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ABSTRACT

We construct the S matrix for bound state (gauge-invariant) scattering for nonlinear sigma models defined on the manifold $SU(n)/S(U(p)\otimes U(n-p))$ with fermions. It is not possible to compute gauge non-singlet matrix elements. In the present language they are not submitted to sufficiently many constraints derived from higher conservation laws.

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Two dimensional nonlinear sigma models are important objects in field theory(1). When they are defined on symmetric spaces they also share the property of integrability(2), which remains valid when the models are coupled to fermions in some restricted way(3,4,5). However, in quantum case there may be anomalies(5), which turn out to cancel against the Adler-Bardeen anomaly in the case where fermions are present(3,4). In general it is possible to compute S-matrix elements(3,7,8). The case of Grassmannians was studied in full detail in ref. (5). This is an interesting case, since higher conservation laws are not enough to fix up the S-matrix. In this paper, S-matrix elements are computed for gauge singlets, which was the missing ingredient (since gauge dependent quantities do not make sense), and we should not bother about computing S-matrix elements containing those objects.

We shall fix attention on the model given by the Lagrangean density

$$L = \overline{D_\mu z}\, D_\mu z + i\bar\psi \not{D} \psi \qquad (1)$$

where z_i^α transforms under $SU(n)$ as

$$z_i^\alpha = g^{ij} z_j^\alpha \quad .$$

and under gauge $U(p) = SU(p)\otimes U(1)$ as

$$z_i^\alpha = z_i^\beta h^{\beta\alpha} \quad .$$

The fermion field is an n-plet ψ_i^a, transforming in the same way as z under g^{ij}, and a is a flavor index running from 1 to n (7,9).

The current

$$J_{\mu\, ij} = z_i \overleftrightarrow{D}_\mu \bar{z}_j + i\bar{\psi}_j \gamma_\mu \psi_i \tag{2}$$

is the Noether current associated to $SU(n)$ rotations. We define also the matter current

$$j^M_{\mu\, ij} = i\bar{\psi}_j \gamma_\mu \psi_i \quad .$$

They satisfy

$$\partial_\mu (J_\nu + j^M_\nu) - \partial_\nu (J_\mu + j^M_\mu) + 2[J_\mu, J_\nu] = 0 \tag{3}$$

permitting the definition of the conserved charge

$$Q = \int dy_1\, dy_2\, \epsilon(y_1 - y_2)(J_0 + j^M_0)(t_1 y_2)(J_0 + j^M_0)(t_1 y_2) - \int dy\, J_1(t,y) \, . \tag{4}$$

This nonlocal charge is conserved after quantization (see (5)).

We define asymptotic fields, compute the asymptotic charges (actually care must be taken in the definition of such objects in terms of free fields - see (10), (11)) and the asymptotic gauge singlets respectively by

$$z^{i\alpha}_{\substack{in\\out}} = \int d\mu(p) \left\{ b^{i\alpha}_{\substack{in\\out}}(p)\, e^{-ip.x} + \left(d^{i\alpha}_{\substack{in\\out}} \right)^\dagger (p)\, e^{ipx} \right\} \tag{5}$$

where $\quad d\mu(p) = \dfrac{dp_1}{2\pi\, 2p_0}$

$$Q^{ij}_{\substack{in\\out}} = \frac{1}{n} \int d\mu(p_1)\, d\mu(p_2)\, \epsilon(p_1 - p_2) : (b^{\dagger i\alpha}(p_1)\, b^{\alpha k}(p_1)$$

$$- a^{\dagger \alpha k}(p_1)\, d^{i\alpha}(p_1))(b^{\dagger k\beta}(p_2)\, b^{\beta j}(p_2) - d^{\dagger \beta j}(p_2)\, d^{k\beta}(p_2)) :$$

$$- \frac{1}{i\pi} \int d\mu(p)\, \ell n\, \frac{p^0 + p}{m} : (b^{\dagger i\alpha}(p)\, b^{\alpha j}(p) - d^{\dagger \alpha j}(p)\, d^{i\alpha}(p)) \, . \tag{6}$$

$$\left| \Pi_{c_1 c_2} \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right\rangle = 2\pi^2 \int d\mu(k_1) d\mu(k_2)\, \delta(q - k_1 - k_2) d^{\dagger \beta c_1}(k_1) d^{\dagger \beta c_2}(k_2) |0\rangle \tag{7}$$

Although there are difficulties in those definitions, we take them as a practical way to built the constraints upon the asymptotic fields. This procedure turns out to be correct a posteriori.

The ansatz for the S matrix is given by the most general SU(n) invariant expression, with a tensorial structure in eight indices given by

$$\langle \Pi_{e_1e_2}(\vartheta_1)\,\Pi_{f_1f_2}(\vartheta_2)\,|\,\Pi_{c_1c_2}(\vartheta_1')\,\Pi_{d_1d_2}(\vartheta_2')\rangle \;=$$

$$= \Big\{\sigma_1(\vartheta)\delta_{c_1c_2}\delta_{d_1d_2}\delta_{e_1e_2}\delta_{f_1f_2} + \sigma_2(\vartheta)\delta_{c_1d_2}\delta_{c_2d_1}\delta_{e_1f_2}\delta_{e_2f_1}$$

$$+ \sigma_3(\vartheta)\delta_{c_1c_2}\delta_{d_1d_2}\delta_{e_1f_2}\delta_{e_2f_1} + \sigma_4(\vartheta)\delta_{c_1d_2}\delta_{c_2d_1}\delta_{e_1e_2}\delta_{f_1f_2}$$

$$+ \sigma_5(\vartheta)\delta_{c_1c_2}\delta_{d_1f_1}\delta_{d_2f_2}\delta_{e_1e_2} + \sigma_6(\vartheta)\delta_{d_1d_2}\delta_{c_1e_1}\delta_{c_2e_2}\delta_{f_1f_2}$$

$$+ \sigma_7(\vartheta)\delta_{c_1c_2}\delta_{d_1e_1}\delta_{d_2e_2}\delta_{f_1f_2} + \sigma_8(\vartheta)\delta_{d_1d_2}\delta_{e_1f_1}\delta_{c_2f_2}\delta_{e_1e_2}$$

$$+ \sigma_9(\vartheta)\delta_{c_1c_2}\delta_{d_1e_1}\delta_{d_2f_2}\delta_{e_2f_1} + \sigma_{10}(\vartheta)\delta_{d_1d_2}\delta_{c_1e_1}\delta_{c_2f_2}\delta_{e_2f_1}$$

$$+ \sigma_{11}(\vartheta)\delta_{c_1c_2}\delta_{d_1f_1}\delta_{d_2e_2}\delta_{e_1f_2} + \sigma_{12}(\vartheta)\delta_{d_1d_2}\delta_{c_1f_1}\delta_{c_2e_2}\delta_{e_1f_2}$$

$$+ \sigma_{13}(\vartheta)\delta_{c_1f_1}\delta_{c_2d_1}\delta_{d_2f_2}\delta_{e_1e_2} + \sigma_{14}(\vartheta)\delta_{c_1e_1}\delta_{c_2d_1}\delta_{d_2e_2}\delta_{f_1f_2}$$

$$+ \sigma_{15}(\vartheta)\delta_{c_1d_2}\delta_{c_2f_2}\delta_{d_1f_1}\delta_{e_1e_2} + \sigma_{16}(\vartheta)\delta_{c_1d_2}\delta_{c_2e_2}\delta_{d_1e_1}\delta_{f_1f_2}$$

$$+ \sigma_{17}(\vartheta)\delta_{c_1e_1}\delta_{c_2d_1}\delta_{d_2f_2}\delta_{e_2f_1} + \sigma_{18}(\vartheta)\delta_{c_1d_2}\delta_{c_2e_2}\delta_{d_1f_1}\delta_{e_1f_2}$$

$$+ \sigma_{19}(\vartheta)\delta_{c_1f_1}\delta_{c_2d_1}\delta_{d_2e_2}\delta_{e_1f_2} + \sigma_{20}(\vartheta)\delta_{c_1d_2}\delta_{c_2f_2}\delta_{d_1e_1}\delta_{e_2f_1} \;+$$

$$+ \sigma_{21}(\vartheta)\delta_{c_1e_1}\delta_{c_2e_2}\delta_{d_1f_1}\delta_{d_2f_2} + \sigma_{22}(\vartheta)\delta_{c_1f_1}\delta_{c_2f_2}\delta_{d_1e_1}\delta_{d_2e_2}$$

$$+ \sigma_{23}(\vartheta)\delta_{c_1e_1}\delta_{c_2f_2}\delta_{d_1f_1}\delta_{d_2e_2} + \sigma_{24}(\vartheta)\delta_{c_1f_1}\delta_{c_2e_2}\delta_{d_1e_1}\delta_{d_2f_2}\Big\}$$

$$\times\, \delta(\vartheta_1-\vartheta_1')\,\delta(\vartheta_2-\vartheta_2') + \Big\{\vartheta_1' \leftrightarrow \vartheta_2',\ (e_1e_2) \leftrightarrow (f_1f_2)\Big\}\ . \qquad (8)$$

Imposition of conservation of the nonlocal charge is by now a straightforward though extremely tedious computation (see, for instance, (7,8,10). The only difficulty resides in the fact that each field $\pi(\vartheta)$ is a bound state, therefore we should rather write $\pi(\vartheta,\vartheta')$, where $\frac{\vartheta+\vartheta'}{2}$ is the rapidity, and $\vartheta-\vartheta'=\alpha$ is the position of a pole in the elementary fields S-matrix[12]. After a hard work one finds as a result, all amplitudes in terms of σ_{21}; the relations turn out to be the following $(\vartheta=\varphi\pi i,\ \lambda=\alpha\pi i)$

$$\sigma_1(\vartheta) \;=\; \frac{(2/n)^2}{(1-\lambda)^2}\left(1+\frac{(2/n)^2}{\varphi(1-\varphi-\lambda)}\right)\sigma_{21}(\vartheta) \qquad (9a)$$

$$\sigma_2(\vartheta) \;=\; \frac{(2/n)^2}{(1-\varphi-\lambda)(1-\varphi+\lambda)}\;\sigma_{21}(\vartheta) \qquad (9b)$$

$$\sigma_3(\vartheta) \;=\; \sigma_4(\vartheta) \;=\; 0 \qquad (9c)$$

$$\sigma_5(\vartheta) \;=\; -\,\frac{2/n}{1-\lambda}\left(1-\frac{(2/n)^2}{\varphi(1-\varphi-\lambda)}\right)\vartheta_{21}(\vartheta) \qquad (9d)$$

$$\sigma_6(\vartheta) = -\frac{2/n}{1-\varphi+\lambda}\left(1+\frac{(2/n)^2}{(\varphi-\lambda)(1-\varphi-\lambda)}\right)\sigma_{21}(\vartheta) \tag{9e}$$

$$\sigma_7(\vartheta) = \sigma_8(\vartheta) = -\frac{(2/n)^3}{(\varphi-\lambda)(1-\varphi-\lambda)}\sigma_{21}(\vartheta) \tag{9f}$$

$$\sigma_9(\vartheta) = \sigma_{10}(\vartheta) = \sigma_{13}(\vartheta) = \sigma_{14}(\vartheta) = \sigma_{19}(\vartheta) = \sigma_{20}(\vartheta) = 0 \tag{9g}$$

$$\sigma_{11}(\vartheta) = \sigma_{15}(\vartheta) = \frac{(2/n)^2}{(1-\varphi-\lambda)\varphi}\sigma_{21}(\vartheta) \tag{9h}$$

$$\sigma_{12}(\vartheta) = \sigma_{16}(\vartheta) = \frac{(2/n)^2}{\varphi(1-\varphi-\lambda)}\sigma_{21}(\vartheta) \tag{9i}$$

$$\sigma_{17}(\vartheta) = \frac{2/n}{1-\varphi+\lambda}\sigma_{21}(\vartheta) \tag{9j}$$

$$\sigma_{18}(\vartheta) = -\frac{2/n}{1-\varphi-\lambda}\sigma_{21}(\vartheta) \tag{9k}$$

$$\sigma_{22}(\vartheta) = \frac{(2/n)^2}{\varphi^2}\sigma_{21}(\vartheta) \tag{9ℓ}$$

$$\sigma_{23}(\vartheta) = -\frac{2/n}{\varphi}\sigma_{21}(\vartheta) \tag{9m}$$

$$\sigma_{24}(\vartheta) = -\frac{2/n}{\varphi}\sigma_{21}(\vartheta) \quad . \tag{9n}$$

The very same result can be obtained in another way as a crosscheck. Suppose one has a factorizable S-matrix for the elementary fields as in reference (5):

$$\begin{aligned}\langle z^{\alpha}_{c_1}(\vartheta_1)\, z^{\beta}_{c_2}(\vartheta_2)\,|\,z^{\gamma}_{c_2}(\vartheta_3)\, z^{\delta}_{c_4}(\vartheta_4)\rangle = \Big\{ & u_1\,\delta_{c_2c_4}\delta^{\beta\delta}\delta_{c_1c_3}\delta^{\alpha\gamma} \\ & + u_2(\vartheta)\delta_{c_2c_4}\delta^{\beta\gamma}\delta_{c_1c_3}\delta^{\alpha\gamma} + u_3(\vartheta)\delta_{c_2c_3}\delta^{\beta\delta}\delta_{c_1c_4}\delta^{\alpha\gamma} \\ & + u_4(\vartheta)\delta_{c_2c_3}\delta^{\beta\gamma}\delta_{c_1c_4}\delta^{\alpha\delta}\Big\}\,\delta(\vartheta_1-\vartheta_3)\,\delta(\vartheta_2-\vartheta_3) \\ & + (c_3 \leftrightarrow c_4\,,\ \gamma \leftrightarrow \delta\,,\ \vartheta_3 \leftrightarrow \vartheta_4) \quad ,\end{aligned} \tag{10}$$

and

$$\begin{aligned}\langle z^{\alpha}_{c_1}(\vartheta_1)\, \bar{z}^{\beta}_{c_2}(\vartheta_2)\,|\,z^{\gamma}_{c_3}(\vartheta_3)\, \bar{z}^{\delta}_{c_4}(\vartheta_4)\rangle = \Big\{ & t_1(\vartheta)\delta_{c_2c_4}\delta_{c_2c_3}\delta^{\beta\delta}\delta^{\alpha\gamma} \\ & + t_2(\vartheta)\delta^{\alpha\beta}\delta^{\gamma\delta}\delta_{c_1c_3}\delta_{c_2c_4} + t_3(\vartheta)\delta_{c_1c_2}\delta_{c_3c_4}\delta^{\beta\delta}\delta^{\alpha\gamma} \\ & + t_4(\vartheta)\delta_{c_1c_2}\delta_{c_3c_4}\delta^{\alpha\beta}\delta^{\gamma\delta}\Big\}\,\delta(\vartheta_1-\vartheta_3)\,\delta(\vartheta_2-\vartheta_4) \quad .\end{aligned} \tag{11}$$

where $\vartheta = \vartheta_1 - \vartheta_2$. Factorization equations are

$$\begin{aligned} u_3(\vartheta) &= -\frac{2\pi i}{n\vartheta}\, u_1(\vartheta) \\ u_4(\vartheta) &= -\frac{2\pi i}{n\vartheta}\, u_2(\vartheta) \\ t(\vartheta) &= u(i\pi - \vartheta) \quad . \end{aligned} \tag{12}$$

We define bound states fusing elementary fields(12), writting the amplitudes σ_i appearing in the ansatz (8) in terms of u's and t's from eqs. (10,11) above. We have a few constraints on the above equations, namely (12), and the σ's written in terms of u's and t's.

It is a matter of muscles to use them and verify the equations of factorization (9).

As a result, we checked that it is possibel to obtain a closed expression for all amplitudes, in terms of one of them, namely $\sigma_{21}(\vartheta)$, which can be fixed by unitarity and by the bound state spectrum. We shall suppose trivial spectrum, since this is already a bound state solution, and get the minimal S-matrix. This result should be checked by a 1/n expansion of the model. Unitarity provides the equation

$$\sigma_{21}(\vartheta)\sigma_{21}(-\vartheta) = \left\{\frac{\vartheta^2}{\vartheta^2 + \left(\frac{2\pi}{n}\right)^2}\right\}^2 \tag{13}$$

which gives as a minimal solution

$$\sigma_{21}(\vartheta) = \left\{\frac{\Gamma(-\frac{i\vartheta}{2\pi}+\frac{1}{n})\,\Gamma(-\frac{i\vartheta}{2\pi}+\frac{1}{2})\,\Gamma(\frac{1}{2}+\frac{i\vartheta}{2\pi}+\frac{1}{n})\,\Gamma(1+\frac{i\vartheta}{2\pi})}{\Gamma(-\frac{i\vartheta}{2\pi})\,\Gamma(-\frac{i\vartheta}{2\pi}+\frac{1}{n}+\frac{1}{2})\,\Gamma(\frac{1}{2}+\frac{i\vartheta}{2\pi})\,\Gamma(1+\frac{i\vartheta}{2\pi}+\frac{1}{n})}\right\}^2 . \tag{14}$$

The supersymmetric problem can be solved analogously, although no further insight can be obtained writting down the

solution. The question we propose to have an answer is the existence of a factorizable solution to the above gauge theory. The position of the bound state pole α remained as a free parameter. There is no bound state in the solution (14), since this is already a composite scattering.

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