UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-695

CLASSICAL MECHANICS - ON THE DEDUCTION OF LAGRANGE'S EQUATIONS

S.F. Cortizo
Instituto de Física, Universidade de São Paulo

Fevereiro/1988

CLASSICAL MECHANICS—ON THE DEDUCTION OF LAGRANGE'S EQUATIONS

S. F. Cortizo

Instituto de Física da Universidade de São Paulo,
São Paulo, Brasil

Abstract: A deduction of Lagrange's equations from Newton's Laws is presented. We tried to follow in the footseps of J. L. Lagrange's original deduction (which uses d'Alembert's principle) employing an updated mathematical formalism.

Postal address: R. Minas Gerais, 428, ap 9-A,
CEP 01244, São Paulo, SP, Brasil.

(0) INTRODUCTION

For the past few decades, the mathematical basis of classical mechanics has been going through a process of reformulation (take for instance Whittaker's work[2] as opposed to the work of Abraham and Marsden's[3]).

We aim here at expanding this reformulation to the deduction of Lagrange's equations, which is found in his renowed treatise "Mécanique Analityque"[1]. With this book, Lagrange laid the foundations of analytical mechanics, which was later developed by Hamilton, Jacobi, Poisson and others. That work is still important today for it relates the basic concepts of classical mechanics —space, time, mass, force— to those found in analytical formulations (Lagrangian and Hamiltonian mechanics).

We will procede as follows: in section (1) we summarize the basic notions of Newtonian mechanics in a few definitions and propositions. In section (2) we introduce the concept of constraint (holonomic) on a system of particles and then we "classify" the various forces of constraint using theorem 2.5. This theorem gives us three important corollaries: the first one (2.6) enables us to define the d'Alembertian force of constraint (the one specified in the classical formalism by d'Alembert's principle); the second (2.14) establishes one of the several versions of Lagrange's equations; and the third (2.15) is an expression for the force of constraint which does not require the use of Lagrange's multipliers (remark 2.17). Finally, in section (3) we analyze the case of conservative forces, which completes the deduction of Lagrange's equations from Newtonian mechanics.

(1) NEWTONIAN MECHANICS

We will now gather a few basic concepts of Newtonian mechanics —space, mass points and force fields— in a definition that suits our objectives.

CONVENTION. We assume all objects and mappings dealt with here to be of class C^∞. Moreover, we shall omit inclusions in the compositions of mappings.

1.1 Definition. *We say that (N, g, M, F) is a* **system of particles** *if:*

(i) (N,g) is an **Euclidean space**—*that is, an affine manifold N of finite dimension, with a Riemannian metric g that is invariant through affine translations;*

(ii) $M: TN \to TN$ is a tensor on N, symmetric, positive definite — via g— and invariant through affine translations; and

(iii) $F: TN \to TN$ is a mapping (not necessarily tensorial) that preserves the base point ($\tau_N \circ F = \tau_N$, where $\tau_N: TN \to N$ is the tangent bundle of N).

1.2 Proposition. *If (N, g, M, F) is a system of particles then the tensor m defined by $m(X, Y) = g[X, M(Y)]$ is an Euclidean metric on N.*

Proof. It follows from the fact that M is g-symmetric, positive definite and invariant through affine translations. ■

1.3 Definition. *Given a system of particles (N, g, M, F), define its* **kinetic energy** $K: TN \to \mathbb{R}$ *by* $K(v) = \frac{1}{2}m(v, v)$.

1.4 Proposition. *Let (N, g, M, F) be a system of particles and $c: (-\varepsilon, \varepsilon) \to N$ a curve on N. If $v: (-\varepsilon, \varepsilon) \to TN$ is the velocity of c and $a: (-\varepsilon, \varepsilon) \to TN$ is the acceleration of c, then:*

(i) $\frac{d}{dt}(M \circ v) = M \circ a$*; and*

(ii) if $F \circ v = M \circ a$ *then* $\frac{d}{dt}(K \circ v) = g(v, F \circ v)$.

Proof. (i) follows from the fact that M is invariant through affine translations; (ii) follows from the definition of K:

$$\frac{d}{dt}(K \circ v) = \frac{d}{dt}\left[\frac{1}{2}m(v, v)\right] = m(v, a) = g(v, M \circ a) = g(v, F \circ v). \quad ■$$

These definitions and propositions can be better understood with example 1.5 below. In general, if (N, g, M, F) is a system of particles, we call N the *space of configurations* (of the system), M the *mass tensor* (for it assigns a *velocity* $v \in TN$ to the *momentum* $M(v) \in TN$), F the *force field* (these force depend on the position and on the velocity of the system), and K the (total) *kinetic energy* of the system. This way, proposition 1.4(ii) states that if the motion c of the system satisfies *Newton's law* ($F \circ v = M \circ a$), then the temporal variation of the *kinetic energy* is equal to the *work* $g(v, F \circ v)$ of force F on the system.

1.5 Example. In definition 1.1, if we take:

(i) $(N, g) = \mathbb{R}^{3n}$ with its usual metric;

(ii) $M(\vec{x}_1, \ldots, \vec{x}_n, \vec{v}_1, \ldots, \vec{v}_n) = (\vec{x}_1, \ldots, \vec{x}_n, m_1\vec{v}_1, \ldots, m_n\vec{v}_n)$, where $m_1, m_2, \ldots, m_n$ are real, positive constants; and

(iii) $F: \mathbb{R}^{6n} \to \mathbb{R}^{6n}$ a mapping from TN into TN ($T\mathbb{R}^{3n} \cong \mathbb{R}^{6n}$) such that $(\vec{x}_1, \ldots, \vec{x}_n, \vec{v}_1, \ldots, \vec{v}_n) \mapsto (\vec{x}_1, \ldots, \vec{x}_n, \vec{f}_1, \ldots \vec{f}_n)$;

then (N, g, M, F) is a system of particles, in which

$$K(\vec{x}_1, \ldots, \vec{x}_n, \vec{v}_1, \ldots, \vec{v}_n) = \sum_{i=1}^{n} \frac{1}{2} m_i \|\vec{v}_i\|^2,$$

and, for any curve $c: (-\varepsilon, \varepsilon) \to N$,

$$(F \circ v = M \circ a) \iff (\vec{f}_i = m_i \vec{a}_i, \quad i = 1, \ldots, n).$$

Proposition 1.2 shows that we are dealing simultaneously with two metrics on N (which are differents, unless M is the identity). This is *inevitable* and the two metrics (g and m, according to our notation) must not be mistaken: the first one —g, which we call *spatial metric*— gives us the *work* $g(v,f)$ of a *force* $f \in T_xN$ on a *state* $v \in T_xN$ ($x \in N$); and the second metric —m, which we call *metric of masses*— gives us the *kinetic energy* K of the system ($K(v) = \frac{1}{2}m(v,v)$, $v \in TN$).

(2) CONSTRAINED SYSTEMS

We need some notation before we introduce the definition of *constraint (holonomic)* on a system of particles.

2.1 Notation. *Let* (N,g) *an Euclidean space,* $Q \subset N$ *a submanifold (boundaryless) of* N*, and* $c\colon(-\varepsilon,\varepsilon) \to Q$ *a curve on* Q*. Let* $TN|Q$ *denote the union of the* T_qN*, with* $q \in Q$*; and* $\pi\colon TN|Q \to Q$ *denote the vector bundle defined in the obvius way. As previously stated by convention, we shall omit inclusions* ($i\colon Q \to N$, $Ti\colon TQ \to TN$, $i'\colon TQ \to TN|Q$, $i''\colon TN|Q \to TN$, *etc) in the compositions and, moreover, we will use* $v\colon(-\varepsilon,\varepsilon) \to TQ$ *and* $a\colon(-\varepsilon,\varepsilon) \to TN|Q$ *to indicate the velocity and the acceleration (respec.) of a curve* $c\colon(-\varepsilon,\varepsilon) \to Q \subset N$.

2.2 Definition. *A pair* (Q, F_{con}) *is a* **constraint** *on a system of particles* (N,g,M,F_{ext}) *if:*

(i) $Q \subset N$ *is a boundaryless submanifold of* N *(not necessarily of codimension 1); and*

(ii) $F_{con}\colon TQ \to TN|Q$ *is a mapping that preserves the base point* ($\pi \circ F_{con} = \tau_Q$) *and such that for any given* $v_0 \in TQ$ *there is a curve* $c\colon(-\varepsilon,\varepsilon) \to Q$ *with* $v(0) = v_0$ *and* $(F_{con} + F_{ext}) \circ v = M \circ a$.

If (Q, F_{con}) *is a constraint, then we say that* F_{con} *is a* **force of constraint** *on* Q.

Our current aim is theorem 2.5.

2.3 Definition. *Let* (N,h) *be an Euclidean space* ($h \in \{g,m\}$) *and* $Q \subset N$ *be a submanifold (boundaryless) of* N*. Define* $\|^h\colon TN|Q \to TQ$ *by the orthogonal projection —via* h*— of* T_qN *onto* T_qQ ($q \in Q$) *and* $\perp^h\colon TN|Q \to TN|Q$ *by the projection —via* h*— of* T_qN *onto* $(T_qQ)^\perp \subset T_qN$ ($q \in Q$)*. In other words, vectors* $\|^h(X)$ *and* $\perp^h(X)$ *are the components of vector* $X \in T_qN$ *that are, respectively, parallel and normal to* T_qQ*, via metric* h.

2.4 Notation. *If* Q *is a manifold, let* $\mathfrak{X}(Q)$ *denote the set of* **vectors fields** *on* Q *(sections of* $\tau_Q\colon TQ \to Q$*), and* $\tilde{\mathfrak{X}}(Q)$ *denote the set of* **vector fields that depend on the velocity** *on* Q*, that is:*

$$\tilde{\mathfrak{X}}(Q) = \{X\colon TQ \to TQ \mid \tau_Q \circ X = \tau_Q\}.$$

We will now state the main theorem of this presentation.

2.5 Theorem. *Let* (N,g,M,F_{ext}) *be a system of particles and* $Q \subset N$ *a submanifold of* N *(not necessarily of codimension 1). For each field* $X \in \tilde{\mathfrak{X}}(Q)$ *there is one, and only one, force of constraint* $F_{con}\colon TQ \to TN|Q$ *such that* $\|^g \circ F_{con} = X$.

Note that theorem 2.5 above "parametrizes", through set $\tilde{\mathfrak{X}}(Q)$, the space of *all* forces of constraint on Q (including those associated to the various forms of *friction* and *viscous forces*, dependent or not on the velocity), which are determined by their component that is tangent —via metric g— to submanifold Q; that is, by their component that is able to do work on the system (prop. 1.4).

2.6 Corollary. *If (N, g, M, F_{ext}) is a system of particles and $Q \subset N$ a submanifold of N then there is one, and only one, force of constraint F_{con} on Q such that $\|^g \circ F_{con} = 0$.*

2.7 Definition. *We call the* **d'Alembertian force of constraint** *on Q the one determined by corollary 2.6 above. We will also say that (Q, F_{con}) is a* **d'Alembertian constraint** *if F_{con} is the d'Alembertian force of constraint on Q; that is, if $\|^g \circ F_{con} = 0$.*

The *d'Alembertian constraints* are the only constraints studied in the classical formalism[1], where the *principle of d'Alembert* and the *principle of virtual velocities* are used to determine the *d'Alembertian force of constraint* on a given restriction of constraint (see remark 2.16 ahead).

The rest of this section will be dedicated to proving theorem 2.5 (which depends on a sequency of lemmas) and other related results.

2.8 Notation. *If (N, h) is an Euclidean space ($h \in \{g, m\}$) and $Q \subset N$ is a submanifold of N, let $\overline{h}$ denote the Riemannian metric defined over Q by inclusion and let $i^*: T^*N|Q \to T^*Q$ denote the pull-back of covectors by the inclusion $i: Q \to N$. Moreover, in a manifold M ($M \in \{N, Q\}$) with Riemannian metric h ($h \in \{g, m, \overline{g}, \overline{m}\}$), let $h^\flat TM \to T^*M$ and $h^\sharp: T^*M \to TM$ denote the mappings defined by $[h^\flat(X)](Y) = h(X, Y)$ and $h^\sharp = (h^\flat)^{-1}$ (lowering and raising —respectively— of indices, via h).*

The following proposition is well known in differential geometry and will be stated (without proof) just to establish the notation.

2.9 Proposition. *If $Q \subset N$ is a submanifold of an Euclidean space (N, h) and $c: (-\varepsilon, \varepsilon) \to Q$ a curve on Q, then —with v and a defined as in 2.1— we have:*

(i) $\|^h \circ a = \nabla^{\overline{h}}_v v$, *and*

(ii) $\perp^h \circ a = \Delta^h \circ v$,

where $\nabla^{\overline{h}}$ is the Levi-Civita connection (on Q) of $\overline{h}$, and $\Delta^h: TQ \to TN|Q$ is a mapping defined only by metric h and by the inclusion $Q \subset N$ (does not depend on curve c).

2.10 Definition. *Let (N, g, M, F_{ext}) be a system of particles and $Q \subset N$ a submanifold of N. Besides the metrics $\overline{g}$ and $\overline{m}$ induced on Q according to 2.8, we also define tensor $\overline{M}: TQ \to TQ$ by $\overline{M} = \|^g \circ (M|_{TQ})$.*

2.11 Lemma. *Tensor $\overline{M}$ has the following properties:*

(i) $\overline{M} = \overline{g}^\sharp \circ \overline{m}^\flat$, *and*

(ii) $\|^g \circ (M|_{TN|Q}) = \overline{M} \circ \|^m$.

Proof. Note that $\|^g = \overline{g}^\sharp \circ i^* \circ (g^\flat|_{TN|Q})$. Then we have, for item (i),

$$\begin{aligned}\overline{M} &= \|^g \circ (M|_{TQ}) \\ &= \overline{g}^\sharp \circ i^* \circ (g^\flat|_{TN|Q}) \circ (M|_{TQ}) \\ &= \overline{g}^\sharp \circ i^* \circ (m^\flat|_{TQ}) \\ &= \overline{g}^\sharp \circ \overline{m}^\flat,\end{aligned}$$

and for item (ii):

$$\begin{aligned}\|^g \circ (M|_{TN|Q}) &= \overline{g}^\sharp \circ i^* \circ (g^\flat|_{TN|Q}) \circ (M|_{TN|Q}) \\ &= \overline{g}^\sharp \circ i^* \circ (m^\flat|_{TN|Q}) \\ &= \overline{g}^\sharp \circ (\overline{m}^\flat \circ \overline{m}^\sharp) \circ i^* \circ (m^\flat|_{TN|Q}) \\ &= (\overline{g}^\sharp \circ \overline{m}^\flat) \circ \|^m \\ &= \overline{M} \circ \|^m. \quad \blacksquare\end{aligned}$$

2.12 Lemma. *If (N, g, M, F_{ext}) is a system of particles, $Q \subset N$ is a submanifold of N and $c: (-\varepsilon, \varepsilon) \to Q$ is a curve on Q, then we have that:*

(i) $\|^g \circ M \circ a = \overline{M} \circ \nabla_v^{\overline{m}} v$, *and*

(ii) $\perp^g \circ M \circ a = M \circ \Delta^m \circ v + \Xi \circ \overline{M} \circ \nabla_v^{\overline{m}} v$,

where $\Xi: TQ \to TN$ *is the tensor defined by* $\Xi = (M \circ \overline{M}^{-1} - id_{TQ})$.

Proof. To prove (i), we use previous lemma (2.11(ii)):

$$(\|^g \circ M) \circ a = (\overline{M} \circ \|^m) \circ a = \overline{M} \circ (\|^m \circ a) = \overline{M} \circ \nabla_v^{\overline{m}} v.$$

To prove item (ii) recall that $\|^h + \perp^h = id_{TN|Q}$ $(h \in \{g, m\})$; therefore

$$\begin{aligned}\perp^g \circ M \circ a &= (id_{TN|Q} - \|^g) \circ M \circ a \\ &= (M - \|^g \circ M) \circ (\perp^m \circ a + \|^m \circ a) \\ &= (M - \overline{M} \circ \|^m) \circ (\Delta^m \circ v + \nabla_v^{\overline{m}} v) \\ &= M \circ \Delta^m \circ v + (M - \overline{M}) \circ \nabla_v^{\overline{m}} v \\ &= M \circ \Delta^m \circ v + (M \circ \overline{M}^{-1} - id_{TQ}) \circ \overline{M} \circ \nabla_v^{\overline{m}} v,\end{aligned}$$

where we used $\|^m \circ \Delta^m = 0$, $\|^g|_{TQ} = id_{TQ}$, and we assumed that $\overline{M}^{-1}$ exists (lemma 2.11(i) guarantees that $\overline{M}$ is invertible because $\overline{g}^\sharp$ and $\overline{m}^\flat$ are). ■

We are ready to prove theorem 2.5, but before we do that we would like to invite the reader to compare the way proposition 2.9 and lemma 2.12 were stated. Note that in 2.12(i) we projected vector $M \circ a$ orthogonally over TQ *via metric* g, but we obtained, as a result, $\overline{M}$ applied to the covariant derivative $\nabla_v^{\overline{m}} v$ *defined from metric* $\overline{m}$ (induced on Q by m). Moreover, in 2.9(ii), the normal component $\perp^h \circ a$ of the acceleration was determined only by the velocity vector v of the curve at this point; whereas in 2.12(ii) the normal component (via g) of $M \circ a$ depends also on the tangential acceleration $\nabla_v^{\overline{m}} v$ (unless Ξ vanishes at the point being considered, which usually does not happen, as stated in proposition 2.19 ahead).

2.13 Proof (of theorem 2.5). We shall prove initially the uniqueness of a force of constraint F_{con} such that $\|^g \circ F_{con} = X$, where $X \in \tilde{\mathfrak{X}}(Q)$ is a given field. If $c: (-\varepsilon, \varepsilon) \to Q$ is a curve that obeys $(F_{con} + F_{ext}) \circ v = M \circ a$ then

$$\|^g \circ M \circ a = \|^g \circ F_{con} \circ v + \|^g \circ F_{ext} \circ v, \text{ and} \tag{1}$$

$$\perp^g \circ M \circ a = \perp^g \circ F_{con} \circ v + \perp^g \circ F_{ext} \circ v, \tag{2}$$

taking lemma 2.12(ii) to equation (2):

$$\perp^g \circ F_{con} \circ v = M \circ \Delta^m \circ v + \Xi \circ \overline{M} \circ \nabla_v^{\overline{m}} v - \perp^g \circ F_{ext} \circ v,$$

by lemma 2.12(i) we have that

$$\perp^g \circ F_{con} \circ v = M \circ \Delta^m \circ v + \Xi \circ (\|^g \circ M \circ a) - \perp^g \circ F_{ext} \circ v,$$

then equation (1) guarantees that

$$\perp^g \circ F_{con} \circ v = M \circ \Delta^m \circ v + \Xi \circ (\|^g \circ F_{con} \circ v + \|^g \circ F_{ext} \circ v) - \perp^g \circ F_{ext} \circ v,$$

recalling that $\|^g \circ F_{con} = X$, and that for any given $v_0 \in TQ$ there is a curve $c: (-\varepsilon, \varepsilon) \to Q$ with $v(0) = v_0$ and $(F_{con} + F_{ext}) \circ v = M \circ a$, we conclude that

$$\perp^g \circ F_{con} = M \circ \Delta^m + \Xi \circ (X + \|^g \circ F_{ext}) - \perp^g \circ F_{ext}. \tag{3}$$

This expression for the normal component of F_{con} proves its uniqueness.

To prove the existence, assume a given $X \in \widetilde{\mathfrak{X}}(Q)$ and define

$$F_{con} = X + \perp^g \circ F_{con}\,,$$

with $\perp^g \circ F_{con}$ given by (3). This F_{con} is a force of constraint because for any $v_0 \in TQ$ there is a curve $c\colon(-\varepsilon,\varepsilon) \to Q$ that satisfies $v(0) = v_0$ and

$$\overline{M} \circ \nabla^{\overline{m}}_v v = (X + \|^g \circ F_{ext}) \circ v$$

(and besides, it is unique if we fix one of the ε's small enough); this curve c also satisfies equation (2) above (just invert the deduction of (3) from (2)); and finally, equations (1) and (2) together guarantee that $(F_{con}+F_{ext})\circ v = M \circ a$. ■

2.14 Corollary. *If (Q, F_{con}) is a constraint on (N, g, M, F_{ext}) then a curve $c\colon(-\varepsilon,\varepsilon) \to Q$ satisfies $(F_{con} + F_{ext}) \circ v = M \circ a$ if and only if*

$$\overline{m}^\flat \circ \nabla^{\overline{m}}_v v = i^* \circ g^\flat \circ (F_{con} + F_{ext}) \circ v\,.$$

Proof. In the previous proof (2.13) the equivalence between equation $(F_{con} + F_{ext}) \circ v = M \circ a$ and

$$\overline{M} \circ \nabla^{\overline{m}}_v v = \|^g \circ (F_{con} + F_{ext}) \circ v$$

was established, then it's enough to recall that $\overline{M} = \overline{g}^\sharp \circ \overline{m}^\flat$ and $\|^g = \overline{g}^\sharp \circ i^* \circ (g^\flat|_{TN|Q})$. ■

2.15 Corollary. *For any force of constraint F_{con} we have that*

$$\perp^g \circ F_{con} = M \circ \Delta^m + \Xi \circ \|^g \circ (F_{con} + F_{ext}) - \perp^g \circ F_{ext}\,,$$

particularly, for any d'Alembertian force of constraint,

$$F_{con} = M \circ \Delta^m + \Xi \circ \|^g \circ F_{ext} - \perp^g \circ F_{ext}\,.$$

Proof. The first expression is exactly the same as equation (3) deduced in proof 2.13, and the second expression follows easily from the definition of d'Alembertian force of constraint (2.7). ■

The first corollary (2.14) establishes *Lagrange's equation* in it's form prior to the introduction of the force potential[4], and will be discussed in the next section.

We have three remarks to make about corollary 2.15:

2.16 Remark. Comparing the expressions stated in corollary 2.15, we see that the normal component of *any* given force of constraint F_{con} on a system of particles (N, g, M, F_{ext}) is *identical* to the *d'Alembertian* force of constraint (on the same submanifold) on another system of particles (N, g, M, F'_{ext}) where $F'_{ext} = F_{ext} + f$, with $f|_{TQ} = \|^g \circ F_{con}$. In other words, we can always consider the component $\|^g \circ F_{con}$ of a non d'Alembertian force of constraint F_{con} as "external force", and then *identify* $\perp^g \circ F_{con}$ with the d'Alembertian force of constraint of the "new" system of particles.

2.17 Remark. The d'Alembertian force of constraint is usually described in terms of the *Lagrange's multipliers*, which are variables determined with the help of the equations of motion. Corollary 2.15 explicitly gives us the force of constraint as a function of the state $v \in TQ$ of the constrained system.

2.18 Remark. Corollary 2.15 describes the normal component of any given force of constraint F_{con} as a sum of three terms:

$$F_1 = M \circ \Delta^m$$
$$F_2 = \Xi \circ \|^g \circ (F_{con} + F_{ext})$$
$$F_3 = -\perp^g \circ F_{ext}$$

The "interpretation" of the terms F_1 and F_3 is clear: F_1 is the *centripetal force* —dependent on the velocity, on the mass, and on the curvature of Q in the direction of the motion— and F_3 is the part of the force of constraint that compensates for the component of the external force that tends to remove the system from the restriction of constraint $Q \subset N$.

The "origin" of the term F_2, in turn, is not so clear since it usually vanishes in "simple" systems (in example 1.5, if all masses m_i are equal then $F_2 = 0$; this follows easily from proposition 2.19 below). The several forces of constraint on a manifold differ, in what regards their normal component, because of term F_2 (which is the only term that depends on $\|^g \circ F_{con}$). The following proposition establishes the conditions in which the term F_2 of the several forces of constraint on a manifold vanishes in a given point, and example 2.20 presents a d'Alembertian constraint in wich $F_{con} = F_2 \neq 0$.

2.19 Proposition. *Let (N, g, M, F_{ext}) be a system of particles, $Q \subset N$ a submanifold of N and $q \in Q$ any given point of Q. The normal components $\perp^g \circ F_{con}$ (via g) of the forces of constraint on Q coincide at point q if and only if T_qQ is invariant through $M|_{T_qN}$; that is, if and only if $M(T_qQ) \subset T_qQ$.*

Proof. By corollary 2.15, the normal components of the forces of constraint coincide at point $q \in Q$ if and only if $\Xi|_{T_qQ} = 0$; since $\Xi = (M \circ \overline{M}^{-1} - id_{TQ})$ (lemma 2.12(ii)), this condition is equivalent to $M|_{T_qQ} = \overline{M}|_{T_qQ}$, which is the same as $M(T_qQ) \subset T_qQ$ (from the definition of $\overline{M} = \|^g \circ (M|_{TQ})$). ■

2.20 Example. Consider the system of particles defined by a pair of mass points with different masses ($m_1 \neq m_2$), that move along a straight line; a constant force (along the line) acts on both mass points:

(i) $(N, g) = \mathbb{R}^2$ with its usual Euclidean metric —$(x^1, x^2) \in \mathbb{R}^2$ is the configuration in which the particles are disposed x^1 and x^2 (respec.) units of length from a fixed origin;

(ii) $M(x^1, x^2, v^1, v^2) = (x^1, x^2, m_1 v^1, m_2 v^2)$, where m_1 and m_2 are real, positive constants, different from one another; and

(iii) $F_{ext}(x^1, x^2, v^1, v^2) = (x^1, x^2, f, f)$, where f is a real, nonzero constant.

Consider now submanifold Q of $N = \mathbb{R}^2$ defined by the constraint condition $x^1 - x^2 = \ell$ $(\ell \in \mathbb{R}_+)$; which describes, for instance, an "ideal rod" (nonextendable and with no mass) of length ℓ connected to the particles.

The tangent spaces to manifold Q (manifold of the constrained configurations) are, obviously, not invariant through M; and therefore the several forces of constraint on Q have not all normal components (via g) coincidental. Moreover, the external forces acting on the particles guarantee that the term F_2 (remark 2.18) of the d'Alembertian force of constraint is not zero; and so, for this force:

$$F_1 = F_3 = 0 \qquad \text{and} \qquad F_{con} = F_2 \neq 0.$$

(3) LAGRANGIAN MECHANICS

We shall now consider systems of particles subject to *conservative* forces[5].

3.1 Definition. *A system of particles (N, g, M, F_{ext}) is* **conservative** *if there is $V: N \to \mathbb{R}$ such that $F_{ext} = -g^\sharp \circ dV \circ \tau_N$. Fix one of the functions V, called* **potential energy**, *and define the* **total energy** *$E: TN \to \mathbb{R}$ of the system by $E = K + V \circ \tau_N$.*

3.2 Proposition (conservation of the energy). *Let (N, g, M, F_{ext}) be a conservative system of particles with potential energy V and total* *energy E. If $c\colon(-\varepsilon,\varepsilon)\to N$ is a curve on N that satisfies $F_{ext}\circ v = M\circ a$ then $\frac{d}{dt}(E\circ v)=0$.*

Proof. Just take $F_{ext} = -g^\sharp\circ dV\circ\tau_N$ to proposition 1.4(ii):

$$\frac{d}{dt}(K\circ v) = g(v, F_{ext}\circ v) = g(v, -g^\sharp\circ dV\circ\tau_N\circ v) = -\frac{d}{dt}(V\circ c),$$

therefore $\frac{d}{dt}(E\circ v) = \frac{d}{dt}(K\circ v) + \frac{d}{dt}(V\circ c) = 0$. ■

3.3 Proposition. *Let (Q, F_{con}) a d'Alembertian constraint on a conservative system of particles (N, g, M, F_{ext}) with potential energy $V\colon N\to\mathbb{R}$. A curve $c\colon(-\varepsilon,\varepsilon)\to Q$ satisfies $(F_{con}+F_{ext})\circ v = M\circ a$ if and only if*

$$\overline{m}^\flat\circ\nabla^{\overline{m}}_v v = -d\overline{V}\circ c,\quad \textit{where}\quad \overline{V} = V|_Q.$$

Proof. By corollary 2.14 , we need only to prove the equivalence between the above equation and

$$\overline{m}^\flat\circ\nabla^{\overline{m}}_v v = i^*\circ g^\flat\circ(F_{con}+F_{ext})\circ v,$$

which is easy because $\|^g\circ F_{con}=0$ and $F_{ext} = -g^\sharp\circ dV\circ\tau_N$ imply that

$$\begin{aligned} i^*\circ g^\flat\circ(F_{con}+F_{ext})\circ v &= -i^*\circ g^\flat\circ g^\sharp\circ dV\circ\tau_N\circ v\\ &= -i^*\circ dV\circ c\\ &= -d\overline{V}\circ c. \quad ■\end{aligned}$$

The condition obtained in the proposition above, *in which metric g is not present*, is equivalent to Lagrange's equation $X_E\in\mathfrak{X}(TQ)$ for the *lagrangian system* $(Q, L = K|_{TQ} - \overline{V}\circ\tau_Q)$. The proof of this statement can be found in Abraham and Marsden[6], as well as the definitions and results necessary for its precise formulation.

The correspondence established by proposition 3.3 above between the Newtonian and Lagrangian formalisms for classical mechanics concludes the program described in the introduction.

References

[1] J. L. Lagrange: *Mécanique Analytique*, Paris 1788.

[2] E. T. Whittaker: *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, fourth edition, Dover, New York 1944.

[3] R. Abraham and J. E. Marsden: *Foundations of Mechanics*, second edition, Benjamin-Cummings, Reading, Massachusetts 1982.

[4] see, for instance, ref. [2], sections 26 (p. 34) and 28 (p. 39).

[5] compare with ref. [2], section 27 (p. 38).

[6] ref. [3], proposition 3.7.4 (p. 226).