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Abstract - Complex angular momentum theory is applied to the problem of high-frequency critical light scattering by a spherical cavity near the critical angle. The main contributions to the scattering arise from a critical domain close to critical incidence. The results are in good agreement with the exact Mie solution.

1. INTRODUCTION

The complex angular momentum (CAM) theory of Mie scattering and its application to the problems of the rainbow and the glory have been reviewed elsewhere[1].

We present here the latest application of CAM theory: the treatment of critical scattering[2]. This is a new diffraction effect found in the transition region around the critical scattering angle for refractive index N relative to the external medium <1 (e.g., for an air bubble in water). The assumptions are $(ka)^{1/3} \gg 1$ and $(1-N)^{1/2}(ka)^{1/3} \gg 1$, where k is the wavenumber in the external medium and a is the radius of the cavity. The main contributions arise from a "critical domain" close to critical incidence, and they lead to a new kind of diffraction integral.

In Mie scattering for $N<1$, the critically incident ray is reflected at a critical scattering angle

$$\Theta_t = \pi - 2\theta_c = \pi - 2\sin^{-1}N \quad . \tag{1}$$

According to ray optics, total reflection takes place for angles of incidence beyond θ_c, i.e., for $\Theta < \Theta_t$.

In the geometrical optics approximation[3], the angular distribution of the scattered intensity goes through a cusp at $\Theta = \Theta_t$. This singularity arises from the abrupt approach of the Fresnel reflectivities to unity at the critical angle[4].

Exact Mie calculations[5] show an oscillatory behavior of the intensity in the total reflection region near Θ_t $(\Theta \lesssim \Theta_t)$. These diffraction fringes have also been observed experimentally[6].

A "physical optics approximation" along the lines of classical diffraction theory has been proposed by Marston[6]. The contribution from surface reflection is treated by a procedure similar to Airy's theory of the rainbow: a Kirchhoff-type approximation is applied to the amplitude distribution along a virtual reflected wavefront. In view of their steep approach to total reflection, the reflectivities are approximated by step functions. This "reflectivity edge" gives rise to an angular distribution of scattered intensity similar to a Fresnel straight-edge pattern, which would account for the diffraction fringes.

The actual angular distribution[5] differs from the Fresnel one: the oscillation amplitude increases as one goes farther away from Θ_t. This reinforcement was explained[7] through interference with directly transmitted rays due to below-critical incidence.

Superimposed on the "slow" oscillations just discussed, the Mie patterns[5] show fine structure, represented by rapid oscillations of relatively smaller amplitude. This arises from interference with "far-side" contributions (in nuclear scattering terminology[8]), mainly from rays that have undergone one internal reflection. The fine structure is unrelated with critical scattering,

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so that it should be subtracted out or averaged over in order to isolate pure critical scattering effects.

The physical optics approximation is in reasonable agreement with the general features of the slow oscillations; however, in the neighborhood of θ_t, the quantitative agreement is poor, specially for $\theta > \theta_t$.

The CAM theory applied for $N < 1$ corrects the deviations shows by the physical optics approximation near θ_t and its results are in good agreement with the exact Mie solution.

The critical domain and the dominant contributions to critical scattering are discussed in Section 2. An outline of the method and the main results are presented in Section 3. Finally, Section 4 lists the relevant conclusions.

2. THE CRITICAL DOMAIN

To discuss critical scattering in terms of CAM theory it is convenient to employ the well-known analogy[9] with Schrödinger scattering of particles with energy $E = k^2$ (in units $\hbar = m = 1$, k = wave number) by a square potential

$$V(r) = V_0 (0 < r < a), \quad = 0 (r > a) \quad . \tag{2}$$

The associated refractive index is

$$N^2 = 1 - (V_0/E) \quad , \tag{3}$$

so that $N < 1$ corresponds to a square barrier ($V_0 > 0$). The effective potential for radial motion is

$$V_{eff}^{\lambda}(r) = V(r) + \lambda^2/r^2 \quad , \tag{4}$$

where λ is the complex angular momentum variable, with physical values $\lambda = \ell + 1/2$ ($\ell = 1, 2, 3, \ldots$) associated with the partial wave terms, and the last term in (4) represents the centrifugal barrier. Therefore, $V_{eff}^{\lambda}(r)$ represents a cusped potential step. The critical angular momentum $\lambda_c = N\beta = \alpha$ (β = size parameter = ka (a = sphere radius)) associated with critical incidence corresponds to an energy level E at the top of the step. There is a critical domain (analogue to the edge domain[1,9] in the $N > 1$ case)

$$\alpha - O(\alpha^{1/3}) \lesssim \lambda \lesssim \alpha + O(\alpha^{1/3}) \quad . \tag{5}$$

For incident rays in the lower critical domain $0 \leq \lambda_c - \lambda \lesssim \alpha^{1/3}$, the radial turning point within the sphere lies very close to the surface, corresponding to rays in a boundary layer that undergo near-total internal reflection. In the upper critical domain $0 \leq \lambda - \lambda_c \lesssim \alpha^{1/3}$, the penetration depth for tunnelling into the sphere is still much larger than the wavelength; correspondingly, the evanescent waves generated by total reflection become inner surface waves, travelling internally along the surface (whispering gallery modes).

In the λ plane, whispering gallery modes are associated with Regge poles[9,10] of $S(\lambda,\beta)$ near $\lambda = \lambda_c = \alpha$. At $\theta = \theta_t$, λ_c is an accumulation point of saddle points associated with different terms of the Debye multiple reflection expansion[1,9]. For $\theta < \theta_t$ and sufficiently far from θ_t, Ludwig[10] proposed including $O(\beta^{1/4})$ saddle point contributions and $O(\beta^{1/4})$ Regge pole contributions; such a representation would be very difficult to evaluate in practice. The critical scattering region was excluded from his treatment.

The dominant contributions to critical scattering in the CAM theory[2] arise from the critical domain (5). The dominant terms from the lower critical domain are the direct reflection and direct transmission Debye terms. The main new effects are contained in the above-critical total reflection term, arising from the upper critical domain. The far-side once internally reflected contribution, which is mainly responsible for the fine-structure oscillations, is given by the WKB approximation[9,11] and it need not be considered any further.

3. RESULTS

A. Preliminary Considerations

The critical region, where our solution is supposed to be valid, is defined as:

$$\varepsilon > 0 : \varepsilon = O(\beta^{-1/3})$$

$$\varepsilon < 0 : \varepsilon = O(\beta^{-1/2}) \qquad (6)$$

$$\varepsilon = \frac{1}{2}(\Theta_t - \Theta) \quad ,$$

where Θ is the scattering angle.

The Debye expansion is used for below-critical incidence. For above-critical incidence we use expressions without making this expansion. After deforming judiciously[2] the paths in the λ-plane, the dominant contributions to the critical scattering are obtained as integrals on the real axis of the CAM plane. In the below-critical (above-critical) terms λ_c is taken as the upper (lower) limit in the integrals, as a consequence, only the part of the range of the saddle point which is in the lower (upper)-critical domain is considered in each term. The dominant contributions are obtained in lowest order approximation.

B. The Dominant Contributions

The below-critical direct transmission term is the interference term included in the physical optics approximation[7], where it was evaluated by the stationary phase (WKB) method. Since the critical scattering region is a Fock transition region between 1-ray and 0-ray domains for this term, the WKB approximation is not valid: the evaluation leads[9,11] to

$$S^{<}_{j1}(\beta,\Theta) \approx e^{-\frac{7i\pi}{12}} \left(\frac{\beta}{2\pi \sin\Theta}\right) \exp\left\{-2i\beta(M-\varepsilon N)\right\} \frac{N^{3/2}\eta_j}{\pi M}$$

$$\times \int_0^\infty \frac{\exp\left[2\, e^{-i\frac{\pi}{6}} \frac{\varepsilon}{\gamma'} x\right]}{\left[\mathrm{Ai}\left(e^{\frac{2i\pi}{3}} x\right)\right]^2} dx \quad , \quad (j=1,2) \quad , \qquad (7)$$

where $M = (1-N^2)^{1/2}$, $\gamma' = (2/\alpha)^{1/3}$, $\eta_1 = 1$, $\eta_2 = N^{-2}$, Ai is the Airy function, and $j=1$ $(j=2)$ is associated with perpendicular (parallel) polarization. The contribution (7) is given by an incomplete generalized Fock function, containing only part of the range of the direct transmission saddle point.

The below critical reflection term was not taken into account in the physical optics approximation. It is given[2,9,11] by

$$S^{<}_{j0}(\beta,\Theta) \approx e^{\frac{3i\pi}{4}} \left[\frac{N\sin(\frac{\Theta}{2})}{2\pi\sin\Theta}\right]^{1/2} \beta \exp\left\{-2i\beta\sin(\frac{\Theta}{2})\right\}$$

$$\times \int_{-\infty}^{Z=0} \left[\frac{1+\Delta}{1-\Delta}\right] \exp(-iv^2)\, dv \quad , \quad (j=1,2) \quad , \qquad (8)$$

where

$$Z = \gamma'\left[\left(\beta\sin(\frac{\Theta}{2})\right)^{1/2} v + \beta\cos(\frac{\Theta}{2}) - \alpha\right] \quad ,$$

$$v = \left[\frac{2}{\beta\sin(\Theta/2)}\right]^{1/2} \left(\lambda - \beta\cos(\frac{\Theta}{2})\right) \quad , \qquad (9)$$

$$\Delta = e^{-i\frac{\pi}{6}} \frac{N\gamma'\eta_j}{M} \ell n'\, \mathrm{Ai}\left(e^{-\frac{2i\pi}{3}} Z\right)$$

($\ell n'$ denotes the logarithmic derivative). The contribution (8) is given by an incomplete Fresnel-Fock integral, containing only part of the range of the direct reflection saddle point.

The above critical total reflection term, containing the new diffraction effects associated with critical scattering, is given by[2]

$$S^{>}_{j}(\beta,\Theta) \approx e^{-i\frac{\pi}{4}} \left[\frac{MN}{2\pi\sin\Theta}\right]^{1/2} \beta\exp\left[-2i\beta(M-\varepsilon N)\right] P_F(x,y) \quad , \qquad (10)$$

$$(j=1,2) \quad ,$$

where

$$P_F(x,y) = \int_0^\infty \exp\left\{-i\left[u^2 - xu - \left(\frac{y}{\sqrt{2}}\right)^{4/3}\right.\right.$$

$$\left.\left.\times\, \eta_j\, \ell n'\, \mathrm{Ai}\left(2^{4/3}\frac{u}{y^{2/3}}\right)\right]\right\} du \quad , \qquad (11)$$

is a new type of diffraction integral, y depends only on N and β, and x is proportional to $\theta - \theta_t$.

If we neglect the variation of the last term in the exponent, replacing it by a constant, we get a Fresnel integral, as in the physical optics approximation[6]. Since $y \; \beta^{-1/4}$, the argument of the Airy function is $\gg 1$ for $\beta \gg 1$, except near $u = 0$, so that one may employ the asymptotic approximation

$$\ln' \mathrm{Ai}(z) \approx -\sqrt{z} \quad , \qquad z \gg 1 \quad . \tag{12}$$

This corresponds to the "plane surface limit", in which the effects spherical curvature are neglected. In this limit, setting $u = t^2$, (11) with $j = 1$ becomes

$$P_F(x,y) \approx 2 \int_0^{\infty} \exp\left[-i(t^4 - xt^2 + yt)\right] t \, dt \quad . \tag{13}$$

Pearcey's integral[12], associated with the cusp diffraction catastrophe[13], is given by

$$P(x,y) = \int_{-\infty}^{\infty} \exp\left[i(t^4 + xt^2 + yt)\right] dt \quad . \tag{14}$$

Thus, $\partial P/\partial y$ is related with $P_F(x,y)$ given by (13).

In this plane surface limit, the y term in the exponent gives rise to a shifted Fresnel-like pattern. For each above-critical ray, this shift corresponds to the Goos-Hänchen lateral shift[14].

In the present case, we have the spherical analogue of this shift which is a Goos-Hänchen angular displacement $\Delta\vartheta$. A ray with angle of incidence above θ_c tunnels along the surface thorugh an extra angle $\Delta\vartheta$ as an inner surface wave (evanescent wave) before reemerging at the angle of reflection.

To obtain $\Delta\vartheta$, one may employ the concept of angular displacement in a scattering process[15], which is analogous to the Wigner time delay[16,17], applied to the conjugate pair angular momentum and angle. For an "angular momentum wave packet" centered around λ_0, the angular displacement ϑ is given by

$$\vartheta = 2 \, d\eta(k,\lambda_0)/d\lambda \quad , \tag{15}$$

where $\eta(k,\lambda)$ is the scattering phase shift as a function of the (continuous) angular momentum λ. The Goss-Hänchen effect appears as an additional angular displacement arising from the last term in (11).

We propose to call the new diffraction integral (11) the Pearcey-Fock half-range integral, because of its connection both with generalized Fock functions and with Pearcey's integral.

4. CONCLUSIONS

The combined effect of the dominant CAM terms (below-critical direct reflection and transmission terms and above-critical total reflection term) was compared with the exact Mie solution within the critical scattering region, for $\beta = 10^3$ and $\beta = 10^4$. The results[2] are in good agreement with the "slow" component of the Mie solution, which, as explained before, represents the critical scattering effects (fine structure arises from the far-side contribution). We conclude that CAM theory also accounts for critical scattering.

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