UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
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NOETHER'S THEOREM: A TETRAD FORMULATION

H. Fleming

Instituto de Física, Universidade de São Paulo

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Instituto de Física, Universidade de São Paulo
C.P. 20516, 01498 São Paulo, SP, Brasil

ABSTRACT

The Noether theorem for classical field theories including gravity and Dirac spinors is presented in a simple, unified way, following an idea of Jackiw. Classical index notation is used throughout.

1. INTRODUCTION

The theorem of Emmy Noether [1] connecting symmetries and conservation laws is a result of great importance and beauty. In 1972 Roman Jackiw published a gem of a proof [2] of Noether's theorem which made the whole question transparent as it never had been before. The neatness of his arguments allowed for an extension of his results to curved spacetimes which turned out to be relatively straightforward [3], as compared to older approaches. Also, Jackiw's definition of a symmetry led us to investigate the conserved currents associated with the isometries of the spacetime, whereas the standard treatments [4] take a symmetry to be a general coordinate transformation. While we do get conserved quantities, the standard treatments end up just with Bianchi identities.

In Ref. (3) we considered tensor fields, excluding spinors. In this paper we offer a more general formulation, in terms of tetrads, which allow for the treatment of both tensors and Dirac spinors. No new results are presented here. This is a more systematic and, I believe, clearer treatment of a classical topic in theoretical physics.

2. THE FLAT CASE

The classical action which describes our system (for the moment restricted to flat spacetime) is written

$$S = \int d^4x\, L(\phi, \partial_\mu \phi) \tag{1}$$

$L(\phi, \partial_\mu \phi)$ being the Lagrangian density, a function of some fields ϕ and of their derivatives $\partial_\mu \phi$. To start with, ϕ will be a scalar, in order to reveal most clearly the structure of the theorem.

An infinitesimal transformation of the fields

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta\phi(x) \tag{2}$$

induces an infinitesimal variation $\delta L(x)$ in the Lagrangian. The transformation is a symmetry when it can be shown, <u>without using the equations of motion</u>, that

$$\delta L(x) = \partial_\mu \Lambda^\mu \tag{3}$$

where Λ^μ is some 4-vector. That is to say, δL has the form given in Eq. (3) for all field configurations, not only for those which are solutions of the equations of motion.

Example: coordinate translations.

Consider the infinitesimal coordinate transformations

$$x'^\mu = x^\mu + \varepsilon^\mu$$

ε^μ being an infinitesimal constant 4-vector. It induces on $\phi(x)$ a transformation $\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta\phi(x)$, to be

computed now. Taking $\phi(x)$ to be scalar under translations, one has

$$\phi'(x') = \phi(x) \quad . \tag{4}$$

Power expansion on ε^μ gives

$$\phi'(x') = \phi'(x) + (x'-x)^\lambda \partial_\lambda \phi(x)$$

or

$$\phi'(x') = \phi'(x) + \varepsilon^\lambda \partial_\lambda \phi(x) \tag{5}$$

which, combined with (4), gives

$$\delta\phi(x) = \phi'(x) - \phi(x) = -\varepsilon^\lambda \partial_\lambda \phi \quad . \tag{6}$$

Suppose

$$L = \frac{1}{2}\, \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2}\, \phi^2 \tag{7}$$

so that

$$\delta L = \partial^\mu \phi \partial_\mu \delta\phi - m^2 \phi \delta\phi \quad . \tag{8}$$

Using (6),

$$\delta L = -\varepsilon^\lambda \partial_\lambda \left\{ \frac{1}{2}\, \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2}\, \phi^2 \right\}$$

$$\delta L = -\varepsilon^\lambda \partial_\lambda L \quad . \tag{9}$$

Finally, as ε^{λ} is a constant,

$$\delta L = \partial_{\lambda}(-\varepsilon^{\lambda} L) \tag{10}$$

which shows that translations are symmetries of the system described by the Lagrangian (7). This kind of variation, $\delta\phi(x)$, is called the form variation of the field.

Noether's theorem asserts that to each continuous symmetry there corresponds a current which satisfies a continuity equation, or, equivalently, a quantity which is conserved. Furthermore, it gives an explicit expression for that current. Suppose $\delta\phi$ is the symmetry transformation. Then there is Λ^{μ} such that

$$\delta L = \partial_{\mu}\Lambda^{\mu} \quad . \tag{3}$$

An independent computation of δL, now using the equations of motion will now be done:

$$\delta L = \frac{\partial L}{\partial\phi}\,\delta\phi + \frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\partial_{\mu}\delta\phi \tag{11}$$

and the equations of motion are

$$\frac{\partial L}{\partial\phi} = \partial_{\mu}\,\frac{\partial L}{\partial(\partial_{\mu}\phi)} \quad . \tag{12}$$

Using (12) into (11) gives

$$\delta L = \partial_{\mu}\left[\frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\delta\phi\right] \quad . \tag{13}$$

Subtracting (13) from (3) one gets

$$\partial_{\mu}\left\{\Lambda^{\mu} - \frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\delta\phi\right\} = 0 \quad . \tag{14}$$

This is Noether's theorem. The 4-vector

$$j^{\mu} = \Lambda^{\mu} - \frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\delta\phi \tag{17}$$

is the Noether current associated to the symmetry $\delta\phi$.

It is a simple matter to see that the conservation law associated, in this sense, to translations, is

$$\partial_{\mu}T^{\mu}{}_{\nu} = 0 \tag{16}$$

where

$$T^{\mu}{}_{\nu} = \frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\partial_{\nu}\phi - \delta^{\mu}{}_{\nu}L \tag{17}$$

is the canonical energy-momentum tensor.

Notice that we need not consider, contrary to the standard treatments, two different Noether theorems, one for finite Lie groups, and another one for Lie groups with infinitely many parameters. Our treatment encompasses both cases.

3. THE CURVED CASE

An infinitesimal coordinate transformation in curved spacetime,

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{18}$$

induces on a scalar field $\phi(x)$ the same form variation we have met before,

$$\delta\phi(x) = -\xi^{\lambda}\partial_{\lambda}\phi \ . \tag{19}$$

Let us compute the form variation induced on the metric tensor $g^{\mu\nu}(x)$. From

$$g'^{\mu\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x) \tag{20}$$

which characterizes it as a second order tensor field, it follows, for the infinitesimal transformations (18), that

$$g'^{\mu\nu}(x') = g^{\mu\nu}(x) + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{21}$$

and, by Taylor expansion around x^{μ},

$$g'^{\mu\nu}(x') = g'^{\mu\nu}(x) + \xi^{\lambda}\partial_{\lambda} g^{\mu\nu}(x) \ . \tag{22}$$

Using both, one arrives at

$$\delta g^{\mu\nu}(x) = -\xi^{\lambda}\partial_{\lambda} g^{\mu\nu} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{23}$$

or, equivalently,

$$\delta g^{\mu\nu}(x) = \xi^{\mu;\nu} + \xi^{\nu;\mu} \ . \tag{24}$$

Vector fields $\xi^{\mu}(x)$ which satisfy $\delta g^{\mu\nu} = 0$, that is, which generate transformations (18) which do not change the form of the metric field, are called Killing fields. Therefore, a Killing field is characterized by

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0 \ . \tag{25}$$

A transformation of type (18) with $\xi^{\mu}(x)$ Killing, is called an isometry of the spacetime. Observers connected by such a transformation observe identical metric relationships in spacetime, and hence the same gravitational field.

We have still to determine the form variation of a Dirac spinor. This is a somewhat harder task. To do that, we start now a detour, in the course of which we will obtain the Dirac equation in the presence of a gravitational field.

4. THE DIRAC EQUATION

In order to study Dirac spinors we introduce, as usual, tetrads, that is, local reference frames attached to each point of spacetime. They are formed by vectors $e^a_\mu(x)$, the latin index specifying the particular vector, while the greek index specifies a component of that vector. For details we refer the reader to Refs. (5) and (6), which we follow rather closely.

Consider a tetrad rotation

$$e'^{\mu a}(x) = \ell^a{}_b(x)\, e^{\mu b}(x) \tag{26}$$

$$e'^\mu{}_a(x) = \ell_a{}^b(x)\, e^\mu{}_b(x) \tag{27}$$

with the defining property

$$e'^{\mu a}\, \eta_{ab}\, e'^{\nu b} = e^{\mu a}\, \eta_{ab}\, e^{\nu b} , \tag{28}$$

η_{ab} being the metric tensor of Minkowski spacetime. From this it follows that

$$\ell^a{}_c\, \ell_a{}^d = \delta_c{}^d \tag{29}$$

and

$$\ell_a{}^d\, \ell^m{}_d = \delta_a{}^m . \tag{30}$$

The matrices $\ell^a{}_b$, therefore, are finite-dimensional representations of the Lorentz group. The Dirac spinors are therefore required to transform, under tetrad rotations, as

$$\psi'(x) = L\psi(x) \tag{31}$$

with L defined, as in the flat case, by

$$L^{-1}\gamma^a L = \ell^a{}_b\, \gamma^b , \tag{32}$$

the γ^a being the usual (constant) Dirac matrices. Now, the Dirac equation will certainly contain the term $\partial_\mu\psi$, which, however, does not transform as a spinor. In fact, from Eq.(31) one has that

$$\partial_\mu \psi' = L(x)\, \partial_\mu\psi + \partial_\mu L\, \psi(x)$$

whereas we would like to have something like

$$D'_\mu \psi'(x) = L(x)\, D_\mu \psi(x) . \tag{33}$$

So, the hard part of the task of extending the Dirac equation to curved spacetimes is that of discovering the "covariant derivative" D_μ. This is masterly explained in Ref. (7), Section 3-17.. It turns out that the Dirac Lagrangian density in flat spacetime,

$$\mathcal{L}(x) = -\frac{1}{2}\, \psi^+ \gamma^0 \left[\gamma^\mu \left(\frac{1}{i}\right) \partial_\mu + m\right] \psi \tag{34}$$

must be modified to

$$L(x) = -\frac{1}{2}\,\psi^{+}(x)\,\gamma^{0}\left[\gamma^{a} e_{a}{}^{\mu}(x)\left(\frac{1}{i}\right) D_{\mu} + m\right]\psi(x) \quad , \tag{35}$$

with

$$D_{\mu} = \partial_{\mu} - \frac{i}{4}\,\omega_{a\mu b}\,\sigma^{ab} \tag{36}$$

$$\sigma^{ab} = \frac{i}{2}\left[\gamma^{a}, \gamma^{b}\right] \tag{37}$$

$$\omega_{a\mu c} = \frac{1}{2}\, e_{\mu}{}^{b}\Big[e_{\lambda a}(e_{c}{}^{\nu}\partial_{\nu} e_{b}{}^{\lambda} - e_{b}{}^{\nu}\partial_{\nu} e_{c}{}^{\lambda}) + e_{\nu c}(e_{b}{}^{\lambda}\partial_{\lambda} e_{a}{}^{\nu} - e_{a}{}^{\lambda}\partial_{\lambda} e_{b}{}^{\nu}) - e_{\nu b}(e_{a}{}^{\lambda}\partial_{\lambda} e_{c}{}^{\nu} - e_{c}{}^{\lambda}\partial_{\lambda} e_{a}{}^{\nu})\Big] . \tag{38}$$

The action is, then,

$$S = \int d^{4}x\; e(x)\, L(x) \quad , \tag{39}$$

where

$$e(x) = \det e_{a}{}^{\mu}(x) = \sqrt{-g} \quad . \tag{40}$$

By construction this action is invariant under both coordinate transformations and tetrad rotations, and

$$L^{-1}(\partial_{\mu} - \frac{i}{4}\,\omega'_{a\mu b}\,\sigma^{ab})L = \partial_{\mu} - \frac{i}{4}\,\omega_{a\mu b}\,\sigma^{ab} \quad . \tag{41}$$

The Dirac equation is obtained from (39), and reads

$$(i\gamma^{a} e_{a}{}^{\mu}(x) D_{\mu} - m)\,\psi(x) = 0 \quad . \tag{42}$$

5. FORM VARIATION OF DIRAC SPINORS AND TETRADS

Under coordinate transformations, spinors transform the same way as scalars, namely,

$$\psi'(x') = \psi(x) \quad . \tag{43}$$

If, however, one assigns to them the form variation of scalars,

$$\delta\psi(x) = -\xi^{\lambda}\,\partial_{\lambda}\psi \quad , \tag{44}$$

the problem arises that $\delta\psi(x)$ is not a spinor (it has incorrect properties of transformations under tetrad rotations). The solution is, however, simple:

$$\delta\psi = -\xi^{\lambda} D_{\lambda}\,\psi \quad . \tag{45}$$

This is a spinor, and has, for instance, the property that

$$\delta(\bar{\psi}\psi) = -\xi^{\lambda}\,\partial_{\lambda}(\bar{\psi}\psi) \quad ,$$

as it should, $\bar{\psi}\psi$ being a scalar.

The form variation of tetrads follow from that of the metric tensor by using the relation

$$g^{\mu\nu} = \eta_{ab}\, e^{a\mu}\, e^{b\nu} \quad . \tag{46}$$

This gives

$$\delta g^{\mu\nu} = \eta_{ab}(e^{a\mu}\,\delta e^{b\nu} + e^{b\nu}\,\delta e^{a\mu}) \quad . \tag{47}$$

Now, under the infinitesimal coordinate transformation

$$x'^{\lambda} = x^{\lambda} + \xi^{\lambda}(x) \quad ,$$

the form variation of $g^{\mu\nu}$ is

$$\delta g^{\mu\nu} = -\xi^{\lambda}\,\partial_{\lambda} g^{\mu\nu} + g^{\mu\nu}\,\partial_{\lambda}\xi^{\nu} + g^{\lambda\nu}\,\partial_{\lambda}\xi^{\mu} \quad , \tag{48}$$

which, using (46), reads

$$\eta_{ab}\left\{e^{a\mu}\,\delta e^{b\nu} + e^{b\nu}\,\delta e^{a\mu}\right\} =$$

$$= \eta_{ab}\left\{-\xi^{\lambda}(\partial_{\lambda} e^{a\mu})e^{b\nu} - \xi^{\lambda} e^{a\mu}\partial_{\lambda} e^{b\nu} + e^{a\mu} e^{b\lambda}\,\partial_{\lambda}\xi^{\nu} + \right.$$

$$\left. + e^{a\nu} e^{b\lambda}\,\partial_{\lambda}\,\xi^{\mu}\right\} \tag{49}$$

wherefrom it follows that

$$\delta e^{b\nu} = -\xi^{\lambda}\partial_{\lambda} e^{b\nu} + e^{b\lambda}\,\partial_{\lambda}\,\xi^{\nu} \tag{50}$$

In Section 3 we characterized an isometry by the property

$$\delta g^{\mu\nu} = 0 \quad ,$$

or

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0 \quad ,$$

if the transformation is

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \quad .$$

It is easy to see that an equivalent condition is

$$\frac{\partial x'^{\lambda}}{\partial x^{\nu}} = e_a{}^{\lambda}(x')\,e^a{}_{\nu}(x) \quad . \tag{51}$$

(See Ref. (5), §116.8 for a proof. For infinitesimal transformation this gives Eq. (25)). In a tetrad formulation the most convenient way to characterize an isometry is

$$\delta e^{a\mu}(x) = 0 \quad , \tag{52}$$

which leads, through use of (50), to

$$-\xi^{\lambda}\partial_{\lambda} e^{a\mu}(x) + e^{a\lambda}(x)\,\partial_{\lambda}\xi^{\mu} = 0 \quad , \tag{53}$$

which is (51) for infinitesimal transformations.

6. SYMMETRIES IN CURVED SPACETIME

We consider here two Lagrangian densities which are general enough to cover most cases, the Klein-Gordon and the Dirac Lagrangians,

$$L_{KG} = \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{m^2}{2}\,\phi^2 \tag{54}$$

and

$$L_D = -\frac{1}{2}\,\psi^+\gamma^0\left[\gamma^a e_a^{\ \mu}(x)\left(\frac{1}{i}\right)D_\mu + m\right]\psi \tag{55}$$

to which non-derivative interaction terms may be added at will.

Starting with the scalar case, the invariant action

$$S = \int d^4x\,\sqrt{-g}\,L(g^{\mu\nu},\phi,\partial_\mu\phi)$$

responds to variations of the fields as follows

$$\delta S = \int d^4x\,\sqrt{-g}\left\{\frac{\partial L}{\partial\phi}\,\delta\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\,\partial_\mu\delta\phi\right\} + \\ + \frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu} \quad . \tag{56}$$

For the last piece consult, for instance, Ref. (5). $T_{\mu\nu}$ is the symmetric energy-momentum tensor. For solutions of the equations of motion, $T^{\mu\nu}{}_{;\nu} = 0$, as a consequence of invariance

under general coordinate transformations. Using Eqs. (19) and (24) one has

$$\delta S = \int d^4x\,\sqrt{-g}\left\{-\xi^\lambda\left(\frac{\partial L}{\partial\phi}\,\partial_\lambda\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\,\partial_\lambda\partial_\mu\phi\right) - \right. \\ \left. -\frac{1}{2}\,(\partial^\mu\xi^\lambda + \partial^\lambda\xi^\mu)\,\frac{\partial L}{\partial(\partial^\mu\phi)}\,\partial_\lambda\phi\right\} + \frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu} \quad . \tag{57}$$

If $\xi^\lambda(x)$ is a Killing field, then $\delta g^{\mu\nu} = 0$, or

$$\partial^\mu\xi^\nu + \partial^\nu\xi^\mu = \xi^\lambda\partial_\lambda g^{\mu\nu} \quad ,$$

so that (57) becomes

$$\delta S = -\int d^4x\,\sqrt{-g}\,\xi^\lambda\left[\frac{\partial L}{\partial\phi}\,\partial_\lambda\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\,\partial_\lambda\partial_\mu\phi + \frac{1}{2}\,\partial_\lambda g^{\mu\nu}\,\frac{\partial L}{\partial(\partial^\mu\phi)}\,\partial_\nu\phi\right]$$

which, for Lagrangians whose derivative terms coincide with Klein-Gordon's, is

$$\delta S = -\int d^4x\,\sqrt{-g}\,\xi^\lambda\partial_\lambda L \quad . \tag{58}$$

The Killing equations imply

$$\xi^\mu{}_{;\mu} = 0 \quad ,$$

that is,

$$\partial_\lambda (\sqrt{-g}\ \xi^\lambda) = 0 \quad , \tag{59}$$

so that (58) may be rewritten as

$$\delta S = \int d^4x\ \partial_\lambda (-\sqrt{-g}\ \xi^\lambda L) \quad . \tag{60}$$

In Section 2 we defined a symmetry in the case of flat spacetime. The extension to a curved spacetime is simple. An infinitesimal transformation of the fields

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = \phi(x) + \delta\phi(x) \\ \psi(x) &\rightarrow \psi'(x) = \psi(x) + \delta\psi(x) \\ g^{\mu\nu}(x) &\rightarrow g'^{\mu\nu}(x) = g^{\mu\nu}(x) + \delta g^{\mu\nu}(x) \end{aligned} \tag{61}$$

is a symmetry if the induced variation δS can be written, <u>without the use of the equations of motion</u>, as

$$\delta S = \int d^4x\ \partial_\mu \Lambda^\mu \quad ,$$

where Λ^μ is some vector density. We immediately see from Eq. (60) that all Killing fields of the metric $g_{\mu\nu}$ generate symmetries of the action S, provided that L is a scalar based on the Klein-Gordon Lagrangian.

We now extend this result to include Lagrangians based on the Dirac Lagrangian.

The action is now written

$$S = \int d^4x\ e(x)\ L(\psi(x)\ ,\ \psi^+(x)\ ,\ D_\mu\psi(x)) \tag{62}$$

and its response to variations of all fields is

$$\delta S = \int d^4x\ e(x) \left\{ \frac{\partial L}{\partial\psi^+}\ \delta\psi^+ + \frac{\partial L}{\partial\psi}\ \delta\psi + \frac{\partial L}{\partial(D_\mu\psi)}\ \delta D_\mu\psi \right\} \tag{63}$$

where we ignored variations of $e_a{}^\mu(x)$, because they will vanish for isometries, the transformations we are investigating. Consider the scalar density

$$\left\{ \frac{\partial L}{\partial\psi^+}\ \delta\psi^+(x) + \frac{\partial L}{\partial\psi}\ \delta\psi(x) + \frac{\partial L}{\partial(D_\mu\psi)}\ \delta D_\mu\ \psi(x) \right\} e(x) \tag{64}$$

where $\delta\psi^+(x)$, $\delta\psi(x)$ and $\delta D_\mu\psi(x)$ are form variations, given by

$$\begin{aligned} \delta\psi &= -\xi^\lambda D_\lambda \psi \\ \delta\psi^+ &= -\xi^\lambda (D_\lambda\psi)^+ \\ \delta(D_\mu\psi) &= D_\mu\delta\psi \quad \text{(for isometries)} . \end{aligned}$$

To simplify our computation, recall that spinors transform in a linear homogeneous way under both coordinate changes and tetrad rotations. Hence, if two spinors coincide for a particular choice of coordinates and tetrads, they coincide in general. Now, it is possible to choose coordinates and tetrads in such a way that, locally, $\omega_{a\mu b} = 0$. With this choice, (64) reads (except for an irrelevant divergence)

$$\left\{- \xi^\lambda \left[\frac{\partial L}{\partial \psi^+} \partial_\lambda \psi^+ + \frac{\partial L}{\partial \psi} \partial_\lambda \psi + \frac{\partial L}{\partial(\partial_\mu \psi)} \partial_\lambda \partial_\mu \psi\right] - \frac{\partial L}{\partial(\partial_\mu \psi)} \partial_\lambda \psi \partial_\mu \xi^\lambda\right\} e(x) .$$

But

$$\frac{\partial L}{\partial(\partial_\mu \psi)} \partial_\lambda \psi = \frac{i}{2} \psi^+ \gamma^0 e_a{}^\mu \gamma^a \partial_\lambda \psi$$

so that

$$\frac{\partial L}{\partial(\partial_\mu \psi)} \partial_\lambda \psi \partial_\mu \xi^\lambda = \frac{i}{2} \psi^+ \gamma^0 \gamma^a \partial_\lambda \psi e_a{}^\mu \partial_\mu \xi^\lambda \qquad (66)$$

and, because ξ^λ is Killing,

$$e_a{}^\mu \partial_\mu \xi^\lambda = \xi^\mu \partial_\mu e_a{}^\lambda \quad ,$$

which, used in (66), gives rise to

$$\frac{\partial L}{\partial(\partial_\mu \psi)} \partial_\lambda \psi \partial_\mu \xi^\lambda = \frac{i}{2} \psi^+ \gamma^0 \gamma^a \partial_\mu \psi \xi^\lambda \partial_\lambda e_a{}^\mu$$

$$= \xi^\lambda \frac{\partial L}{\partial e_a{}^\mu} \partial_\lambda e_a{}^\mu \quad . \qquad (67)$$

Therefore, (65) is written

$$- \xi^\lambda \left\{\frac{\partial L}{\partial \psi^+} \partial_\lambda \psi^+ + \frac{\partial L}{\partial \psi} \partial_\lambda \psi + \frac{\partial L}{\partial(\partial_\mu \psi)} \partial_\lambda \partial_\mu \psi + \frac{\partial L}{\partial e_a{}^\mu} \partial_\lambda e_a{}^\mu\right\} e(x) \qquad (68)$$

or, finally,

$$- e(x) \xi^\lambda \partial_\lambda L \qquad (69)$$

But, from the Killing equations,

$$\partial_\lambda (e \xi^\lambda) = 0 \quad ,$$

which is Eq. (59). Then,

$$\delta S = \int d^4x \, \partial_\lambda (- e \xi^\lambda L) \qquad (70)$$

showing that also for Lagrangians of the Dirac type, all infinitesimal isometries are symmetries of the action.

We could also have invoked the following general argument: all Lagrangian densities are scalars, so that their form variations are those of a scalar,

$$\delta L = - \xi^\lambda \partial_\lambda L \quad .$$

Therefore, if ξ^λ is Killing,

$$\delta S = \int d^4x \, e(x) (-\xi^\lambda \partial_\lambda L)$$

and

$$\partial_\lambda (e(x)\ \xi^\lambda) = 0 \quad ,$$

so that

$$\delta S = \int d^4x\ \partial_\lambda (-e\ \xi^\lambda L) \quad ,$$

which is the same as Eq. (70).

7. NOETHER'S THEOREM

We now add to the action the gravitational contribution. Let it read

$$S = \int d^4x \sqrt{-g} \left\{ - \frac{1}{16\pi k} R + L \right\} \tag{71}$$

where <u>k</u> is Newton's constant. The Lagrangian density L may now be a scalar composed of spinors. This case had been excluded from Ref. (3), so that we concentrate, here, on its study.

The symmetric energy-momentum tensor corresponding to Dirac's action (Eq. (35)) reads

$$T_{\mu\nu} = \frac{1}{2} \bar{\psi} \gamma^a \frac{1}{2i} (e_{\mu a} \partial_\nu + e_{\nu a} \partial_\mu) \psi -$$
$$- \frac{1}{4} \frac{1}{2} \bar{\psi} \gamma^a \sigma^{bc} \psi \frac{1}{2} (e_{\mu a} \omega_{b\nu c} + e_{\nu a} \omega_{b\mu c}) + g_{\mu\nu} L \tag{72}$$

and is obtained by computing the functional derivative of the "matter" action with respect to $e_{\mu a}$.

To compute the Noether currents we must now obtain two expressions for the form variation of S (Eq. (71)), and then subtract them.

By varying all fields we have

$$\delta S = \int d^4x \sqrt{-g} \left\{ - \frac{1}{16\pi k} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{1}{2} T_{\mu\nu} \right\} \delta g^{\mu\nu} +$$
$$+ \int d^4x\ e(x)\ \xi^\lambda \partial_\lambda L \tag{73}$$

and, for ξ^λ Killing,

$$\delta S = \int d^4x\ \partial_\lambda (-e\ \xi^\lambda L) \quad , \tag{74}$$

the same as Eq. (74). No use has been made of the equations of motion.

A second expression for δS is obtained in the following way:

$$\delta S = \int d^4x \sqrt{-g} \left\{ - \frac{1}{16\pi k} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{1}{2} T_{\mu\nu} \right\} \delta g^{\mu\nu} +$$
$$+ \int d^4x\ e(x) \left\{ \frac{\partial L}{\partial \psi^+} \delta\psi^+ + \frac{\partial L}{\partial \psi} \delta\psi + \frac{\partial L}{\partial (\partial_\mu \psi)} \partial_\mu \delta\psi \right\} \tag{75}$$

where $\delta g^{\mu\nu}$, $\delta\psi^+$, $\delta\psi$ are the form variations corresponding to a Killing ξ^λ. Using now the equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi k T_{\mu\nu} \tag{76}$$

$$\frac{\partial L}{\partial \psi^+} = 0 \tag{77}$$

and

$$\frac{\partial L}{\partial \psi} = \frac{1}{e} \partial_\mu \left\{ e \frac{\partial L}{\partial(\partial_\mu \psi)} \right\} = 0 \tag{78}$$

we get, from (75),

$$\delta S = \int d^4x \, \partial_\mu \left\{ e \frac{\partial L}{\partial(\partial_\mu \psi)} \partial\psi \right\} \tag{79}$$

or, more explicitly,

$$\delta S = \int d^4x \, \partial_\mu \left\{ e \frac{\partial}{\partial(\partial_\mu \psi)} (-\xi^\lambda D_\lambda \psi) \right\} , \tag{80}$$

where use was made of Eq. (45). Subtracting (80) from (74), one arrives at

$$\int d^4x \, \partial_\mu \left\{ e \xi^\lambda \left[\frac{\partial L}{\partial(\partial_\mu \psi)} D_\lambda \psi - \delta^\mu{}_\lambda L \right] \right\} = 0 , \tag{81}$$

that is,

$$\partial_\mu \left\{ e \xi^\lambda \left[\frac{\partial L}{\partial(\partial_\mu \psi)} D_\lambda \psi - \delta^\mu{}_\lambda L \right] \right\} = 0 , \tag{82}$$

which are Noether's conservation laws. The term inside brackets is a generalization, to curved spacetimes, of the canonical energy-momentum tensor of Dirac fields. It is not symmetric. Let us denote it by $\theta^\mu{}_\lambda$. Then

$$\partial_\mu \left\{ e \, \xi_\lambda \, \theta^{\mu\lambda} \right\} = 0 \tag{83}$$

or

$$\partial_\mu \left\{ e \, \xi_\lambda \left([\theta^{\mu\lambda} + \theta^{\lambda\mu}] + [\theta^{\mu\lambda} - \theta^{\lambda\mu}] \right) \right\} = 0 \tag{84}$$

which is

$$\partial_\mu \left\{ e \, \xi_\lambda T^{\mu\lambda} \right\} + \partial_\mu \left\{ e \, \xi_\lambda A^{\mu\lambda} \right\} = 0 \tag{85}$$

where $T^{\mu\lambda}$ is the symmetric energy-momentum tensor of Eq. (72), and

$$A^{\mu\lambda} \equiv \frac{1}{2} (\theta^{\mu\lambda} - \theta^{\lambda\mu}) . \tag{86}$$

Now, the first term of Eq. (85) equals

$$e(\xi_\lambda T^{\mu\lambda})_{;\mu} = e(\xi_{\lambda;\mu} T^{\mu\lambda}) = 0 , \tag{87}$$

as $\xi_{\lambda;\mu}$ is antisymmetric (ξ_λ is Killing!), and $T^{\mu\lambda}$ is symmetric (recall that $T^{\mu\lambda}{}_{;\mu} = 0$). Then Eq. (85) gives

$$\partial_\mu \left\{ e\, \xi_\lambda\, T^{\mu\lambda} \right\} = 0 \tag{88}$$

and

$$\partial_\mu \left\{ e\, \xi_\lambda\, A^{\mu\lambda} \right\} = 0 \quad . \tag{89}$$

For scalars one obtains only Eq. (88), as, in this case, $A^{\mu\lambda} = 0$. Eq. (88) are the Noetherian conservation laws; Eq. (89) are identities which, though occasionally useful, have no particular significance. The simplest example is related to translations in Minkowski spacetime. One has

$$x'^\mu = x^\mu + \varepsilon^\mu \quad ,$$

with constant ε^μ. So, the Noetherian conservation law gives

$$\partial_\mu T^{\mu\lambda} = 0 \quad ,$$

as it should. As, accordingly to Eq. (72),

$$T^{\mu\lambda} = \frac{1}{2} \left\{ -\frac{1}{2}\, \bar\psi(x) \gamma^a \frac{1}{i} \left(e_a{}^\mu D^\lambda + e_a{}^\lambda D^\mu \right) \psi(x) - g^{\mu\lambda} L \right\}$$

whereas

$$A^{\mu\lambda} = -\frac{1}{4i}\, \bar\psi(x)\, \gamma^a \left[e_a{}^\mu D^\lambda - e_a{}^\lambda D^\mu \right] \psi(x) \quad ,$$

and as, in Minkowski spacetime, $e_a{}^\mu$ can be chosen as $\delta_a{}^\mu$,

one has

$$A^{\mu\lambda} = \frac{1}{4i}\, \bar\psi(x) \left\{ \gamma^\mu \partial^\lambda - \gamma^\lambda \partial^\mu \right\} \psi(x) \quad ,$$

and the identity (89) gives

$$\partial_\mu \left\{ \bar\psi(x) \left(\gamma^\mu \partial^\lambda - \gamma^\lambda \partial^\mu \right) \psi(x) \right\} = 0 \quad ,$$

which can independently be shown to hold.

Next in simplicity come Lorentz transformations in Minkowski spacetime. They correspond to

$$\xi^\lambda(x) = \omega^\lambda{}_\alpha\, x^\alpha \quad ,$$

with $\omega_{\lambda\alpha} = -\omega_{\alpha\lambda}$, and give rise to the Noetherian conservation laws

$$\partial_\mu \left\{ x^\nu\, T^{\mu\lambda} - x^\lambda\, T^{\mu\nu} \right\} = 0$$

as well as to the identities

$$\partial_\mu \left\{ \bar\psi(x) \left(x_\nu \gamma_\lambda - x_\lambda \gamma_\nu \right) \partial^\mu \psi - \bar\psi(x) \gamma^\mu \left(x_\nu \partial_\lambda - x_\lambda \partial_\nu \right) \psi \right\} = 0 \quad ,$$

which come from Eq. (89).

8. CONCLUSIONS

By using tetrads we included fermions in our treatment of Noether's theorem without undue complication of the formalism. Noether's conservation laws (Eq. (83)) were rewritten as Eq. (88) because of the symmetry of the energy-momentum tensor, which is inherited from the Ricci tensor through Einstein's equations (Eq. (76)). More general couplings of spin to gravity lead to Ricci tensors that are no longer symmetric[7] (Einstein-Cartan theories[8], for instance). Our treatment is easily extended to these situations, and then Eq. (83), involving the natural generalization of the canonical energy-momentum tensor, is the final result.

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