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ABSTRACT

Using the conventional toroidal coordinate system Laplace's equation for the magnetic scalar potential due to toroidal helical currents is solved. The potential is written as a sum of an infinite series of functions. Each partial sum represents the potential within some accuracy. The effect of the winding law is analysed in the case of small curvature.

1. INTRODUCTION

Magnetic field configurations due to a toroidal helical field have been considered for plasma confinement in stellarators (Ohkawa et al., 1980). In these devices the pitch of the magnetic field lines reverses inside the plasma. The pitch reversal is mostly due to external helical windings. Toroidal effects are not taken into acocunt by Ohkawa et al. (1980). According to Sharp et al. (1979) some effects of toroidicity can be important for small aspect ratios $\left(\frac{R_0}{b} \geq 4\right)$ and negligible when $\frac{R_0}{b} \gtrsim 12$. Also, the influence of toroidicity is altered if the helical winding pitch is modulated.

In torsatrons magnetic field configuration is partly due to regularly spaced toroidal helical conductors carrying equal currents in the same direction. Gourdon et al. (1970) suggest that choosing conveniently the pitch of the helical winding, one coil could be enough to produce the required stable magnetic field configuration. For these and other reasons analytical expressions for the toroidal helical field due to specified windings have become necessary. Approximate analytical expressions for the toroidal helical field can be obtained considering helically symmetric system bent into a torus (La Haye et al. (1981), Morozov and Solov'ev (1966) and Sometani et al. (1983)). The resultant system is neither helical nor axisymmetric. Therefore, bending a straight system leads to a complicated equation in terms of a local polar coordinate system. Approximate Laplace's equation for the magnetic scalar potential can be used near the torus. This approximation yields a problem: the boundary conditions for the magnetic field outside the torus become undefined. Sometani et al. (1983) in their calculations exclude solutions of the approximate equation outside the winding region that diverge at an infinite distance from the torus axis. The reason does not seem obvious.

In this work, exact Laplace's equation for the magnetic scalar potential is solved using a more natural coordinate system. A particularly simple winding law in this coordinate system is adopted. The potential is written as an infinite series of functions. Each term is of the order of a power of the inverse aspect ratio.

In the case of small curvature system, the first order toroidal effect on the magnetic fiels is specially analysed. Also, the effect of a non-uniform winding law is considered in this approximation. Both effects can be of the same order of magnitude.

2. TOROIDAL HELICAL CURRENTS

In this work, a number of thin conductors wound on a circular torus, carrying currents I in alternating directions is considered.

The toroidal helical winding is characterized by the major and minor radii of the torus R_o and b, respectively, by the number of periods of the helical field in the poloidal and toroidal directions m_o and n_o and by a winding law.

The most appropriate coordinate system seems to be the usual toroidal coordinate system defined by ξ, ω and φ (see figure 1):

$$r = \frac{R_o' \sinh\xi}{\cosh\xi - \cos\omega} \quad \text{and} \quad z = \frac{R_o' \sin\omega}{\cosh\xi - \cos\omega} \tag{2.1}$$

where r, φ and z are the polar coordinates (Appendix A).

If $\xi = \xi_o$ defines the toroidal surface, then:

$$\cosh\xi_o = \frac{R_o}{b} \quad \text{and} \quad R_o' = R_o\sqrt{1 - \frac{b^2}{R_o^2}} \quad . \tag{2.2}$$

Here, a linear relation between ω and φ is taken as the winding law:

$$m_o\omega + n_o\varphi = \text{constant} \quad . \tag{2.3}$$

Also a new toroidal helical coordinate system is defined by ξ, ω and u where $u = m_o\omega + n_o\varphi$.

The surface current density due to a thin helical toroidal conductor with current I has the components:

$$i_\varphi = \frac{m_o I}{R_o'}(\cosh\xi_o - \cos\omega)\,\delta(m_o\omega + n_o\varphi)$$

and

$$i_\omega = -\frac{n_o I}{r}\,\delta(m_o\omega + n_o\varphi) \quad . \tag{2.4}$$

i_φ and i_ω have periodicity $\frac{2\pi}{m_o}$, $\frac{2\pi}{n_o}$ and 2π in the variables ω, φ and u, respectively. Therefore, the δ-function can be expanded in Fourier series:

$$\delta(u) = \frac{1}{2\pi}\left(1 + 2\sum_{N=1}^{\infty}\cos Nu\right) \quad . \tag{2.5}$$

The resulting current density is

$$i_\varphi = \frac{m_o I}{2\pi R'_o} (\cosh\xi_o - \cos\omega)\left(1 + 2\sum_{N=1}^{\infty} \cos Nu\right) . \qquad (2.6)$$

Two simple ways may be considered of producing an n-th harmonic field $\begin{pmatrix}\sin n\omega\\ \cos n\omega\end{pmatrix}$:

(i) by a pair of conductors with current flow I in opposite directions. The winding must be such that $m_o = n$. In this case the current dnesity i_φ is given by:

$$i_\varphi = \frac{m_o I}{R'_o} (\cosh\xi_o - \cos\omega)\left[\delta(m_o\omega + n_o\varphi) - \delta(m_o(\omega - \frac{\pi}{m_o}) + n_o\varphi)\right]; \qquad (2.7)$$

(ii) by n pairs of conductors with currents $\pm I$ alternately. n must be some multiple of m_o. In this case:

$$i_\varphi = \frac{m_o I}{R'_o} (\cosh\xi_o - \cos\omega)\left[\sum_{K=0}^{n-1} \delta(m_o(\omega - K\frac{2\pi}{n}) + n_o\varphi) - \sum_{K=0}^{n-1} \delta(m_o(\omega - K\frac{2\pi}{n} - \frac{\pi}{n}) + n_o\varphi)\right] . \qquad (2.8)$$

The resultant current density is:

$$i_\varphi = \frac{2 m_o I}{\pi R'_o} (\cosh\xi_o - \cos\omega) \sum_{N=1}^{\infty} \cos Nu \qquad (2.9)$$

where $N = 2p+1$; $p = 0,1,2,\ldots\infty$ in the case (i) and

$$i_\varphi = 2n \frac{m_o I}{\pi R'_o} (\cosh\xi_o - \cos\omega) \sum_{N=1}^{\infty} \cos Nu \qquad (2.10)$$

where $N = (2p+1)\frac{n}{m_o}$; $p = 0,1,2,\ldots\infty$ in the case (ii).

i_ω is expressed in terms of $\cos Nu$ in similar fashion. In both cases

$$\cos Nu = \cos\left[(2p+1)\, n(\omega + \frac{n_o}{m_o}\varphi)\right]$$

and the field presents an infinite number of harmonics. Near the axis, the predominant contribution to the magnetic field comes from the lowest harmonic term.

3. BOUNDARY CONDITIONS

In toroidal coordinates, boundary conditions on the magnetic field at the surface of the torus are given by:

$$B^e_\omega - B^i_\omega = -\mu_o i_\varphi$$

$$B^e_\varphi - B^i_\varphi = -\mu_o i_\omega$$

and $$B^e_\xi - B^i_\xi = 0 . \qquad (3.1)$$

Quantities in the region outside the helical winding are given a suffix e and those in the region inside the helical winding a suffix i .

Magnetic field may be described in the two regions by the scalar potential $\Phi(\xi,\omega,\varphi) : \vec{B} = \text{grad}\,\Phi$.

Here, the field of a single harmonic:

$$i_\varphi = \frac{m_o I}{\pi R_o'} (\cosh\xi_o - \cos\omega) \cos Nu$$

and (3.2)

$$i_\omega = -\frac{n_o I}{\pi r} \cos Nu \quad , \quad N \neq 1$$

is deduced.

Boundary conditions for the scalar potential in toroidal helical coordinate system (ξ,ω,u) become:

$$\frac{\partial\Phi^e}{\partial\xi} - \frac{\partial\Phi^i}{\partial\xi} = 0$$

$$\frac{\partial\Phi^e}{\partial u} - \frac{\partial\Phi^i}{\partial u} = -\frac{\mu_o I}{\pi} \cos Nu \qquad (3.3)$$

$$\frac{\partial\Phi^e}{\partial\omega} - \frac{\partial\Phi^i}{\partial\omega} = 0 \quad .$$

The zero'th harmonic $i_\varphi = \frac{m_o I}{2\pi R_o'} (\cosh\xi_o - \cos\omega)$ produces zero field inside the winding and $i_\omega = -\frac{n_o I}{2\pi r}$ produces the toroidal field $B^i_\varphi = \frac{\mu_o}{4\pi} 2 n_o I$.

4. THE EXACT SOLUTION

In toroidal coordinate system, Laplace's equation is:

$$\frac{\partial}{\partial\omega} \frac{\sinh\xi}{\cosh\xi - \cos\omega} \frac{\partial\Phi}{\partial\omega} + \frac{\partial}{\partial\xi} \frac{\sinh\xi}{\cosh\xi - \cos\omega} \frac{\partial\Phi}{\partial\xi} + \frac{\partial}{\partial\varphi} \frac{1}{\sinh\xi(\cosh\xi - \cos\omega)} \frac{\partial\Phi}{\partial\varphi} =$$

$$= 0 \quad . \qquad (4.1)$$

Introducing a complex function $F(\xi,\omega,\varphi)$ related

to Φ by the expression:

$$\Phi = \mathrm{Re}(\cosh\xi - \cos\omega)^{\frac{1}{2}} F(\xi,\omega,\varphi) \qquad (4.2)$$

where Ref means real part of f , we can write for F the equation

$$\frac{\partial^2 F}{\partial\omega^2} + \frac{1}{\sinh\xi} \frac{\partial}{\partial\xi} \sinh\xi \frac{\partial F}{\partial\xi} + \frac{1}{4} F + \frac{1}{\sinh^2\xi} \frac{\partial^2 F}{\partial\varphi^2} = 0 \quad . \qquad (4.3)$$

Boundary conditions (3.3) suggest solutions of the form:

$$F = \sum_\nu F_\nu(\cosh\xi)\, e^{i\nu\omega}\, e^{iNu} \quad . \qquad (4.4)$$

$F_\nu(\cosh\xi)$ is found to satisfy Legendre's associated equation:

$$\frac{d}{d\cosh\xi} (1 - \cosh^2\xi) \frac{dF_\nu}{d\cosh\xi} + \left\{\left[(\nu + m_o N)^2 - \frac{1}{4}\right]\right.$$

$$\left. - \frac{n_o^2 N^2}{1 - \cosh^2\xi}\right\} F_\nu = 0 \quad . \qquad (4.5)$$

If divergent functions are excluded in each region we obtain the following expressions for the interior and exterior fields:

$$F^e_\nu = A^e_\nu\, P(\cosh\xi)$$

and (4.6)

$$F^i_\nu = A^i_\nu\, Q(\cosh\xi)$$

were P and Q are the associated Legendre functions:

$$P \equiv P^{n_o N}_{m_o N + \nu - \frac{1}{2}}$$

$$Q \equiv Q^{n_o N}_{m_o N + \nu - \frac{1}{2}} \quad .$$

The constants A^e_ν and A^i_ν can be determined using the boundary conditions:

$$\sum e^{i\nu\omega}(F^e_\nu - F^i_\nu) = -\frac{1}{i}\frac{\mu_o I}{\pi N}(\cosh\xi_o - \cos\omega)^{-\frac{1}{2}}$$

and (4.7)

$$\sum_\nu e^{i\nu\omega}\left(\frac{\partial}{\partial\xi}F^e_\nu - \frac{\partial}{\partial\xi}F^i_\nu\right) = \frac{1}{i}\frac{\mu_o I}{2\pi N}\frac{\sinh\xi_o}{(\cosh\xi_o - \cos\omega)^{\frac{3}{2}}}$$

deduced from (3.3).

On the boundary, $\cosh\xi_o = \dfrac{R_o}{b}$.

The right hand side of each of the equations (4.7) is expanded in powers of the inverse aspect ratio and in Fourier series in ω . (4.7) become:

$$F^e_\nu - F^i_\nu = -\frac{1}{i}\frac{\mu_o I}{\pi N}\sum_{\substack{s=|\nu|+2K\\ K=0,1,2,\ldots\infty}} C_{s,\nu}\left(\frac{b}{R_o}\right)^{s+\frac{1}{2}}$$

(4.8)

$$\frac{1}{\sinh\xi_o}\left(\frac{\partial}{\partial\xi}F^e_\nu - \frac{\partial}{\partial\xi}F^i_\nu\right) = \frac{1}{i}\frac{\mu_o I}{\pi N}\sum_{\substack{s=|\nu|+2K\\ K=0,1,2,\ldots\infty}} C_{s,\nu}\frac{2s+1}{2}\left(\frac{b}{R_o}\right)^{s+\frac{3}{2}}$$

where

$$C_{s,\nu} = \frac{(2s-1)!!}{2^{2s}\left(\frac{s-\nu}{2}\right)!\left(\frac{s+\nu}{2}\right)!} \quad .$$

If the functions (4.6) are to satisfy the conditions (4.8), the expressions for the potential inside the winding region must be:

$$\Phi = \sum_{s=0}^{\infty}\Phi_s \qquad (4.9)$$

$$\Phi_s = \frac{\mu_o I}{\pi N}(\cosh\xi - \cos\omega)^{\frac{1}{2}}\left(\frac{b}{R_o}\right)^s \times$$

$$\times \sum_\nu \left(\frac{b}{R_o}\right)^{\frac{3}{2}}\frac{C_{s,\nu}}{-W}\left[\frac{2s+1}{2}P\left(\frac{R_o}{b}\right) + \frac{R_o}{b}P'\left(\frac{R_o}{b}\right)\right] \times$$

$$\times\, Q(\cosh\xi)\sin(Nu + \nu\omega)$$

where $\nu = -s, -s+2, \ldots, s-2, s$

and $-W = P'Q - PQ'$.

Near the surface each function Φ_s is of the order of $\left(\frac{b}{R_o}\right)^s$ (Appendix B).

5. SMALL CURVATURE APPROXIMATION

In the case of small curvature the scalar potential can be represented by:

$$\Phi \simeq \Phi_o + \Phi_1 \qquad (5.1)$$

$$\Phi_o = \left(\frac{\cosh\xi - \cos\omega}{\cosh\xi_o}\right)^{\frac{1}{2}}\frac{\mu_o I}{2\pi N}\left(\frac{e^{\xi_o}}{e^{\xi}}\right)^{m_o N + \frac{1}{2}}\sin Nu$$

$$\Phi_1 = \left(\frac{\cosh\xi - \cos\omega}{\cosh\xi_o}\right)^{\frac{1}{2}} \frac{\mu_o I}{2\pi N}\,\frac{b}{4R_o}\left\{\frac{m_o N+2}{m_o N+1}\left(\frac{e^{\xi_o}}{e^{\xi}}\right)^{m_o N+\frac{3}{2}} \sin(Nu+\omega) + \frac{m_o N}{m_o N-1}\left(\frac{e^{\xi_o}}{e^{\xi}}\right)^{m_o N-\frac{1}{2}} \sin(Nu-\omega)\right\} .$$

Terms of the order $\geq \left(\frac{b}{R_o}\right)^2$ are neglected.

The potential can be deduced for a non-linear winding law of the form:

$$m_o(\omega + \delta_o \sin\omega) + n_o\varphi = \text{constant} . \tag{5.2}$$

The surface current density due to a single coil becomes:

$$i_\varphi = \frac{m_o I}{R_o'}(\cosh\xi_o - \cos\omega)(1+\delta_o\cos\omega)\delta(m_o(\omega+\delta_o\sin\omega)+n_o\varphi)$$

and

$$i_\omega = -\frac{n_o I}{R_o'}\delta(m_o(\omega+\delta_o\sin\omega)+n_o\varphi) \tag{5.3}$$

instead of (2.7).

Expression (2.10) must be substituted by:

$$i_\varphi = 2n\frac{m_o I}{\pi R_o'}(\cosh\xi_o - \cos\omega)(1+\delta_o\cos\omega)\sum_{N=1}^{\infty}\cos(Nu + m_o N\delta_o\sin\omega) . \tag{5.4}$$

Using the formula:

$$\exp(iX\sin\vartheta) = J_o(X) + \sum_{K=1}^{\infty}(\pm 1)^K J_K(X)\, e^{\pm iK\vartheta}$$

and taking account of only the first two side-bands (5.4) becomes:

$$i_\varphi \simeq i_\varphi^{(o)} + i_\varphi^{(\pm 1)} \tag{5.5}$$

where

$$i_\varphi^{(o)} = 2n\,J_o(m_o N\delta_o)\frac{m_o I}{\pi R_o'}(\cosh\xi_o - \cos\omega)\cos N(m_o\omega + n_o\varphi)$$

and

$$i_\varphi^{(\pm 1)} = 2n\left[\frac{\delta_o}{2}J_o(m_o N\delta_o) \pm J_1(m_o N\delta_o)\right]\frac{m_o I}{\pi R_o'} \times$$

$$\times \cos(N(m_o \pm \tfrac{1}{N})\omega + n_o\varphi) .$$

J_o and J_1 are the cylindrical Bessel functions.

The potential due to this system of currents is:

$$\begin{aligned}\Phi \simeq \left(\frac{\cosh\xi - \cos\omega}{\cosh\xi_o}\right)^{\frac{1}{2}}\frac{n\mu_o I}{\pi N}\Bigg\{&\left(\frac{e^{\xi_o}}{e^{\xi}}\right)^{m_o N+\frac{1}{2}} J_o(m_o N\delta_o)\sin Nu \\ &+ \left(\frac{e^{\xi_o}}{e^{\xi}}\right)^{m_o N+\frac{3}{2}}\left[\frac{m_o N+2}{m_o N+1}\,\frac{b}{4R_o}\,J_o(m_o N\delta_o)\right. \\ &\left.+ \frac{m_o N}{m_o N+1}\left[\frac{\delta_o}{2}J_o(m_o N\delta_o) + J_1(m_o N\delta_o)\right]\right]\sin(Nu+\omega) \\ &+ \left(\frac{e^{\xi_o}}{e^{\xi}}\right)^{m_o N-\frac{1}{2}}\left[\frac{m_o N}{m_o N-1}\,\frac{b}{4R_o}\,J_o(m_o N\delta_o)\right. \\ &\left.+ \frac{m_o N}{m_o N-1}\left[\frac{\delta_o}{2}J_o(m_o N\delta_o) - J_1(m_o N\delta_o)\right]\right]\sin(Nu-\omega)\Bigg\} .\end{aligned} \tag{5.6}$$

The magnetic field can be obtained by:

$$B_\omega = \frac{\cosh\xi - \cos\omega}{R_o'}\,\frac{\partial\Phi}{\partial\omega}$$

and

$$B_\xi = \frac{\cosh\xi - \cos\omega}{R_o'}\,\frac{\partial\Phi}{\partial\xi} . \tag{5.7}$$

In our approximation B_ω becomes:

$$B_\omega \simeq B_{\omega_0} + B_{\omega_1} \qquad (5.8)$$

with

$$B_{\omega_0} = n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{m_0 N-1} J_0(m_0 N \delta_0)\cos Nu$$

$$B_{\omega_1} = -n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{m_0 N}\left[\frac{b}{4R_0}\left(2-\frac{1}{m_0 N}\right) J_0(m_0 N \delta_0)\right.$$

$$\left.-\left(\frac{\delta_0}{2} J_0(m_0 N \delta_0) + J_1(m_0 N \delta_0)\right)\right]\cos(Nu+\omega)$$

$$- n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{m_0 N-2}\left\{\frac{b}{4R_0}\left[\left(3-\frac{1}{m_0 N}\right)\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^2 - 1\right] J_0(m_0 N \delta_0)\right.$$

$$\left.-\left[\frac{\delta_0}{2} J_0(m_0 N \delta_0) - J_1(m_0 N \delta_0)\right]\right\}\cos(Nu-\omega) \quad .$$

B_{ω_0} contains a first order toroidal effect. B_ξ can be obtained in a similar way.

Near the minor axis the magnetic field can be approximated by:

$$B_\omega \simeq B_{\omega_1} \simeq n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{m_0 N-2}\left[\frac{b}{4R_0} J_0(m_0 N \delta_0)\right.$$

$$\left.+ \frac{\delta_0}{2} J_0(m_0 N \delta_0) - J_1(m_0 N \delta_0)\right]\cos(Nu-\omega) \quad .$$

It does not vanish in the case $m_0 N = 2$.

If the approximation:

$$e^{-\xi} \simeq e^{-\xi_0}\frac{\rho}{b} \quad \text{and} \quad \omega \simeq \pi - (\vartheta + \epsilon \sin\vartheta)$$

where $\epsilon = \frac{\rho^2 + b^2}{2R_0 \rho}$ (Appendix A) is used, near the toroidal surface, the magnetic field can be expressed in terms of local polar coordinates by:

$$B_{\omega_0} \simeq - B^C_\vartheta - B^T_{\vartheta_1} \quad \text{and} \quad B_{\omega_1} \simeq - B^T_{\vartheta_2} \qquad (5.9)$$

where

$$- B^C_\vartheta = (-1)^{m_0 N} n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{\rho}{b}\right)^{m_0 N-1} J_0(m_0 N \delta_0) \times$$

$$\times J_0(m_0 N \epsilon)\cos(m_0 N\vartheta - n_0 N\varphi)$$

$$- B^T_{\vartheta_1} = \pm(-1)^{m_0 N} n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{\rho}{b}\right)^{m_0 N-1} J_0(m_0 N \delta_0) \times$$

$$\times J_1(m_0 N \epsilon)\cos(m_0 N\vartheta - n_0 N\varphi \pm \vartheta)$$

and

$$- B^T_{\vartheta_2} = (-1)^{m_0 N} n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{\rho}{b}\right)^{m_0 N} J_0((m_0 N+1)\epsilon)\cos(m_0 N\vartheta - n_0 N\varphi + \vartheta)$$

$$\times\left\{\left[\frac{b}{4R_0}\left(2-\frac{1}{m_0 N}\right) J_0(n_0 N \delta_0) - \left[\frac{\delta_0}{2} J_0(m_0 N \delta_0)\right.\right.\right.$$

$$\left.\left.\left.+ J_1(m_0 N \delta_0)\right]\right]\right\} + (-1)^{m_0 N} n\, m_0 \frac{\mu_0 I}{\pi b}\left(\frac{\rho}{b}\right)^{m_0 N-2} \times$$

$$\times J_0((m_0 N-1)\epsilon)\cos(m_0 N\vartheta - n_0 N\varphi - \vartheta)$$

$$\times\left\{\left[\frac{b}{4R_0}\left[\left(3-\frac{1}{m_0 N}\right)\frac{\rho^2}{b^2} - 1\right] J_0(m_0 N \delta_0) - \right.\right.$$

$$- \left[\frac{\delta_o}{2} J_o(m_o N \delta_o) - J_1(m_o N \delta_o)\right]\Bigg\}$$

B^C_ϑ corresponds to the cylindrical approximation.

The corresponding scalar potential is:

$$\phi = \left(\frac{R_o}{r}\right)^{\frac{1}{2}} (-1)^{m_o N+1} \frac{n \mu_o I}{\pi N} \Bigg\{\Bigg[J_o(m_o N \delta_o) J_o(m_o N\epsilon) \left(\frac{\rho}{b}\right)^{m_o N}$$

$$\times \sin N(m_o \vartheta - n_o \varphi)\Bigg] - \left(\frac{\rho}{b}\right)^{m_o N+1} \Bigg\{\Bigg[\frac{b}{4R_o} \frac{m_o N+2}{m_o N+1} J_o(m_o N \delta_o)$$

$$+ \frac{m_o N}{m_o N+1} \left[\frac{\delta_o}{2} J_o(m_o N \delta_o) + J_1(m_o N \delta_o)\right]\Bigg] J_o((m_o N+1)\epsilon)$$

$$- \frac{b}{\rho} J_1(m_o N\epsilon)\Bigg\} \sin(N(m_o \vartheta - n_o \varphi) + \vartheta)$$

$$- \left(\frac{\rho}{b}\right)^{m_o N-1} \Bigg\{\Bigg[\frac{b}{4R_o} \frac{m_o N}{m_o N-1} J_o(m_o N \delta_o) + \frac{m_o N}{m_o N-1} \left[\frac{\delta_o}{2} J_o(m_o N \delta_o)\right.$$

$$\left. - J_1(m_o N \delta_o)\right]\Bigg] J_o((m_o N-1)\epsilon)$$

$$+ \frac{\rho}{b} J_1(m_o N \epsilon)\Bigg\} \sin(N(m_o \vartheta - n_o \varphi) - \vartheta)\Bigg\} \quad . \qquad (5.10)$$

The first harmonic field B_{ω_o} contains a first order toroidal effect. The magnitudes of the side bands are found to be strongly dependent on the winding law parameter δ_o.

The winding law (5.2) in terms of polar coordinates becomes:

$$\vartheta + \left(\frac{b}{R_o} - \delta_o\right) \sin \vartheta - \frac{n_o}{m_o} \varphi = \text{const} \quad . \qquad (5.11)$$

The value of the winding law parameter $\delta_o = \frac{b}{R_o}$ would describe a linear relation between ϑ and φ as in Sometani's model.

La Haye et al. (1981) studied the effect of the winding law on the magnetic surface structure of a configuration having a large toroidal plasma current and a set of helical currents. A winding law of the form $\vartheta + \epsilon \sin\vartheta - \frac{n_o}{m_o} \varphi$ = constant was considered for different values of the parameter ϵ. From the results of numerical calculations of the magnetic field using the Biot-Savart law they concluded that the best magnetic surface structure occurs around $\epsilon \simeq 0.18$. The surfaces of constant $|\vec{B}|$ would be nearly circular and nearly centred about the minor axis of the torus.

The values of the minor and major radii of the system considered are $b = 0{,}235$ m and $R_o = 1{,}24$ m. This would correspond to taking $\delta_o \simeq 0$ in the equation (5.11), that is to say a linear relation between the toroidal angular coordinates.

The use of toroidal coordinate system in order to get a series solution of Laplace's equation has an advantage over the other approach, by bending a straight cilinder into a torus: the boundary conditions for the approximate solutions of approximate Laplace's equation outside the torus are not clear. The problem is complicated by the fact that the contribution to the side band potential by a non-uniform winding law may be of the same order of magnitude as the other toroidal effects.

Comparing results obtained using different methods may help to understand the nature of distortion caused by bending a straight system.

APPENDIX A

Relations between toroidal and local polar system:

$$r = R_o - \rho\cos\vartheta \quad ; \quad z = \rho\sin\vartheta \tag{A1}$$

$$\cot\omega = \left(1-\frac{b^2}{R_o^2}\right)^{-\frac{1}{2}}\left[\cot(\pi-\vartheta) + \frac{\rho^2+b^2}{2R_o\,\rho\sin\vartheta}\right] \tag{A2}$$

$$\coth\xi = \frac{1-\frac{\rho}{R_o}\cos\vartheta + \frac{\rho^2-b^2}{2R_o^2}}{\left(1-\frac{b^2}{R_o^2}\right)^{\frac{1}{2}}\left(1-\frac{\rho}{R_o}\cos\vartheta\right)} \quad . \tag{A3}$$

(A3) may be substituted more conveniently by:

$$e^{-2\xi} = e^{-2\xi_o}\,\frac{1-\frac{\rho}{R_o}\cos\vartheta - \frac{b}{2R_o}e^{\xi_o}\left(1-\frac{\rho^2}{b^2}\right)}{1-\frac{\rho}{R_o}\cos\vartheta - \frac{b}{2R_o}e^{-\xi_o}\left(1-\frac{\rho^2}{b^2}\right)} \quad . \tag{A4}$$

If $\frac{b}{R_o} \ll 1$ (large aspect ratio) and $\frac{\rho}{b} \gg \frac{b}{R_o}$ (not near the torus axis) we find

$$e^{-\xi} \simeq e^{-\xi_o}\frac{\rho}{b} \quad ; \quad \omega \simeq \pi - (\vartheta + \epsilon\sin\vartheta) \quad ; \quad \epsilon = \frac{\rho^2+b^2}{2R_o\,\rho} \tag{A5}$$

and for the unit vectors:

$$\vec{e}_\xi \simeq -\vec{e}_\rho$$
$$\vec{e}_\omega \simeq -\vec{e}_\vartheta \quad . \tag{A6}$$

APPENDIX B

Legendre functions:

$$P^{n_oN}_{m_oN+\nu-\frac{1}{2}}(\cosh\xi) = \frac{\Gamma((m_o+n_o)N+\nu+\frac{1}{2})}{\Gamma((m_o-n_o)N+\nu+\frac{1}{2})}\;\frac{2^{-2n_oN}}{\Gamma(1+n_oN)}\;e^{-(m_oN+\nu+\frac{1}{2})\xi}$$

$$\times\ (1-e^{-2\xi})^{n_oN} \times F(n_oN+\tfrac{1}{2}\,,\,(m_o+n_o)N+\nu+\tfrac{1}{2}\,;\,1+2n_oN\,;\,1-e^{-2\xi}) \tag{B1}$$

$$Q^{n_oN}_{m_oN+\nu-\frac{1}{2}}(\cosh\xi) = \sqrt{\pi}\,(-1)^{n_oN}\;\frac{\Gamma((m_o+n_o)N+\nu+\frac{1}{2})}{\Gamma(m_oN+\nu+1)}\;e^{-(m_oN+\nu+\frac{1}{2})\xi}$$

$$\times\ (1-e^{-2\xi})^{n_oN} \times F(n_oN+\tfrac{1}{2}\,,\,(m_o+n_o)N+\nu+\tfrac{1}{2}\,;\,m_oN+\nu+1\,;\,e^{-2\xi}) \quad . \tag{B2}$$

F are hypergeometric functions.

Expressions (4.11) may be written:

$$\Phi_s = \left(\frac{b}{R_o}\right)^s\left(\frac{\cosh\xi-\cos\omega}{\cosh\xi_o}\right)^{\frac{1}{2}}\frac{\mu_oI}{2\pi N}\sum_\nu D_{s,\nu}\,f_\nu(\xi)\sin(Nu+\nu\omega) \tag{B3}$$

$$f_\nu(\xi) = \left(\frac{e^{\xi_o}}{e^{\xi}}\right)^{m_oN+\nu+\frac{1}{2}} q_\nu(\xi)$$

$$q_\nu(\xi) = \frac{1}{E_o}\,e^{(m_oN+\nu+\frac{1}{2})\xi}\;Q(\cosh\xi)$$

$$= (1-e^{-2\xi})^{n_oN}\sum_{K=0}^{\infty}\frac{E_K}{E_o}\,e^{-2\xi K}$$

$$E_o = (-1)^{n_oN}\sqrt{\pi}\;\frac{\Gamma((m_o+n_o)N+\nu+\frac{1}{2})}{\Gamma(m_oN+\nu+1)}$$

$$\frac{E_K}{E_o} = \frac{1}{K!}\;\frac{\Gamma(n_oN+K+\frac{1}{2})}{\Gamma(n_oN+\frac{1}{2})}\;\frac{\Gamma((m_o+n_o)N+\nu+K+\frac{1}{2})}{\Gamma((m_o+n_o)N+\nu+\frac{1}{2})}\;\frac{\Gamma(m_oN+\nu+1)}{\Gamma(m_oN+\nu+K+1)}$$

$$D_{s,\nu} = -C_{s,\nu} \frac{2E_o}{\cosh\xi_o} \frac{1}{W} \left(\frac{2s+1}{2} P + \cosh\xi_o P'\right) e^{-\xi_o(m_o N+\nu+\frac{1}{2})}$$

$$-W = P'Q - PQ' = \frac{(-1)^{n_o N}}{\sinh^2\xi_o} \frac{\Gamma((m_o+n_o)N+\nu+\frac{1}{2})}{\Gamma((m_o-n_o)N+\nu+\frac{1}{2})} .$$

If terms of the order of $\left(\frac{b}{R_o}\right)^2$ are neglected, we have:

$$D_{s,\nu} \simeq C_{s,\nu} \frac{m_o N+\nu+s}{m_o N+\nu} \quad ; \quad m_o N+\nu \neq 0$$

and

$$D_{s,\nu} \simeq 2C_{s,\nu}(1+s\xi_o) \quad ; \quad m_o N+\nu = 0 .$$

Near the surface ($\xi \simeq \xi_o$) each function Φ_s is of the order of $\left(\frac{b}{R_o}\right)^s$.

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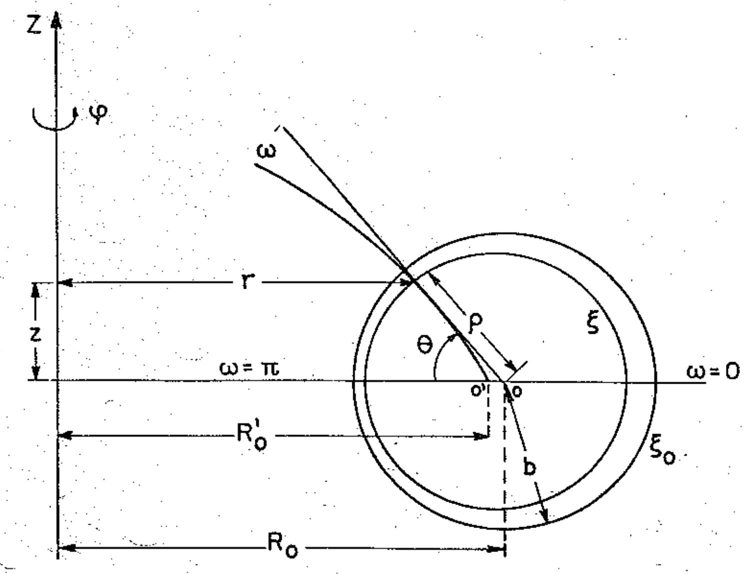

Fig. 1 - Coordinate system.