy Glauber approximation. We start first with the Glauber approximation. In this case the wave functions $|\psi_{\vec{k}}^{(+)}(\lambda V)\rangle$ and $\langle\psi_{\vec{k}'}^{(-)}(\lambda V)|$ are given by

$$\langle r|\Psi_{\vec{k}}^{(+)}(\lambda V)\rangle = \langle z,\vec{b}\,|\Psi_{\vec{k}}^{(+)}(\lambda V)\rangle = e^{i\vec{k}\cdot\vec{r}}\exp\Big[i\int_{-\infty}^{z}\Delta k^{(\lambda)}(z',b)\,dz'\Big] \tag{40}$$

and

$$\langle\Psi_{\vec{k}'}^{(-)}(\lambda V)|\vec{r}\rangle = e^{-i\vec{k}'\cdot\vec{r}}\exp\Big[-i\int_{z}^{\infty}\Delta k^{(\lambda)}(z',b)\,dz'\Big] \tag{41}$$

the incident momentum $\vec{k}$ is taken to point along the position z-axis, and $\vec{b}$ is the component of $\vec{r}$ perpendicular to Z, which is assumed to vary little along a small-angle trajectory. The exponent of the second factor in $\psi_{\vec{k}}^{(+)}$ is the amount of the generally complex, phase shift accumulated along the trajectory up to the point $(Z,\vec{b})$. The integrand of this phase shift is, in this high energy limit, $\frac{V}{E} << 1$, given by

$$\Delta k^{(\lambda)} = \quad = -\frac{k}{2E}\lambda V(z,b) \tag{42}$$

and thus is linear in λ. Note that $V(z,b)$ is generally complex, and k represents the asymptotic wave number, $k=\sqrt{\frac{2\mu}{\hbar^2}E}$.

With the above expressions, we have for $\langle\vec{k}'|T|\vec{k}\rangle$, Eq. (12) the following

$$\langle\vec{k}'|T|\vec{k}\rangle = \int_0^1 d\lambda\int_{-\infty}^{\infty}dz\int d\vec{b}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\exp\Big[i\int_{-\infty}^{\infty}\Delta k^{(\lambda)}(z',b)\,dz'\Big]V(z,b)$$

$$= \int_{-\infty}^{\infty}dz\int d\vec{b}\, e^{i\vec{q}\cdot\vec{r}}\exp\Big[-i\frac{\lambda k}{2E}\int_{-\infty}^{\infty}V(z',b)\,dz'\Big]V(z,b)$$

$$= \int_{-\infty}^{\infty}dz\int d\vec{b}\, e^{i\vec{q}\cdot\vec{r}}\,\frac{\exp\Big[-i\frac{k}{2E}\int_{-\infty}^{\infty}V(z',b)\,dz'\Big]-1}{-\frac{ik}{2E}\int_{-\infty}^{\infty}V(z',b)\,dz'}\,V(z,b)$$

$$= \frac{i2E}{k}\int d\vec{b}\, e^{i\vec{q}_{//}\cdot\vec{b}}\Big\{\exp\Big[-\frac{ik}{2E}\int_{-\infty}^{\infty}V(z',b)\,dz'\Big]-1\Big\} \tag{43}$$

which is of course the usual Glauber expression for the T-matrix element. Note that q_z is assumed zero. Clearly, we have apparently not accomplished anything new in so far as obtaining a better approximation to high energy scattering (e.g. a wider angular range validity). However, as we shall show in details in the second paper of this series, with a slight modification of the method of evaluation of the integrals appearing in Eq. (12), we will be able to obtain a high energy approximation which has a wider angular range of validity. This arises from the symmetrical form of the integrand in Eq. (12). Further the approximation can be extend to lower energies.

We now consider an interaction potential, which is nonlocal, but separable, namely

$$V(r,r') = V_0\, g(r)\, g(r') \tag{44}$$

For simplicity we consider only s-wave scattering. The wave function $\psi^{(+)}_{\vec{k}}(\lambda V)$, then becomes

$$\psi^{(+)}_{k}(\lambda V) = \sin kr + \frac{\lambda V_0 \langle G_0^{(+)} g\rangle (r) \langle g \sin kr\rangle}{1 - \lambda V_0 \langle g\, G_0^{(+)} g\rangle} \tag{44}$$

and

$$\psi^{(-)*}_{k}(\lambda V) = \sin kr + \frac{\lambda V_0 \langle G_0^{(+)} g\rangle (r) \langle g \sin kr\rangle}{1 - \lambda V_0 \langle g\, G_0^{(+)} g\rangle} \tag{46}$$

where the notation < > implies an integration over r, $\langle G_g^{(+)}\rangle(r) \equiv \int_0^\infty G^{(+)}(r,r')g(r')dr'$ and $G(r,r')$ is the $\ell=0$, Green function given by $-\frac{i}{k}\, e^{ikr_>} \sin kr_<$.

With Eqs. (45) and (46), we have for the T-matrix element

$$\langle k|T|k\rangle = V_0 \int_0^1 d\lambda\, \frac{\langle g \sin kr\rangle \langle g \sin kr\rangle}{(1-\lambda V_0 \langle g G_0^{(+)} g\rangle)^2}$$

$$= V_0 \frac{1}{V_0 \langle g G_0^{(+)} g\rangle} \langle g \sin kr\rangle^2 \cdot \left[\frac{1}{1 - V_0\langle g G_0^{(+)} g\rangle} - 1\right]$$

$$= V_0 \frac{\langle g \sin kr\rangle^2}{1 - V_0 \langle g G_0^{(+)} g\rangle} \tag{47}$$

which is clearly the expression obtained by directly applying Eq. (1).

The third case we consider is a delta function potential,

$$V(r) = a V_0\, \delta(r-a) \tag{48}$$

Then for s-wave scattering, we have

$$\psi^{(+)}_k(r) = \sin kr + \lambda a V_0\, G^{(+)}(r,a)\, \psi^{(+)}_k(a) \tag{49}$$

or

$$\psi^{(+)}_k = \sin kr + \frac{\lambda a V_0\, G^{(+)}(r,a) \sin ka}{1 - \lambda a V_0\, G^{(+)}(a,a)} \tag{50}$$

and similarly

$$\psi_k^{(-)}(\lambda V)^{*} = \sin kr + \frac{\lambda a V_0 G^{(+)}(r,a) \sin ka}{1-\lambda a V_0 G^{(+)}(a,a)} = \psi_k^{(+)}(\lambda V) \qquad (51)$$

The T-matrix element in this case is just

$$\begin{aligned} \langle k|T|k\rangle &= V_0 \int_0^1 \left(\psi_k^{(+)}(\lambda V)\right)_{r=a} d\lambda \\ &= V_0 \int_0^1 \frac{\sin^2 ka}{(1-\lambda a V_0 G^{(+)}(a,a))^2}\, d\lambda \\ &= V_0 \frac{\sin^2 ka}{1-aV_0 G^{(+)}(a,a)} \end{aligned} \qquad (52)$$

which is again as one might have obtained using directly the conventional expression for $\langle k|T|k\rangle = \langle k|V|\psi_k^{(+)}\rangle$.

We turn now to a slightly more involved example, namely the square well potential.

Before considering this example let us derive some general properties of the scattering amplitude that follows from our symmetric expression for spherically symmetric potentials. By using the usual partial wave decomposition one can write

$$f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2i\delta_\ell}-1)\, P_\ell(\cos\theta) \qquad (53)$$

The phase shifts δ_ℓ can be written in two

equivalent forms

$$\begin{aligned} e^{i\delta_\ell}\sin\delta_\ell &= -\frac{2\mu k}{\hbar^2}\int_0^\infty V(r)\, R_\ell(r)\, j_\ell(kr)\, r^2\, dr \\ &= -\frac{2\mu k}{\hbar^2}\int_0^1 d\lambda \int_0^\infty V(r)\, R^2_{\ell,\lambda}(r)\, r^2\, dr \end{aligned} \qquad (54)$$

where the second expression follows from the partial wave decomposition of the wave functions (13,14) and then substituting this decomposition into (6). The function $R_{\ell\lambda}(r)$ in (54) is the solution of the equation

$$R''_{\ell,\lambda}(r) + \frac{2}{r}R'_{\ell,\lambda}(r) + \left[\frac{2\mu}{\hbar^2}(E-\lambda V(r)) - \frac{\ell(\ell+1)}{r^2}\right]R_{\ell,\lambda}(r) = 0 \qquad (55)$$

$R_\ell(r)$ and $j_\ell(r)$ are just particular cases of $R_{\ell\lambda}(r)$, namely

$$R_\ell(r) = R_{\ell,\lambda=1}(r) \qquad (56)$$

$$j_\ell(r) = R_{\ell,\lambda=0}(r) \qquad (57)$$

If one defines $\chi_{\ell,\lambda}(r)$ as

$$\chi_{\ell,\lambda}(r) = r\, R_{\ell,\lambda}(r) \qquad (58)$$

then, from (54) follows the following identity

$$\int_0^1 d\lambda \int_0^\infty V(r)\, \chi^2_{\ell,\lambda}(r)\, dr = \int_0^\infty V(r)\, \chi_{\ell,\lambda=1}(r)\, \chi_{\ell,\lambda=0}(r)\, dr \qquad (59)$$

A stronger version of the above identity is

$$\int_0^1 d\lambda\, \chi^2_{\ell,\lambda}(r) = \chi_{\ell,\lambda=1}(r)\, \chi_{\ell,\lambda=0}(r) = \chi_\ell(r)\, j_\ell(r) \qquad (60)$$

which is exactly identical to Eq. (17), with $f_\ell^{(\lambda)} \equiv \chi_{\ell,\lambda}$. This is a quite interesting property satisfied by the radial wave-function $\chi_{\ell\lambda}(r)$. This property is equivalent to saying that the average of $\chi^2_{\ell\lambda}(r)$ over the whole range of values of λ is equal to the product $\chi_{\ell\lambda=1}(r) \cdot \chi_{\ell\lambda=0}$.

The equivalence of the two expressions for the phase shifts leads to properties of the wave function that in some simple cases leads to, as far as we know, unsuspected properties of spherical functions. We will derive some explicit properties for the spherical Bessel functions. In order to achieve this, let us consider the spherically symmetric potential defined by

$$V(r) = \begin{cases} -V_0 & r < R \\ 0 & r > R \end{cases} \qquad (61)$$

Let us consider first S-waves. In this case the solutions of the Schrödinger equation (55), with $\chi(r)$ defined by (58) is

$$\chi_\lambda(r) = A_\lambda \sin k(\lambda)\, r \qquad r < R \qquad (62)$$

whereas

$$\begin{aligned} \chi_\lambda(r) &= B_\lambda \sin k(0) r + C_\lambda \cos k(0) r \\ &= \rho_0 \sin\left(kr + \delta(\lambda)\right) \qquad r > R \end{aligned} \qquad (63)$$

where

$$k(\lambda) = \sqrt{\frac{2\mu}{\hbar^2}\,(E + \lambda V_0)} \qquad (64)$$

By imposing the usual boundary conditions at r=R and the appropriate normalization one gets

$$A_\lambda = \frac{1}{\sqrt{k(0) \sin^2 k(\lambda) R + k^2(\lambda) \cos^2 k(\lambda) R}} \qquad (65)$$

From (62) and (65) it follows that

$$\int_0^1 d\lambda \frac{\sin^2 k(\lambda)\, r}{k^2(0) \sin^2 k(\lambda) R + k^2(\lambda) \cos^2 k(\lambda) R} =$$

$$= \frac{\sin k(1)\, r}{\left(k^2(0) \sin^2 k(1) R + k^2(1) \cos^2 k(1) R\right)^{1/2}} \cdot \frac{\sin k(0)\, R}{\left(k^2(0) \sin^2 k(0) R + k^2(0) \cos^2 k(0) R\right)^{1/2}} \qquad (66)$$

In terms of the spherical Bessel function $j_0(kr)$ the expression equivalent to (66) is

$$\int_0^1 d\lambda \frac{1}{1+J_0^2(k(\lambda)R)\left[(R\,k(\lambda))^2-(R\,k(0))^2\right]} J_0^2(k(\lambda)r)$$

$$=\sqrt{\frac{1}{1+J_0^2(k(1)R)\left[(R\,k(1))^2-(R\,k(0))^2\right]}} J_0(k(1)r)J_0(k(0)r) \quad (67)$$

In the general case one can write

$$R_{\ell\lambda}(r) = A_\ell\, J_\ell(k(\lambda)r) \qquad r<R \quad (68)$$

and

$$R_{\ell\lambda}(r) = B_\ell\left(J_\ell(k(0)r)\cos\theta_\ell(\lambda) + n_\ell(k(0)r)\sin\theta_\ell(\lambda)\right) \quad (69)$$
$$r>R$$

By imposing the usual boundary conditions at r=R and the appropriate normalization we gets

$$A_\ell(\lambda) = A_\ell(0)\left[\frac{J_\ell(k(0)R)}{J_\ell(k(\lambda)R)}\sin\theta_\ell(\lambda) + \frac{n_\ell(k(0)R)}{J_\ell(k(\lambda)R)}\cos\theta_\ell(\lambda)\right] \quad (70)$$

$\theta_\ell(\lambda)$ in ( ) is given by

$$\tan\theta_\ell(\lambda) = \frac{n'_\ell(k(0)R)\,y_\ell(\lambda) - n_\ell(k(0)R)}{J_\ell(k(0)R) - J'_\ell(k(0)R)\,y_\ell(\lambda)} \quad (71)$$

with

$$y_\ell(\lambda) = \frac{k(0)\; J_\ell(k(\lambda)R)}{k(\lambda)\; J'_\ell(k(\lambda)R)} \quad (72)$$

where the prime in (71) and (72) stands for derivation with respect to the arguments. The property of the spherical Bessel functions which follows from (60) in this case is

$$\int_0^1 d\lambda\; A_\ell^2(\lambda)\, J_\ell^2(k(\lambda)r)$$

$$= A_\ell(1)\,A_\ell(0)\,J_\ell(k(1)r)\,J_\ell(k(0)r) \quad (73)$$

$A_{\ell(\lambda)}$ appearing in (73) is defined in (70).

We now explicitly check these identities in the case of S-waves by computing the right hand side and the left hand side of (66) in the high energy limit ($E \gg V_0$). In this limit one can write

$$A_\ell^2(\lambda)\sin^2 k(\lambda)r =$$
$$\cong \frac{1}{k^2_{(0)}}\left[\sin^2 k(0)r\left(1-\lambda\gamma\cos^2 k(0)R\right)+\lambda\frac{\gamma\, k(0)r}{k^2_{(0)}}\cos k(0)r\cdot \sin k(0)r\right] \quad (74)$$

where $\gamma = \frac{2mV_0}{\hbar^2}$ and

$$A_0(0)\,A_0(1)\,\sin k(1)r\,\sin k(0)r \cong$$

$$= \frac{1}{k^2_{(0)}}\left[\left(1-\frac{\gamma}{2k^2_{(0)}}\cos^2 k(0)R\right)\sin^2 k(0)r + \frac{k(0)r}{2}\left(\frac{\gamma}{k^2_{(0)}}\right)\cos k(0)r\,\sin k(0)r\right] \quad (75)$$

It is easy to see that (75) is just (74) integrated our λ from 0 to 1 thus comfirming our predictions (66).

Before ending this section, we consider one last aspect of our theory, having to do with the semiclassical approximation. This is valid in situations where the de Brogli wave length of the particle is much shorter than the characteristic length of the interacting system. This necessarily implies a large number of partial waves involved in the ℓ-sum, and thus the replacement of the ℓ-sum by an integral becomes feasible. Thus, from Eq. (15), we have

$$\langle \vec{k}' | T | \vec{k} \rangle = \frac{4\pi}{k^2} \int_0^1 d\lambda \int_0^\infty \hat{\ell}\, d\hat{\ell} \; e^{2i\sigma(\hat{\ell})} \; I^{(\lambda)}(\hat{\ell}) \times$$

$$\times \frac{-1}{(2\pi \hat{\ell} \sin\theta)^{1/2}} \left[e^{-i\hat{\ell}\theta + i\frac{\pi}{4}} - e^{i\hat{\ell}\theta - i\frac{\pi}{4}} \right] \qquad (76)$$

where the asymptotic from of the Legendre function has been used

$$P_\ell(\cos\theta) \xrightarrow[\ell \gg 1]{} \frac{2}{(2\pi \hat{\ell} \sin\theta)^{1/2}} \sin(\hat{\ell}\theta - \pi/4) \qquad (77)$$

$$\hat{\ell} \equiv \ell + 1/2$$

The above expression is valid for $\hat{\ell}^{-1} < \theta < \pi - \hat{\ell}^{-1}$. We should mention, that a more precise statement concerning the

replacement of the ℓ-sum by an integral can be made through the use of the Poisson series. Here we content ourselves taking the leading term of this series, which is precisely Eq. (76).

From the recognition

$$I^{(\lambda)}(\hat{\ell}) = -2i \frac{d\delta^{(\lambda)}(\hat{\ell})}{d\lambda} e^{2i\delta^{(\lambda)}(\hat{\ell})} \qquad (78)$$

where $\delta^{(\lambda)}(\hat{\ell})$ is the usual elastic phase shift for the potential λV, we can proceed in the evaluation of the integrals appearing in Eq. (76) using the method of stationary phase (or more generally the saddle point method). We consider the following integral,

$$I = \int e^{i\phi(x)}\, dx \qquad (79)$$

where the limit of integration are generally infinite and $\phi(x)$ is real on the line of integration and analytic in some region surrounding it. If $\phi(x)$ varies rapidly with x, the integral will be small, but if it has maxima or minima on the line of integration, these extrema will contribute the bulk of the integral, since they are just the points where the integrand does not oscillate. If $\phi'(x_o) = 0$, then near x_o,

$$\phi(x) \simeq \phi(x_0) + \frac{\phi''(x_0)}{2}(x-x_0)^2 \tag{80}$$

Generally, there are more than just one stationary point near which the phase $\phi(x)$ can be expanded as in Eq. (80). The integral I then gives, for well separated stationary points

$$\begin{aligned} I &= \sum_i e^{i\phi(x_i)} \int_{-\infty}^{\infty} e^{i\phi''(x_i)(x-x_0)^2/2} \\ &= \sum_i \sqrt{\frac{2\pi i}{\phi''(x_i)}}\, e^{i\phi(x_i)} \end{aligned} \tag{81}$$

The above result can be extended to the case when the integrand contains a slowly varying function A(x) in additon to the exponential, to yield

$$I = \sum_i \sqrt{\frac{2\pi i}{\phi''(x_i)}}\, A(x_i)\, e^{i\phi(x_i)} \tag{82}$$

The above result is the basis of the method we employ for the evaluation of the $\hat{\ell}$-integrals in Eq. (76). The final expression is[7]

$$\langle k'|T|k\rangle = \frac{4\pi}{k^2}\frac{-1}{\sqrt{2\pi\sin\theta}}\int_0^1 d\lambda\,[N(\lambda,\theta)+F(\lambda,\theta)] \tag{83}$$

where

$$N(\lambda,\theta) = \sum_j i\,\hat{\ell}_j^{1/2}\, e^{2i\sigma(\hat{\ell}_j)}\, I^{(\lambda)}(\hat{\ell}_j)\sqrt{\frac{2\pi}{\Theta'(\hat{\ell}_j)}}\, e^{-i\hat{\ell}_j\theta} \tag{84}$$

$$F(\lambda,\theta) = \sum_k \hat{\ell}_k^{1/2}\, e^{2i\sigma(\hat{\ell}_k)}\, I^{(\lambda)}(\hat{\ell}_k)\sqrt{\frac{2\pi}{\Theta'(\hat{\ell}_k)}}\, e^{i\hat{\ell}_k\theta} \tag{85}$$

where $\hat{\ell}_j$ and $\hat{\ell}_k$ denote, respectively, the stationary phase points associated with the two branches of the Legendre function

$$\frac{d}{d\hat{\ell}}\left(2i\delta^{(\lambda)}(\hat{\ell})+2i\sigma(\hat{\ell})-\hat{\ell}\theta\right)_{\hat{\ell}_j} = 0 \tag{86}$$

and

$$\frac{d}{d\hat{\ell}}\left(2i\delta^{(\lambda)}(\hat{\ell})+2i\sigma(\hat{\ell})+\hat{\ell}\theta\right)_{\hat{\ell}_k} = 0 \tag{87}$$

and Θ' refers to the derivative, with respect to ℓ of the classical deflection function $\Theta(\hat{\ell}_r) \equiv 2\,\frac{d(\delta+\sigma)}{d\ell}]_{\hat{\ell}_r}$. Clearly $\hat{\ell}_j$, $\hat{\ell}_k$ and Θ' depend on λ.

For scattering systems that exhibit strong Coulomb repulsion, such as heavy ions, the above calculation is always valid irrespective to the value of λ (in the range $0<\lambda<1$) which determines the nuclear phase shift. Clearly when two stationary phase points come very close to each other and eventually collide, as occurs in a rainbow scattering, one has to resort to the more powerful uniform approximation[8]. In this paper, we will be content with the asymptotic representation of $\langle\vec{k}'|T|\vec{k}\rangle$, Eq. (76).

The interesting features of Eq. (83) is that the semiclassical T-matrix is now represented as a sum of contributions from all the stationatry points each of which is given as a sum (integral) of contributions arising from potentials varying in strength from 0 to V in a continuous fashion. This raises the very intriguing question whether a strong rainbow scattering can be represented as a sum of several non-rainbow scattering contributions. This would arise if the λ-integral in Eq. (83) is dominated by a λ-stationary phase point. This behaviour, if it arises in general, is extremely interesting as it would obviously imply that a simple, separated stationary phase points representation suffices, even in the presence of a very strong rainbow scattering.

IV. NUMERICAL EXAMPLE: NEUTRON SCATTERING FROM OXYGEN-16

In this section we investigate the behaviour of the integrand $\langle\psi^{(-)}(\lambda V)|V|\psi^{(+)}(\lambda V)\rangle$ as a function of λ in a realistic scattering situation. An important question which we consider is whether there is a region in the λ-integral which dominates the amplitude, and if so how does this localization depend on angle.

In Ref. 6, this question was discussed within a very schematic model and for $\theta=0$. Our purpose here is to perform a more thorough analysis involving the whole angular range and at very low and intermediate energies. Hadron-nucleus scattering is taken up here to represent a typical strongly interacting system, where the full action of the potential is required.

As an example we consider the elastic scattering of neutrons from ^{16}O, at very low and intermediate energies, and use for the purpose the optical model. Within this model, the average potential that represents the interaction between the projectile neutron and the target - ^{16}O nucleus is complex, and can be taken to have a Woods-Saxon form. We use the following

$$V(r) = -\frac{(50+i20)}{1+\exp\left(\frac{r-1.2A^{1/3}}{0.6}\right)}\ \mathrm{Mev} \qquad (88)$$

The values of the parameter V(r) are realistic enough to represent roughly the actual situation.

We have evaluated the integrand $\langle\psi^{(-)}(\lambda V)|V|\psi^{(+)}(\lambda V)\rangle$ of Eq. (6) at E_n = 5.0 MeV and 100.0 MeV. In Figs. (3) and (4), we present the results for $|I^{(\lambda)}_{(\ell)}|$, Eq. (15) as a function of the orbital angular momentum for λ = 0.0, 0.1,..., 1.0. At E = 5 MeV, there is a gradual qualitative evolution of $|I(\ell)|$ as λ increases. For small values of λ, the radial integral drops very rapidly with ℓ.

At intermediate values of λ, a maximum at ℓ=1 is developed. This maximum then disappears in favor of a minimum. At λ close to unity, a flat $I(\ell)$ is seen at small ℓ which then drops rapidly to zero at around ℓ=2. This latter behaviour is expected in cases of strong absorption exemplified by the rather strong imaginary component of the potential. We summarize the above by plotting, in Fig. 3b, the contribution to $I(\ell)$ of ℓ=0, ℓ=1, and ℓ=2 vs λ.

In contrast to the above behaviour of $I^{(\lambda)}(\ell)$ at low energy, its intermediate energy behaviour is quite regular, as a function of λ, as seen in Fig. 4.

The strong absorption nature of the system, exemplified by the constancy of $|I|$ in the small ℓ-region, and the final rapid drop at larger ℓ, is quite clear. In

Fig. 4b, we exhibit the λ-dependence of $I^{(\lambda)}(\ell)$ at ℓ=0, ℓ=4 and ℓ=9.

The larger ℓ-contribution are hardly dependent on λ since they reflect scattering mainly from the centrifugal barrier. For comparison, we also show in Fig. (5), the usual elastic partial wave amplitude vs ℓ, for λ=1.0, namely the genuine full-potential strength case and for λ=0.1.

An important question which we address now, and which was briefly discussed by Tikochinsky[6], is which region in the λ-integral gives the dominant contribution to $\langle\vec{k}'|T|\vec{k}\rangle$. For this purpose, we have evaluated $|\langle\psi^{(-)}(\lambda V)|V|\psi^{(+)}(\lambda V)\rangle|$ over the whole angular region, for several values of λ within the interval $0<\lambda<1$, at two neutron energies E_n = 5.0 MeV and 100.0 MeV. The results are shown in Figs. (6) and (7). Also shown in these figures is the modulus of the elastic element of the T-matrix $|\langle\vec{k}'|T|\vec{k}\rangle|$.

At E_n = 5 MeV, the most important contribution to the scattering amplitude at small angles $0<\theta<50^{\circ}$ seems to be the small λ region, $0<\lambda<0.4$. At intermediate and large angles, there is a destructive interference phenomenon involving basically the whole λ-region. The dips in the elastic scattering amplitude at $\theta \cong 85^{\circ}$ and 15° can be easily interpreted this way. We conclude that at small angles (small momentum transfer) a much reduced potential ($\sim$ 0.25V) suffices for calculating the scattering amplitude in a symmetrical form $\langle\vec{k}'|T|\vec{k}\rangle \sim \langle\psi^{(-)}_{\vec{k}'}(0.2V)|V|\psi^{(+)}_{\vec{k}}(0.2V)\rangle$. This finding extend the result obtained by Ref. 6 to very low energies.

At E_n = 100 MeV, the same feature seen at the small angle scattering at E_n = 5 MeV prevails, in full agreement with Ref. (6), namely the use of Glauber approximation, which requires $\frac{V}{E}$ to be very small, is satisfied here even if V is of the order of E, since we have instead $\frac{\lambda V}{E}$, which is small at small angles. At back angles, the situation is completely reversed, namely in the λ-integral, we have here large λ-values contributing most. This is quite reasonable physically, since in order to scatter the flux all the way to angles close to 180^{o} one certainly needs the action of the full potential.

The above intermediate energy result points to the very interesting possibility of extending the Glauber type representation of the scattering amplitude in such a way as to be valid at large angles. For this purpose, one may use instead of the usual Glauber scattering amplitude, the symmetrized one $\langle\psi^{(-)}_{\vec{k}'}(\lambda V)|V|\psi^{(+)}_{\vec{k}}(\lambda V)\rangle$, with λ being close to unity. We leave the detailed investigation of this point to the second paper of this series.

V. CONCLUSIONS

In this paper we have derived and analyzed the symmetrical expression for the scattering T-matrix, Eq. (6). The λ-integral was evaluated exactly in several solvable scattering cases and the result was found to coincide with the known solutions. A realistic scattering problem involving the interaction of neutrons with ^{16}O was then discussed in detail within the optical model.

It was found that the λ-dependent integrand of Eq. (6) shows several interesting features. At very low energies, the integrand peaks near λ=0 in the forward angle region, implying that the λ-integral can be approximated simply by the integrand evaluated at a small λ. At larger angles, all λ-values seem to contribute roughly equally with varying signs, indicating destructive interference in the elastic scattering amplitude itself.

At higher energies, where Glauber type approximation can be contemplated, we found that whereas the small angle region is still dominated by small λ-values, the large angle region is by far dominated by the large λ-values. This opens up the possibility of approximating the λ-integral, namely the scattering amplitude, by the λ-integrand evaluated at a λ close to unity. The resulting symmetrical form of $\langle\vec{k}'|T|\vec{k}\rangle$, may then be used as a vehicle through which the

Glauber approximation can be extended to large angles. This last point will be fully developed, in the sequel to this paper[9].

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FIGURE CAPTIONS

Figure 1. A schematic diagram showing the behaviour of $|I^{(\lambda)}_{(\ell)}|$ vs λ.

Figure 2. A schematic diagram showing the behaviour of $|\int_0^1 d\lambda I^{\lambda}_{(\ell)}|$ vs ℓ.

Figure 3. The calculated $|I^{(\lambda)}_{(\ell)}|$ for $n + {}^{16}O$ scattering at $E_n = 5$ MeV. a) $|I^{(\lambda)}_{(\ell)}|$ vs ℓ, and b) $|I^{(\lambda)}(\ell)|$ vs λ (see text for details).

Figure 4. Same as Fig. 3 at $E_n = 100$ MeV.

Figure 5. The elastic partial wave amplitude vs ℓ for $\lambda = 1.0$ and $\lambda = 0.1$ for $E_n = 100$ MeV.

Figure 6. The integrand $|\langle\psi^{(-)}_{\vec{k}'}(\lambda V)|V|\psi^{(+)}_{\vec{k}}(\lambda V)\rangle|$ plotted vs θ for $n + {}^{16}O$ scattering at $E_n = 5$ MeV. a) low values of λ, b) high values of λ. The dashed curve is the elastic amplitude $|\langle\vec{k}'|V|\psi^{(+)}_{\vec{k}}(V)\rangle$.

Figure 7. Same as Figure 5 at $E_n = 100$ MeV.

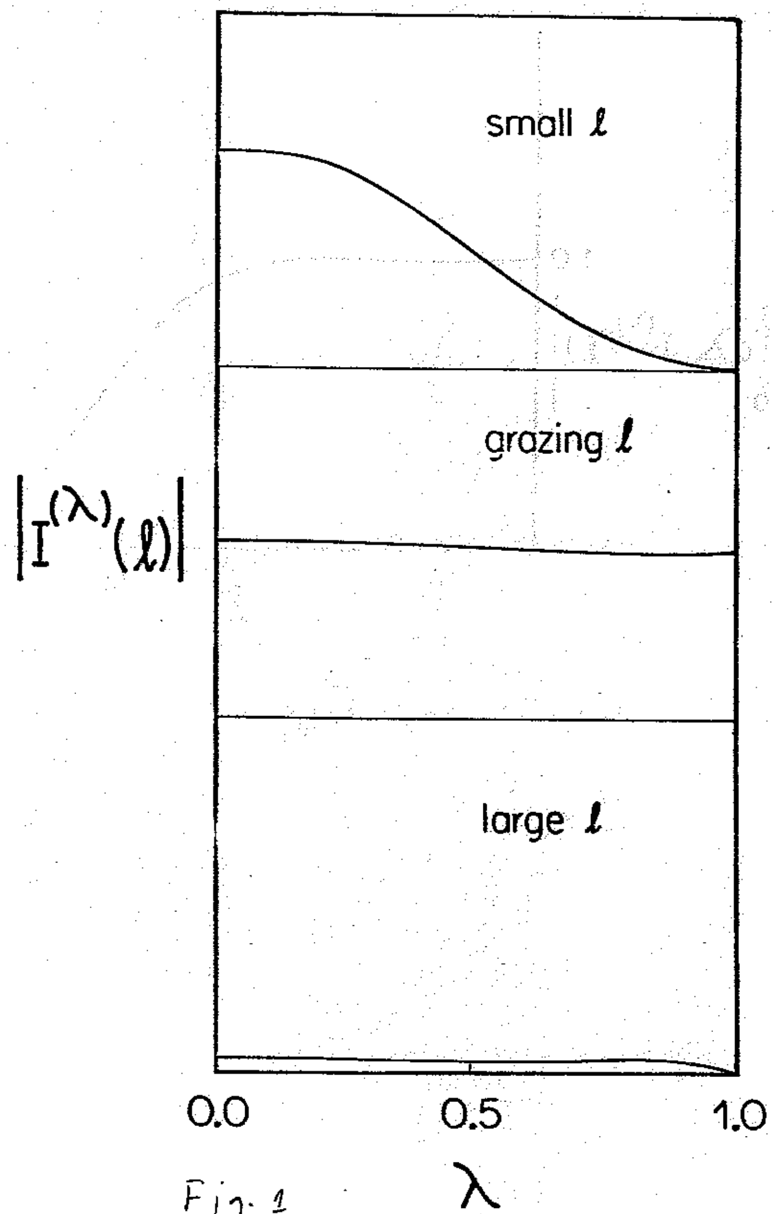
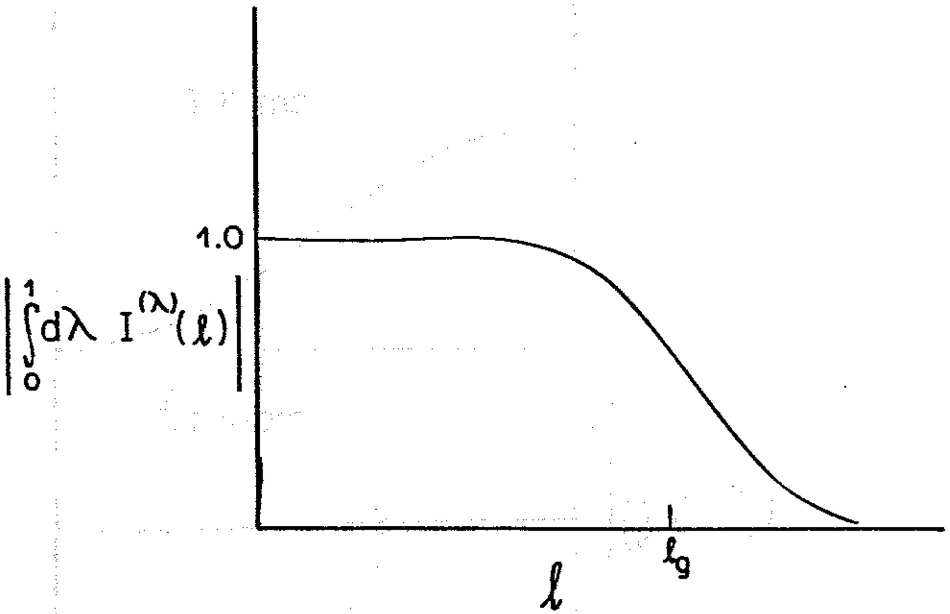
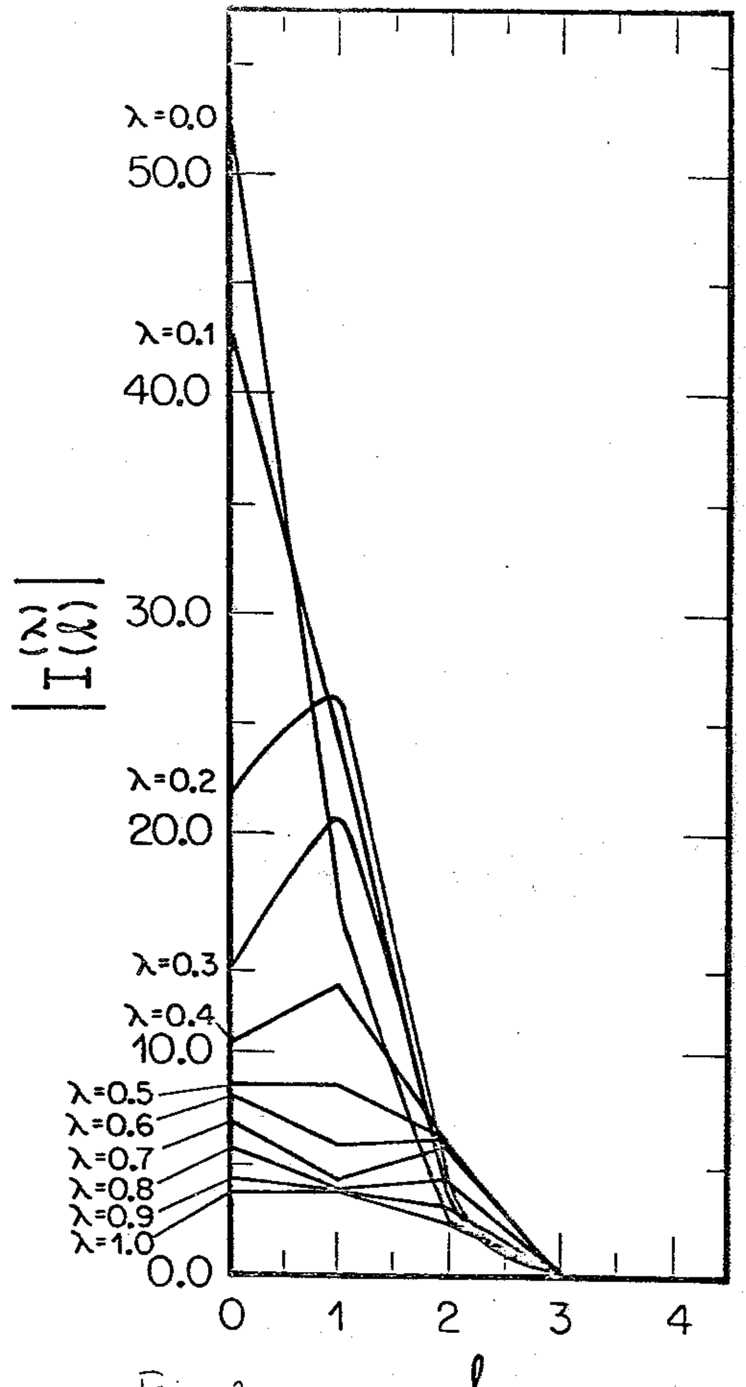


Fig. 1



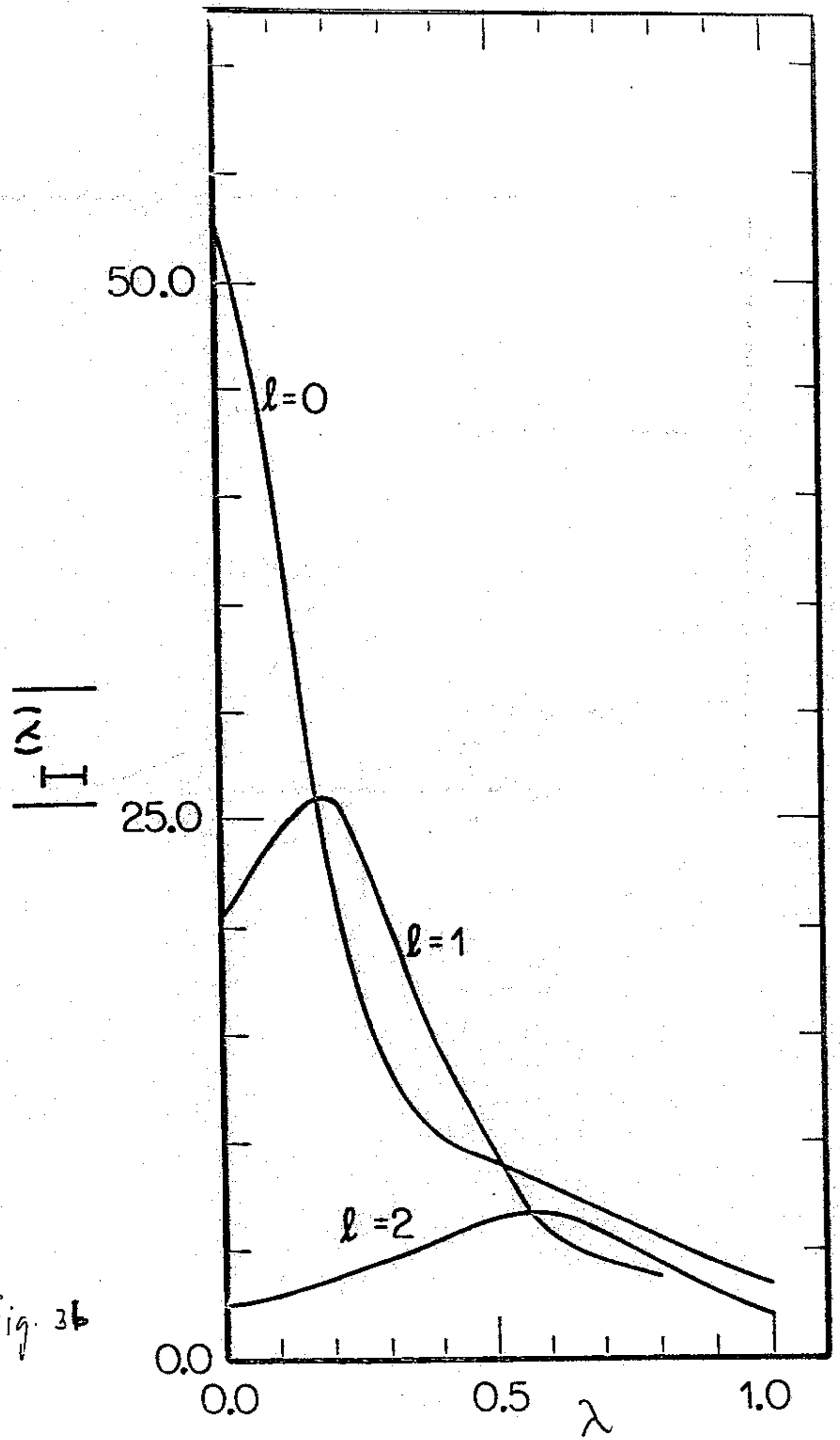
1.0
$\left|\int_0^1 d\lambda\ I^{(\lambda)}(\ell)\right|$
ℓ_g
ℓ

Fig. 2



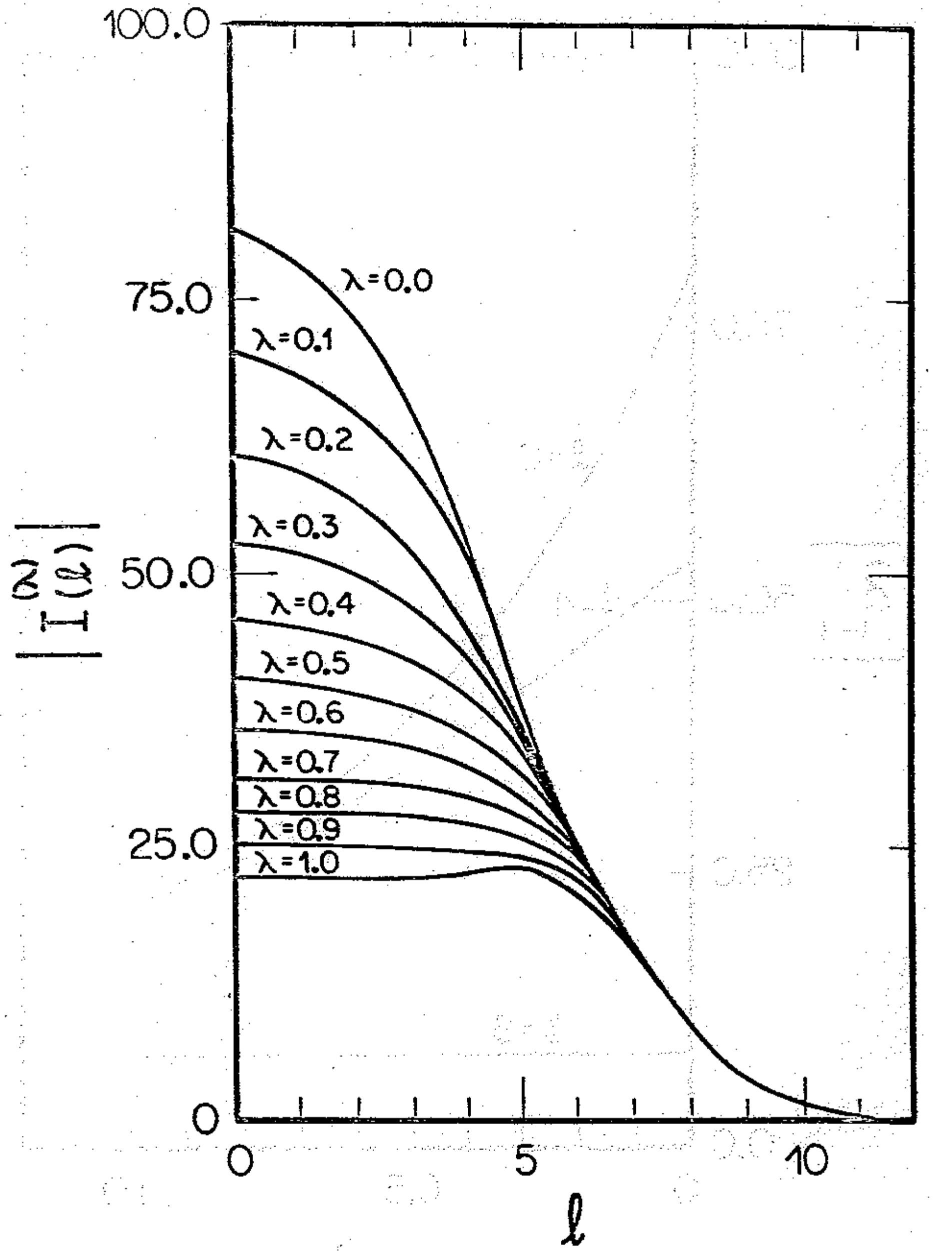
λ=0.0
50.0
λ=0.1
40.0
30.0
$\left|I^{(\lambda)}(\ell)\right|$
λ=0.2
20.0
λ=0.3
λ=0.4
10.0
λ=0.5
λ=0.6
λ=0.7
λ=0.8
λ=0.9
λ=1.0
0.0
0
1
2
3
4
ℓ

Fig. 3 a



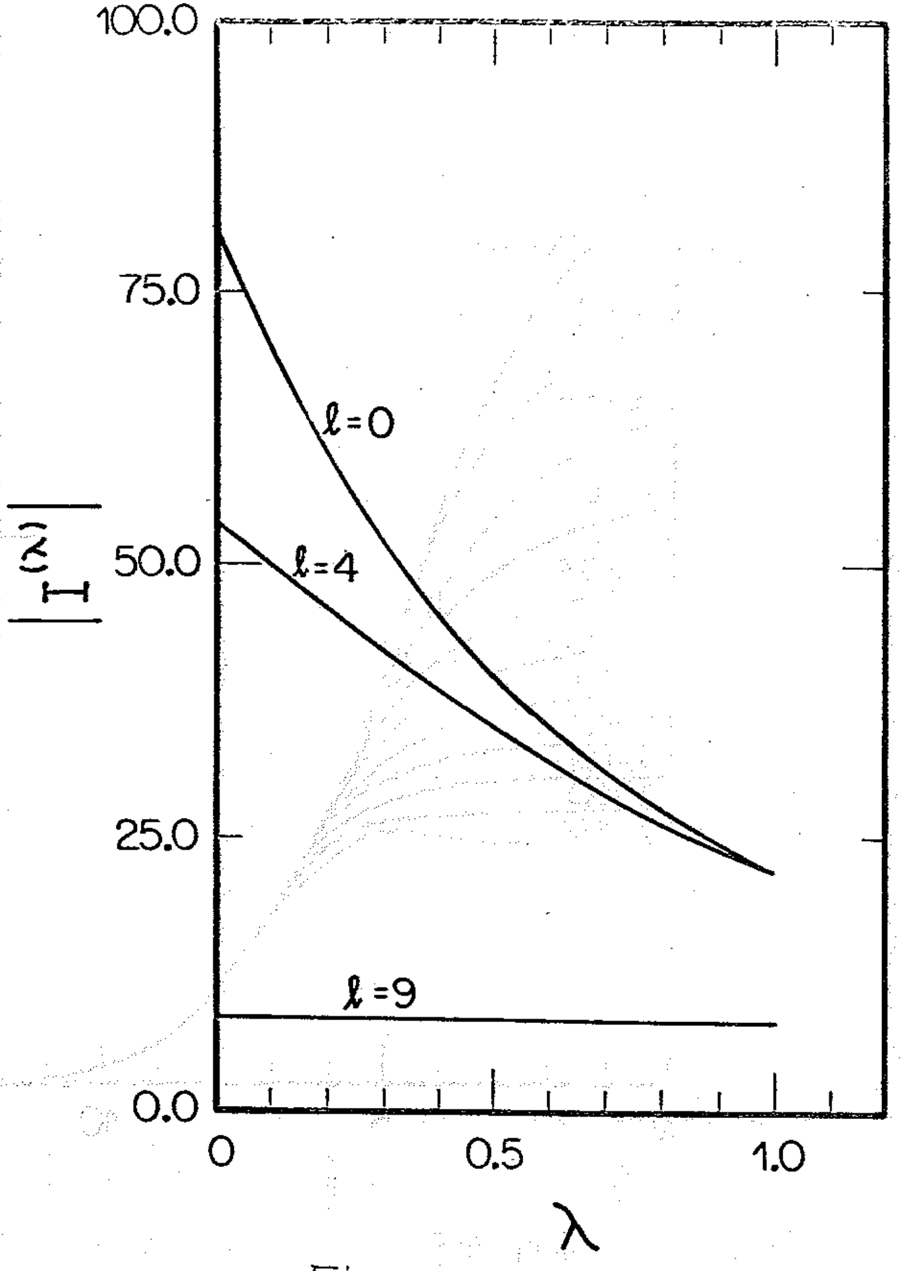
50.0
25.0
0.0
$|I^{(\lambda)}|$
$\ell=0$
$\ell=1$
$\ell=2$
0.0
0.5
1.0
λ

Fig. 3b



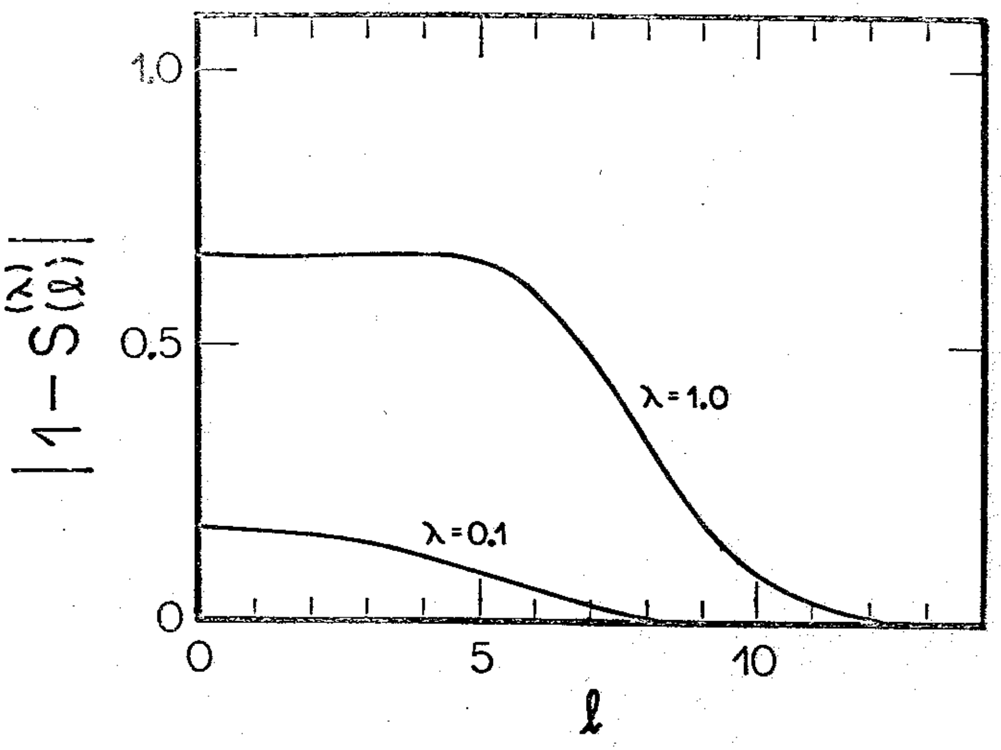
100.0
75.0
50.0
25.0
0
$|I^{(\lambda)}(\ell)|$
λ=0.0
λ=0.1
λ=0.2
λ=0.3
λ=0.4
λ=0.5
λ=0.6
λ=0.7
λ=0.8
λ=0.9
λ=1.0
0
5
10
ℓ

Fig. 4a



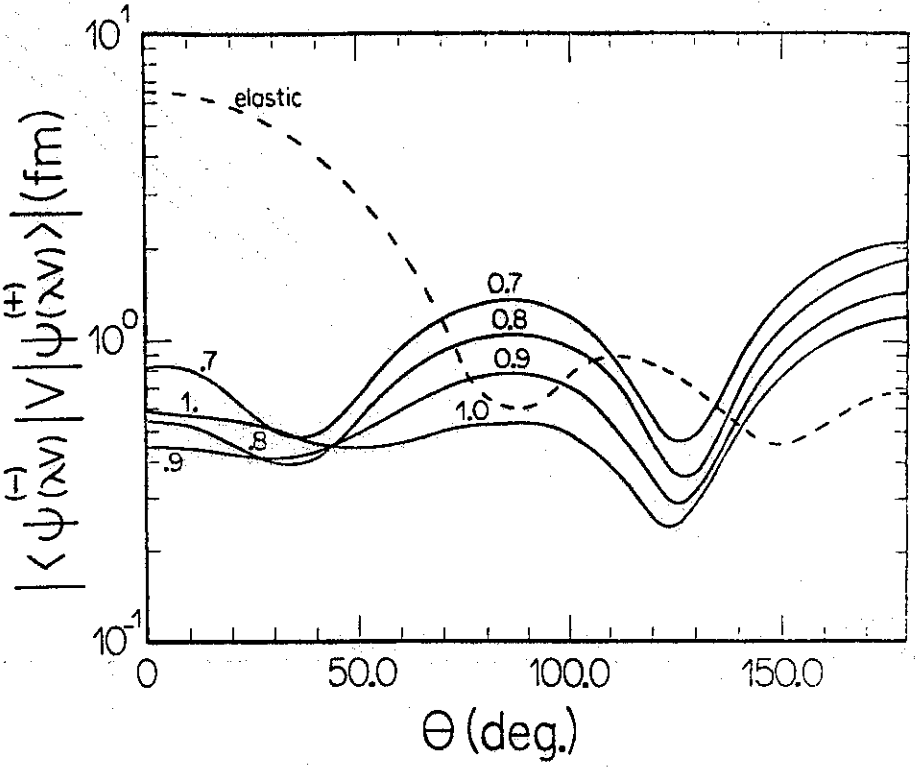
100.0
75.0
50.0
25.0
0.0
0
0.5
1.0
$|I^{(\lambda)}|$
λ
$\ell=0$
$\ell=4$
$\ell=9$

Fig. 4b



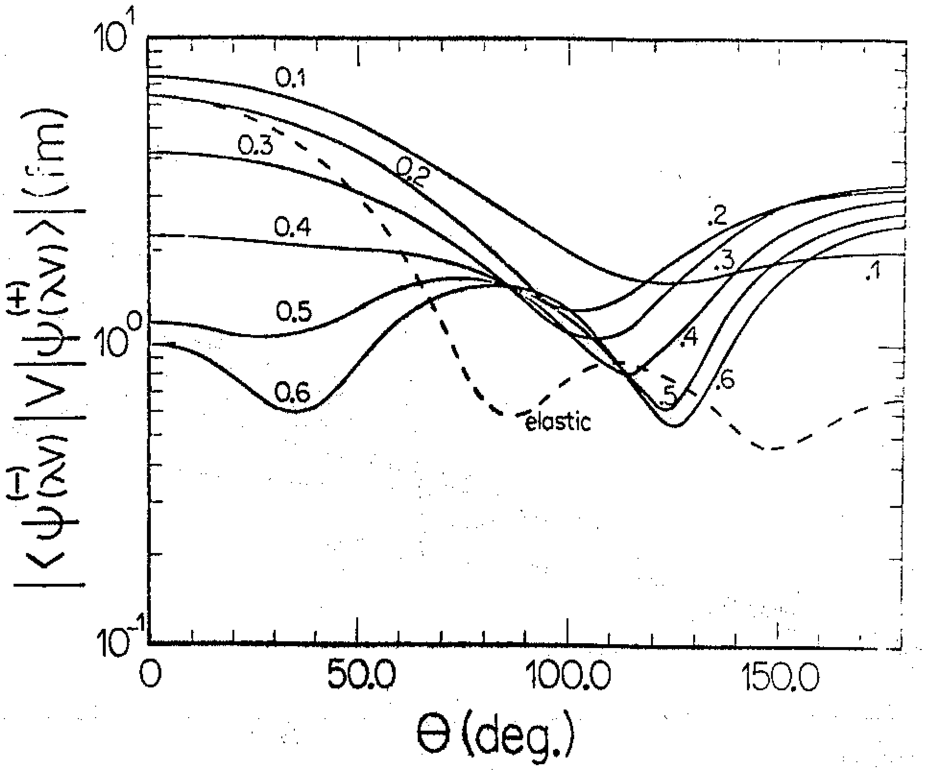
1.0
0.5
0
0
5
10
$|1-S_{(\ell)}^{(\lambda)}|$
ℓ
$\lambda=1.0$
$\lambda=0.1$

Fig. 5



elastic
0.7
0.8
0.9
1.0
.7
1.
.8
.9
$|\langle\psi^{(-)}_{(\lambda\nu)}|V|\psi^{(+)}_{(\lambda\nu)}\rangle|$ (fm)
Θ (deg.)

Fig. 6 a



0.1
0.2
0.3
0.4
0.5
0.6
elastic
.2
.3
.1
.4
.5
.6
$|\langle\psi^{(-)}_{(\lambda\nu)}|V|\psi^{(+)}_{(\lambda\nu)}\rangle|$ (fm)
Θ (deg.)

Fig. 6 b

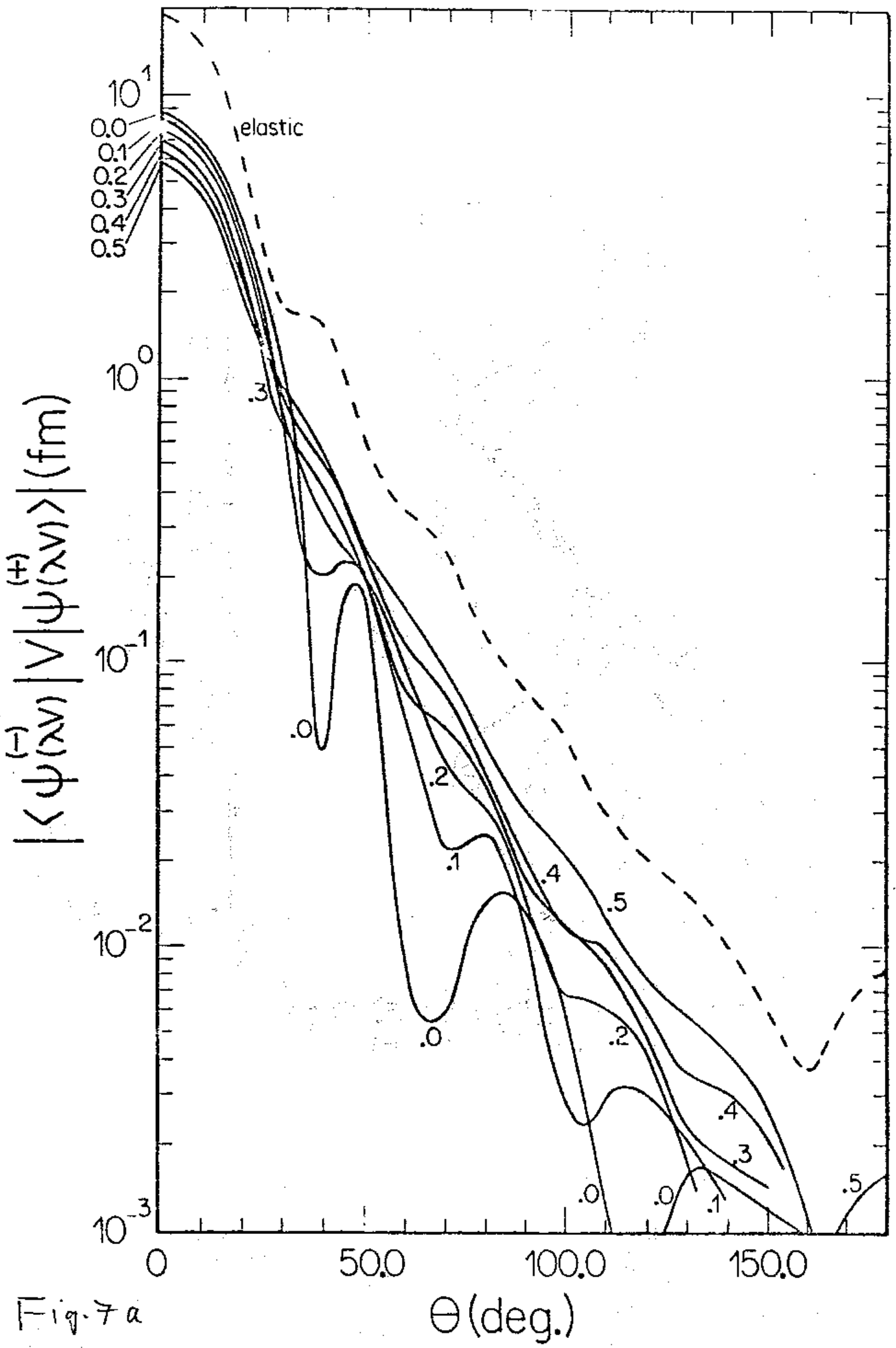

elastic
0.0
0.1
0.2
0.3
0.4
0.5
.3
.0
.2
.1
.4
.5
.0
.2
.4
.3
.0
.0
.1
.5
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
$|\langle\psi^{(-)}_{(\lambda\nu)}|V|\psi^{(+)}_{(\lambda\nu)}\rangle|$ (fm)
0
50.0
100.0
150.0
Θ(deg.)

Fig. 7a

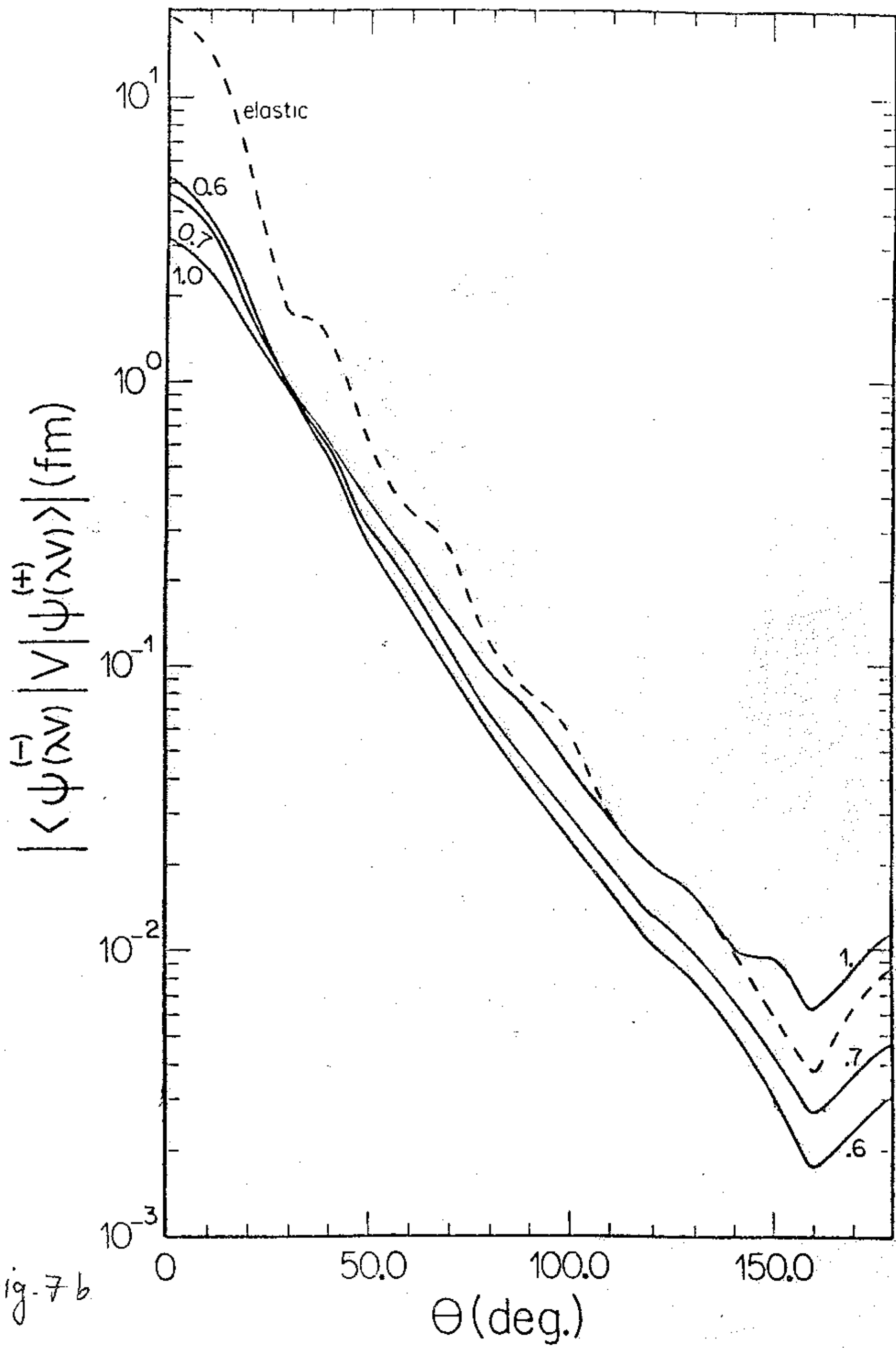

elastic
0.6
0.7
1.0
1.
.7
.6
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
$|\langle\psi^{(-)}_{(\lambda\nu)}|V|\psi^{(+)}_{(\lambda\nu)}\rangle|$ (fm)
0
50.0
100.0
150.0
Θ(deg.)

Fig. 7b