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## SCALAR-TENSOR THEORY OF FOURTH-ORDER GRAVITY

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SUMMARY.- A scalar-tensor theory of fourth-order gravity is considered. Some cosmological consequences, due to the presence of the scalar field, as well as of metric derivatives higher than second order, are analysed. In particular, upperbounds are obtained for the coupling constant $\alpha$ and for the scale factor of the universe, respectively. The discussion is restricted to Robertson-Walker universes.

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## 1. - Introduction.

A few years ago, two papers on fourth-order gravity theory by Stelle [1,2] caused a renewal of interest in higher-derivatives theories. In this sense, it is reasonable to question, ab initio, why should one think it appealing to look into theories with Lagrangians made from metric derivatives higher than second order. The answer is straightforward: Einstein Lagrangian contains non-renormalizable divergences [3-5], whereas fourth-order-gravity theories are usually renormalizable [1]. On the other hand, central to the unification scheme of gauge theories of the fundamental interactions, is the concept of spontaneous symmetry breakdown (SSB). It will be therefore necessary to incorporate, sooner or later, the mechanism of SSB to the theory of gravitation.

Recently, Accioly [7] presented a spontaneously broken symmetric gravity theory of fourth order, whose action is as follows:

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(1) K.S. STELLE: Phys. Rev. D, **16**, 953 (1977).
(2) K.S. STELLE: Gen. Rel. Grav., **9**, 353 (1978).
(3) G.'t HOOFT and M. VELTMANN: Ann. Inst. Henri Poincaré, **20**, 69 (1974).
(4) S. DESER and P. VAN NIEWENHUIZEN: Phys. Rev. D, **10**, 401,411 (1974).
(5) S. DESER , P. VAN NIEWENHUISEN and H.S. TSAO: Phys. Rev.D, **10**, 3337 (1974).
(6) N.H. BARTH and S.M. CHRISTENSEN: Phys. Rev. D, **28**, 1876 (1983).
(7) A.J. ACCIOLY: Lett. al Nuovo Cimento, **44**, 48 (1985).

(here and thereafter we take as usual $\hbar = c = 1$)

(1) $$S = \int\sqrt{-g}\,[R(\frac{1}{2\varkappa} - \frac{\Phi^2}{12}) + \frac{1}{2}\Phi^{,\alpha}\Phi_{,\alpha} - \lambda\Phi^4 + \frac{\Lambda}{\varkappa} + \alpha R^2 + \beta R_{\rho\theta}R^{\rho\theta} + L_m]\,d^4x\ .$$

$\alpha$, $\beta$ and $\lambda$ are dimensionless coupling constants, $\varkappa$ and $\Lambda$ are the Einstein and cosmological constants, respectively. $\Phi$ stands for the scalar field, whereas $L_m$ is the matter Lagrangian density. The field equations obeyed by the theory are (*):

(2) $$\Box\Phi + \frac{\Phi R}{6} + 4\lambda\Phi^3 = 0\ ,$$

(3) $$(\frac{1}{2\varkappa} - \frac{\Phi^2}{12})(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) + \frac{1}{2}\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{4}\Phi^{,\alpha}\Phi_{,\alpha}g_{\mu\nu} + \lambda\Phi^4 g_{\mu\nu} - \frac{1}{12}[\nabla_\nu\nabla_\mu - g_{\mu\nu}\Box]\Phi^2 - \frac{\Lambda}{2\varkappa}g_{\mu\nu} - \beta\Box R_{\mu\nu} - \frac{\alpha}{2}R^2 g_{\mu\nu} - \frac{\beta}{2}R^{\rho\theta}R_{\rho\theta}g_{\mu\nu} + 2\alpha RR_{\mu\nu} + 2\beta R_{\mu\theta\rho\nu}R^{\theta\rho} - (2\alpha + \frac{\beta}{2})g_{\mu\nu}\Box R + (\beta + 2\alpha)\nabla_\nu\nabla_\mu R = -\frac{1}{2}T_{\mu\nu}\ .$$

For future reference, we add to the previous equations the useful relation obtained by tracing Eq. (3) (**):

(*) $\delta\int\sqrt{-g}\,L_m\,d^4x \equiv \frac{1}{2}\int\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu}d^4x$.
(**) $T \equiv T^\alpha{}_\alpha$ .



(4) $$T = \frac{R}{\varkappa} + \frac{4\Lambda}{\varkappa} + 4(3\alpha + \beta)\Box R\ .$$

An exciting point of the theory is the possibility of a phase transition due to temperature dependence of the cosmological "constant", in case radiation and/or neutrinos, i.e., conformally invariant matter, is involved. As a result, there is a change of sign of the Einstein "constant".

Thus, it will be interesting to investigate possible models for the Universe in the aforementioned scalar-tensor theory of fourth-order gravity. We restrict our analysis to Robertson-Walker models because, first of all, there is at present good observational evidence that the actual Universe is fairly homogeneous on a large scale and has been highly isotropic since the epoch in which it became definitely transparent to radiation (8,9). And secondly, because these models are the simplest ones. In particular, we obtain an upperbound for the coupling constant $\alpha$, as well as a value for the scale factor of the Universe.

(8) M.A.H. MacCALLUM, in General Relativity: An Einstein centenary Survey, edited by S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, England, 1979), p.533.
(9) M.A.H. MacCALLUM, in Cargèse Lectures in Physics, Vol. 6 , edited by E. Schatzman(Gordon and Breach, New York,1973), p.61.

## 2. - The model.

We consider a space-time described by the metric

$$ds^2 = a^2(\zeta)\,[d\zeta^2 - \frac{d\eta^2}{1 - k\eta^2} - \eta^2 d\theta^2 - \eta^2 \sin^2\theta\, d\varphi^2], \tag{5}$$

where $k = -1, 0, +1$.

The symmetry of the metric allows us to write $\Phi = \Phi(\zeta)$. As a consequence, the field equations can be written explicitly with the help of the metric (5) as :

$$\frac{1}{a^2}\Phi'' + \frac{2a'}{a^3}\Phi' + (\frac{a''}{a^3} + \frac{k}{a^2})\Phi + 4\lambda\Phi^3 = 0 \tag{6}$$

$$R^0{}_0 \to T^0{}_0 = \rho(\zeta)\ , \tag{7}$$

$$R^1{}_1 = R^2{}_2 = R^3{}_3 \to T^1{}_1 = T^2{}_2 = T^3{}_3 = -\,p(\zeta)\ , \tag{8}$$

where

$$\begin{aligned}\rho(\zeta) = {} & 3\,(\frac{1}{\varkappa} - \frac{\Phi^2}{6})\,(\frac{a'^2}{a^4} + \frac{k}{a^2}) - \frac{\Phi'^2}{2a^2} - \frac{a'\Phi'\Phi}{a^3} - \lambda\Phi^4 + \frac{\Lambda}{\varkappa} \\ & + \frac{36\alpha}{a^2}\,[\frac{2a'a'''}{a^4} - \frac{4a''a'^2}{a^5} - \frac{a''^2}{a^4} - \frac{2a'^2k}{a^4} + \frac{k^2}{a^2}] \\ & + \frac{12\beta}{a^2}\,[\frac{2a'a'''}{a^4} - \frac{2a''a'^2}{a^5} - \frac{a''^2}{a^4} - \frac{4a'^4}{a^6} - \frac{4ka'^2}{a^4} + \frac{k^2}{a^2}]\ ,\end{aligned} \tag{9}$$



$$\begin{aligned}p(\zeta) = {} & (\frac{1}{\varkappa} - \frac{\Phi^2}{6})(-\frac{2a''}{a^3} + \frac{a'^2}{a^4} - \frac{k}{a^2}) + \frac{\Phi\Phi''}{3a^2} + \frac{a'\Phi'\Phi}{3a^3} - \frac{1}{6}\frac{\Phi'^2}{a^2} + \Lambda\Phi^4 - \frac{\Lambda}{\varkappa} \\ & - \frac{12\alpha}{a^2}\,[2\frac{a^{IV}}{a^3} - \frac{10a'''a'}{a^4} - \frac{5a''^2}{a} + \frac{16a''a'^2}{a^5} - \frac{4ka''}{a^3} + \frac{6ka'^2}{a^4} - \frac{k}{a^2}] \\ & - \frac{4\beta}{a^2}\,[2\frac{a^{IV}}{a^3} - \frac{10a'''a'}{a^4} + \frac{14a''a'^2}{a^5} - \frac{5a''^2}{a^4} - \frac{4ka''}{a^3} + \frac{4a'^4}{a^6} + \frac{8ka'^2}{a^4} - \frac{k^2}{a^2}].\end{aligned} \tag{10}$$

In these equations, primes and roman numerals indicate derivatives with respect to $\zeta$ .

The preceding results show us that the symmetry of space-time leads, in a natural way, to the energy-momentum tensor of a perfect fluid at rest.

It is clear from Eq. (3) that the effective Einstein "constant", $\varkappa_{ef} = \varkappa/(1 - \varkappa\Phi^2/6)$, will be a true constant, only in those regions of space-time where $\Phi$ is a constant. So, it will be reasonable to focus our attention on the analysis of solutions with a constant scalar field. If in addition we admit a traceless energy-momentum tensor, we get immediately from Eqs. (2) and (4) that $\Phi^2 = \Lambda/6\lambda$. We have to observe here that the reality of the scalar field depends on the positiveness of the cosmological constant. But, as it is well known, the value of the energy associated to the $\frac{\Lambda}{\varkappa}$ term in Eq. (1) is very small, playing no important part in establishing the global features of the present and past of our Universe. It is worth noticing , en passant, that $\Phi = \pm\sqrt{\Lambda/6\lambda}$ are the possible vacuum expectation values, in case radiation is involved ([7]). Taking into account the former considerations, we get from Eqs. (6), (9) and (10) :

(11) $$\frac{1}{2} a'^2 + \frac{\Lambda}{6} a^4 + \frac{1}{2} ka^2 = C ,$$

(12) $$\rho = 3p = \frac{6C}{a^4} [\varkappa_{ef}^{-1} - 16\Lambda\alpha - 16\beta(2C/a^4 - k/a^2)] ,$$

where C is an integration constant and $\varkappa_{ef} \equiv \dfrac{\varkappa}{1 - \varkappa\Lambda/36\lambda}$ .

In the absence of matter, i.e., C = 0, we find from Eq. (11) that k = -1 and $0 < a < \sqrt{3/\Lambda}$ . Consequently, our model corresponds to an open universe. An upper bound for the scale factor of this universe is given by $\sqrt{3/\Lambda}$ .

If we admit the presence of matter - which means that $C \neq 0$ - three possibilities are then allowed, depending on the values of k. It is not difficult to see that C will be always positive for k = 0 and k = 1. We shall take for granted the positiveness of C, in case k = -1, for physical reasons that will soon become clear. Therefore, C will be assumed positive, irrespective of the values of k.

For k = 0, we must have $0 < a < (6C/\Lambda)^{1/4}$. On the other hand, if k = 1, $0 < a < [\frac{-1+(1+8\Lambda C/3)^{\frac{1}{2}}}{2\Lambda/3}]^{\frac{1}{2}}$ . In case k = -1, $0 < a < [\frac{1+(1+8\Lambda C/3)^{\frac{1}{2}}}{2\Lambda/3}]^{\frac{1}{2}}$ . If tachyonic particles are ruled out from the theory, which is obtained by demanding $\beta > 0$ [10], we get from Eq. (12) an upper bound for the coupling constant α, corresponding to the situations just considered

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[10] B. WHITT: Phys. Rev. D, 32, 379 (1985).



(13) $$\alpha < \frac{\varkappa_{ef}^{-1}}{16\Lambda} .$$

## 3. - Discussion.

Eq. (11) suggests the following mechanical analogy: a particle of unity mass and total energy C, moving under the influence of potential

(14) $$V(a) = \frac{\Lambda}{6} a^4 + \frac{k}{2} a^2 .$$

This interpretation will give us a qualitative insight into the characteristic behaviour of $a = a(\zeta)$.

For the void case, only solutions with k = -1 (region I, Fig. 1) are allowed and these solutions show us an oscillating universe, with expansion and contraction alternating phases, which eventually passes by $a(\zeta) = 0$. The order of magnitude of the "radius of this universe" is:

$$a \lesssim \Lambda^{-\frac{1}{2}} \sim 10^{28} \text{ cm} .$$

We call special attention to the fact that in this case the empty space is not an Einstein-space or, in other words, $R_{\mu\nu} \neq Kg_{\mu\nu}$. This result is entirely due to the presence of the derivatives higher than second order in the Lagrangian of the theory. In fact, it is trivial to show that when $\alpha = 0$, $\beta = 0$, $\Phi$ = cte. , and $T_{\mu\nu} = 0$, Eq. 3 reduces to $R_{\mu\nu} = Kg_{\mu\nu}$ .

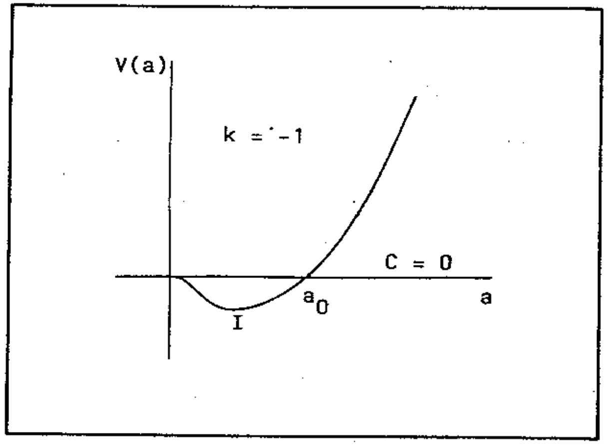


**Fig. 1. - Potential energy as a function of the scale factor. The picture is qualitative and concerns to the vacuum case. $a_0 \sim 10^{28}$ cm .**

In the presence of matter, C > 0, we have again oscillating universes for all possible values of k, k = 0, k = 1 or k = -1 (Fig. 2). The "radii" of such universes depend on the constant C. We point out that the vacuum case can be obtained formally from the situation where matter is present and k = -1, using a limit process. An upper bound for the coupling constant $\alpha$ is given by

$$\alpha \lesssim (\kappa\Lambda)^{-1} \sim 10^{121}$$

This value is totally independent from the parameter k.



Last but not least, we wish to remark that there is a fundamental difference between our theory and the usual Einstein theory. In the first one, the behaviour of $a(\zeta)$ is always oscillatory , irrespective of the values of k(-1, 0, +1), whereas in the Einstein theory, we have endless expanding universes for different values of k. The presence of terms with higher metric derivatives, in the Lagrangian of the theory, as well as the extra field, the scalar $\Phi$, account for the difference.

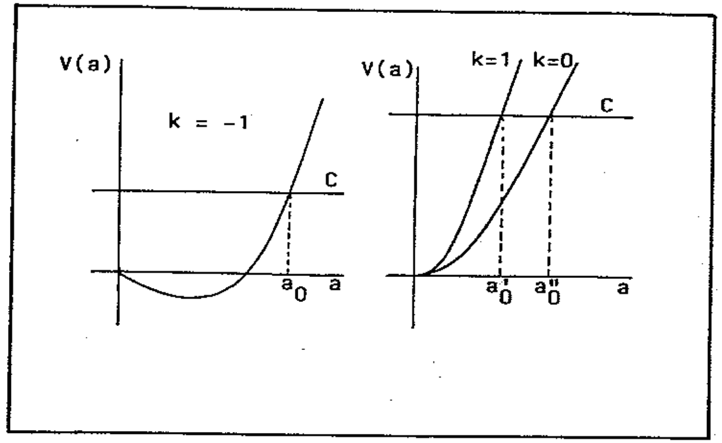


**Fig. 2. - Qualitative picture of the potential energy as a function of the scale factor in the presence of matter (C > 0) for all values of k (-1 , 1 , 0).**

$$a_0 = \left[\frac{1+(1+8\Lambda C/3)^{\frac{1}{2}}}{2\Lambda/3}\right]^{\frac{1}{2}} , \quad a'_0 = \left[\frac{-1+(1+8\Lambda C)/3)^{\frac{1}{2}}}{2\Lambda/3}\right]^{\frac{1}{2}},$$

$$a''_0 = (6C/\Lambda)^{1/4}.$$