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ABSTRACT

We present a procedure to obtain the BRST charge for the representations of the Virassoro algebra. For $c \leqslant 1$ the BRST charge has in general terms containing products of more than three ghosts. It is nilpotent for any allowed value of the central charge and conformal weight of the representation.

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Two dimensional conformal symmetry has many interesting properties due to the fact that it is realized as an infinite dimensional algebra, the Virassoro algebra. Its representations have been studied and classified(1,2) and several applications to two dimensional statistical systems (at their critical points)(3), field theory(4) and strings(5) have been performed. The supersymmetric extensions have also been found(6).

In two dimensions a conformal transformation can be viewed as a general coordinate transformation with analytic parameters. Conformal field theories in two dimensions can therefore be considered as field theories with a local invariance.

The quantization of such theories has been performed at the operator level(4) and it was found that the local fields which form the operator algebra can be classified according to the irreducible representations of the Virassoro algebra(4).

Since conformal symmetry is a local symmetry we can consider the BRST quantization procedure(7,8). At the quantum level, however, the Virassoro algebra acquires a central extension due to the appearance of a Schwinger term. In gauge theories such terms signal the presence of anomalies and conditions must be found such that these Schwinger terms are removed from the algebra in order to have a consistent theory. On the other hand, in conformal theories the Schwinger term is harmless and in fact it is responsible for the rich structure found in these theories. Because of that we cannot apply the straightforward recipe to construct the BRST charge[8], since it starts with the Hamiltonian formulation of the corresponding classical theory and in the classical conformal theory the Schwinger term is, of course, absent.

We then start from the beginning with the quantum theory and consider the physical state conditions as the analogue of the classical constraints in order to build the BRST charge. An important point is that the classical constraints must be first class, or in our case, the physical state conditions must form a closed algebra. As we will see this does not happen for the unitary representations of the Virassoro algebra with $c \leqslant 1$.

Consider a state of conformal weight $\Delta \geqslant 0$. It must satisfy

$$\begin{aligned} L_n|\Delta\rangle &= 0 \quad , \quad n>0 \\ L_0|\Delta\rangle &= \Delta|\Delta\rangle \end{aligned} \tag{1}$$

where L_n are the Virassoro operators satisfying the algebra

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}\,\delta_{n+m,0} \ . \tag{2}$$

For convenience we rewrite eq.(1) as

$$\begin{aligned} \tilde{L}_n|\Delta\rangle = 0 \quad , \quad \tilde{L}_n &\equiv L_n \quad , \quad n>0 \\ \tilde{L}_0 &\equiv L_0 - \Delta \ . \end{aligned} \tag{3}$$

The space of states (the Verma module) is generated by the successive applications of L_{-n} to the state $|\Delta\rangle$

$$\begin{aligned} |-n_1, -n_2, \ldots, -n_N, \Delta\rangle &= L_{-n_1} L_{-n_2} \ldots L_{-n_N}|\Delta\rangle \\ n_1 \geqslant n_2 \geqslant \ldots \geqslant n_N \ . \end{aligned} \tag{4}$$

Let us consider first the case $c>1$. The physical state conditions are given by (1) or (3) and it is easily verified that they have a closed algebra

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} \ . \tag{5}$$

In this case a nilpotent BRST charge is easily constructed by following the standard procedure[8], with the constraint algebra substituted by (5). Introducing the ghosts c_{-m}, $\bar{c}_m$, $m \geqslant 0$, satisfying $\{c_{-m}, \bar{c}_n\} = \delta_{n-m,0}$, $\bar{c}_m|\Delta\rangle = \langle\Delta|c_{-m} = 0$ we have

$$Q = \sum_{m=0}^{\infty} c_{-m}\,\tilde{L}_m - \frac{1}{2}\sum_{m,n=0}^{\infty} (n-m)\,c_{-n}\,c_{-m}\,\bar{c}_{m+n} \ . \tag{6}$$

Notice that the indices n and m are always positive and no normal ordering is necessary. The condition $Q|\Delta\rangle = 0$ gives back the physical state conditions (3).

Up to now we have considered only the conformal transformations associated to one of the analytic conformal parameters, say the one which depends on z. The conformal transformations associated to the other variable z^* are generated by L^*_n, which has the same algebra as L_n (2) and commutes with L_n. The state with conformal weights Δ and Δ^* satisfies

$$\tilde{L}_n|\Delta,\Delta^*\rangle = \tilde{L}^*_n|\Delta,\Delta^*\rangle = 0 \quad . \tag{7}$$

To build the complete BRST charge we need another pair of ghosts $c^*_{-m}, \bar{c}^*_m$ satisfying $\{c^*_{-m}, \bar{c}^*_n\} = \delta_{n-m,0}$, $\bar{c}^*_m|\Delta\rangle = \langle\Delta|c^*_{-m} = 0$ and anticommuting with the ghosts introduced earlier on. We can now define a hermitian operation (which is a complex conjugation plus the usual hermitian operation)

$$(L_n)^\dagger = L^*_{-n} \quad , \quad (c_{-n})^\dagger \equiv c^*_n \quad , \quad (\bar{c}_n)^\dagger \equiv \bar{c}^*_{-n} \quad ,$$

$$(c^*_{-n})^\dagger \equiv c_n \quad , \quad (\bar{c}^*_n)^\dagger \equiv \bar{c}_{-n} \tag{8}$$

such that the BRST charge

$$Q = \sum_{m=0}^{\infty} (c_{-m}\tilde{L}_m + c^*_m \tilde{L}^*_{-m}) - \frac{1}{2}\sum_{n,m=0}^{\infty} (n-m)(c_{-n}c_{-m}\bar{c}_{m+n} - \bar{c}^*_{-m-n}c^*_m c^*_n) \tag{9}$$

is nilpotent and hermitian. Now $Q|\Delta,\Delta^*\rangle = \langle\Delta,\Delta^*|Q = 0$ yields the physical state conditions (7).

For $c \leqslant 1$ it is well known[1,2] that the representations

of the Virassoro algebra are reducible for arbitrary values of c and Δ. To obtain an irreducible representation we must factor out the reducible submodules of the original Verma module[4] and this can be accomplished by imposing the vanishing of Kac's determinant[2]. The roots of Kac's determinant can be labelled by two positive integers n and m and the corresponding values of the conformal weight are given by[1]

$$\Delta_{(n,m)} = \frac{c-1}{24} + \frac{1}{4}(n\alpha_+ + m\alpha_-)^2$$
$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \quad . \tag{10}$$

The reducible submodules are composed of zero norm states so that we impose the vanishing of these states in the physical sector. This means that they must have a closed algebra with the original physical state conditions (3). The conformal weight of the zero norm states is given by $\Delta_{(n,m)} + m.n$. As an example let us consider the level 2 representation. We have to look for a linear combination of states of the form (4), with weight $\Delta+2$, such that it has a closed algebra with $\tilde{L}_n$. Writing the operator $X_{-2} = L_{-2} + aL^2_{-1}$ then we find

$$[\tilde{L}_n,\tilde{L}_m] = (n-m)\tilde{L}_{n+m}$$
$$[\tilde{L}_n,X_{-2}] = [an(n+2)+n+2]\,\delta_{n>1}\tilde{L}_{n-2} + 2a(n+1)\,\delta_{n>0}L_{-1}\tilde{L}_{n-1} + 2\delta_{n,0}X_{-2} \tag{11}$$

if

$$a = \frac{-3}{2(2\Delta+1)} \quad , \quad c = \frac{2\Delta(5-8\Delta)}{2\Delta+1} \quad . \tag{12}$$

Of course, this is equivalent to solving Kac's determinant and the relation between c and Δ in (12) corresponds to the cases $n=1$, $m=2$ and $n=2$, $m=1$ in (10). Now, the algebra (11) has structure constants which are field dependent since the second term in the commutator of $\tilde{L}_n$ and χ_{-2} depends on L_{-1}. The construction of the classical BRST charge for a system with a constraint algebra which has field dependent structure constants has been performed in ref.(9). We can extend the results of ref.(9) to the case where we have an algebra with commutators instead of Poisson brackets, and we find, as in the classical case, that there are higher powers of ghosts in the BRST charge[10]. Introducing the ghosts $c_{-n}, \bar{c}_n$ for $\tilde{L}_n$ and $d_2, \bar{d}_{-2}$ for χ_{-2}, with $\{d_2, \bar{d}_{-2}\} = 1$, $d_2|\Delta\rangle = \langle\Delta|\bar{d}_{-2} = 0$ and with the d ghosts anticommuting with the c ghosts, we find

$$\begin{aligned} Q = & \sum_{n=0}^{\infty} \tilde{L}_n c_{-n} + \chi_{-2} d_2 - \frac{1}{2} \sum_{n,m=0}^{\infty} (n-m) c_{-n} c_{-m} \bar{c}_{m+n} + \\ & + 2a \sum_{n=1}^{\infty} (n+1) c_{-n} \bar{c}_{n-1} L_{-1} d_2 - 2 c_0 d_2 \bar{d}_2 + \\ & + \sum_{n=2}^{\infty} [an(n+1) + (n+2)] c_{-n} \bar{c}_{n-2} d_2 \\ & - a \sum_{n,m=1}^{\infty} (n+1)(m+1) c_{-n} c_{-m} \bar{c}_{n-1} \bar{c}_{m-1} d_2 \quad . \end{aligned} \tag{13}$$

The cubic terms come from the structure constants of the algebra (11) and the last term from the fact that one of the above mentioned structure constants depends on L_{-1}; this last term is essential to prove that (13) is nilpotent. Now $Q|\Delta\rangle = 0$ gives (3) as well as $\chi_{-2}|\Delta\rangle = 0$. To make (13) hermitian we adopt the same procedure as in the $c > 1$ case. Notice that $Q^2 = 0$ does not fix any particular value of c or Δ; we need only the relation between them given by eq.(12).

In the general case, with a zero norm vector $\chi_{-N}|\Delta\rangle$ at level N, we can have in the commutator of $\tilde{L}_n$ with χ_{-N} structure constants with at most $N-1$ powers of L_{-n}. This means that the highest ghost power in Q will be $2N+1$. In the jargon of ref.(9) the theory is of rank N.

We now consider the case in which there are more than one zero norm state. This happens if $\alpha_-/\alpha_+ = -p/q$ with p and q positive integers. For these values of c and Δ we have the unitary representations of the Virassoro algebra[11] and in fact an infinite number of zero norm states. In our former example, if we take $p=3$ and $q=4$ we find $c = 1/2$ and

$$\begin{aligned} & \Delta_{(2,1)} = \Delta_{(5,5)} = \Delta_{(8,9)} = \dots = \\ & = \Delta_{(1,3)} = \Delta_{(4,7)} = \Delta_{(7,11)} = \dots = \frac{1}{2} \quad . \end{aligned} \tag{14}$$

The corresponding zero norm states will be generated by

$X_{-2}, X_{-25}, X_{-72}, \ldots, X_{-3}, X_{-23}, X_{-77}, \ldots$. These new operators however do not have a closed algebra since for example $[X_{-2}, X_{-3}]$ would generate a new zero norm-state at level 5 X_{-5} which is not allowed by (14). This means that we have a second class algebra and some of degrees of freedom represented by L_{-n} are not physical. We can them retain one of the X's as a physical state condition and the others will be imposed strongly so that $X_{-N} = 0$ can be used to write L_{-N} in terms of the other generators and in this way be eliminated from the theory. In our example, if we keep X_{-2} as a physical state condition the BRST charge is still given by (13) with $c = \Delta = 1/2$ and with $L_{-3}, L_{-25}, L_{-28}, \ldots$ being combinations of the other operators. After the elimination of these operators the Virassoro algebra becomes highly non-linear.

It is worth mentioning that this construction of the BRST charge solves an apparent puzzle. As it is well known in string theory the requirement that the BRST charge is nilpotent can be used to find the critical dimension and the Regge slope of the string[12]. A naive BRST quantization of conformal theories taking into account the full Virassoro algebra will always yield the values $c = 26$ and $\Delta = 1$[13] while it is known that all other representations with $c \neq 26$ and $\Delta \neq 1$ can be consistently quantized. As we have shown the correct BRST treatment of conformal theories needs only the physical state conditions and not the full Virassoro algebra.

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